Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–15.
ISSN-0037-8712
doi:10.5269/bspm.67554

Stability of Systems of Rational Difference Equations

Alaa E. Hamza, Abdullah Alshamasi and Abdeljabbar Ghanmi*

ABSTRACT: Motivated by some recent results concerning the stability of second-order systems of nonlinear difference equations, we aim in this paper to investigate the global asymptotic stability of a third-order two-dimensional system. Furthermore, we discuss the convergence of solutions of this system. Moreover, we establish two asymptotic relations for solutions. Finally, many illustrative examples are given

Key Words: Systems of rational difference equations, stability, instability, global asymptotic stability.

Contents

1. Introduction and preliminaries

Difference equations have many applications in a variety of disciplines such as economy, mathematical biology, social sciences, physics, etc. This is why there has been a great interest in studying the behavior of solutions of difference equations or discrete dynamical systems. Surely, the theory of difference equations will continue to play an important role in mathematics as a whole. Although difference equations are sometimes very simple in their forms, they are extremely difficult to understand thoroughly the behavior of their solutions. Rational difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations. It is very interesting to investigate the behavior of solutions of rational difference equations and to discuss the stability of their equilibrium points. Recently, there has been published quite a lot of works concerning the behavior of positive solutions of systems of difference equations [1,2,3,4,5,6,7,9,10]. Kurbanli et al. [1] investigated the behavior of the positive solutions of the following system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1},\ n = 0, 1, \cdots,$$

where the initial conditions are arbitrary real numbers. Qi Wang et al. [2] investigated the positive solutions of the system of two higher-order rational difference equations

$$x_{n+1} = \frac{x_{n-2k+1}}{Ay_{n-k+1}x_{n-2k+1} + \alpha}, y_{n+1} = \frac{y_{n-2k+1}}{Bx_{n-k+1}y_{n-2k+1} + \beta},\ n = 0, 1, \cdots,$$

* Corresponding author.
2010 *Mathematics Subject Classification:* 39A10.
Submitted March 17, 2023. Published September 01, 2025

where k is a positive integer, the parameters A, B, α, β and the initial conditions are positive real numbers. Cinar and Yalcinkaya [3] obtained the positive solution of the difference equation system

$$x_{n+1} = \frac{1}{z_n}, y_{n+1} = \frac{x_n}{x_{n-1}}, z_{n+1} = \frac{1}{z_{n-1}}.$$

Ozban [4] studied the behavior of positive solutions of the difference equation system:

$$x_n = \frac{a}{y_{n-3}}, y_n = \frac{by_{n-3}}{x_{n-q}y_{n-q}}, \ n = 0, 1, \cdots$$

Also, Ozban [5] studied the positive solutions of the system of rational difference equations:

$$x_{n+1} = \frac{1}{y_{n-k}}, y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m+k}}, \ n = 0, 1, \cdots$$

Mehmet Gumus [7] studied the global stability of positive solutions that converge to the equilibrium point of the system of difference equations in the modeling competitive populations in the form

$$x^{(1)}_{n+1} = \frac{\alpha x^{(1)}_{n-2}}{\beta + \gamma \prod_{i=0}^{2} x^{(2)}_{n-i}}, x^{(2)}_{n+1} = \frac{\alpha_1 x^{(2)}_{n-2}}{\beta_1 + \gamma_1 \prod_{i=0}^{2} x^{(1)}_{n-i}}, n = 0, 1, \cdots,$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ are positive numbers and the initial conditions $x^{(1)}_{-i}, x^{(2)}_{-i}$ are arbitrary non-negative numbers for $i \in \{0, 1, 2\}$. Similar nonlinear systems of difference equations were investigated. See for instance, [12,13,14,15,16,17,18,19,20]. Hongmei Bao in [8] investigated the stability of a two-dimensional second-order system of rational difference equations of the form

$$x_{n+1} = \frac{x_n}{A + y_n y_{n-1}}, y_{n+1} = \frac{y_n}{B + x_n x_{n-1}}, \ n = 0, 1, \cdots,$$

where $A, B \in (0, \infty)$ and the initial values $x_{-i}, y_{-i} \in (0, \infty), i = 0, 1$.

Motivated by the above results, our aim in this paper is to study the global asymptotic stability behavior of the third-order system of rational difference equations

$$x_{n+1} = \frac{x_n}{A + y_n y_{n-1} y_{n-2}}, y_{n+1} = \frac{y_n}{B + x_n x_{n-1} x_{n-2}}, \ n = 0, 1, \cdots, \tag{1.1}$$

where $A, B \in (0, \infty)$ and the initial values $x_{-i}, y_{-i} \in [0, \infty), i = 0, 1, 2$. For the theory of stability of difference equations, we refer the reader to the monographs [11], [21] and [22]. We only mention the following theorem because we mainly depend on it in proving our results.

Theorem 1.1 *Let I be an interval. Consider the system*

$$Z_{n+1} = E(Z_n), n = 0, 1, \ldots,$$

where $E(x_1, \ldots, x_k) : I^k \to I^k$ has continuous partial derivatives with respect to $x_i \in I$. Assume $\bar{X} \in I^k$ is an equilibrium point of the equation, that is $\bar{X}$ is a fixed point of E. Then the following statements are true

(i) If all eigenvalues of the Jacobian $J(E)$ of E, evaluated at $\bar{X}$ lie inside the open unit disk, then $\bar{X}$ is stable.

(ii) If at least one eigenvalue lies outside the open unit disk, then $\bar{X}$ is unstable.

For $E = (E_1, E_2, \ldots, E_k)^T$, the Jacobian $J(E)$ of E is defined by

$$J(E) = \begin{pmatrix} \frac{\partial E_1}{\partial x_1} & \frac{\partial E_1}{\partial x_2} & \ddots & \frac{\partial E_1}{\partial x_k} \\ \frac{\partial E_2}{\partial x_1} & \frac{\partial E_2}{\partial x_2} & \ddots & \frac{\partial E_2}{\partial x_k} \\ \ddots & & & \\ \ddots & & & \\ \frac{\partial E_k}{\partial x_1} & \frac{\partial E_k}{\partial x_2} & \ddots & \frac{\partial E_k}{\partial x_k}. \end{pmatrix} \tag{1.2}$$

Consider the k-dimensional general third-order system

$$\begin{cases} x_{n+1}^{(1)} = f_1(x_n^{(1)}, x_{n-1}^{(1)}, x_{n-2}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, x_{n-2}^{(2)}, \ldots, x_n^{(k)}, x_{n-1}^{(k)}, x_{n-2}^{(k)}), \\ x_{n+1}^{(2)} = f_2(x_n^{(1)}, x_{n-1}^{(1)}, x_{n-2}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, x_{n-2}^{(2)}, \ldots, x_n^{(k)}, x_{n-1}^{(k)}, x_{n-2}^{(k)}), \ n = 0, 1, 2, \ldots. \\ \vdots \\ x_{n+1}^{(k)} = f_k(x_n^{(1)}, x_{n-1}^{(1)}, x_{n-2}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, x_{n-2}^{(2)}, \ldots, x_n^{(k)}, x_{n-1}^{(k)}, x_{n-2}^{(k)}), \end{cases} \tag{1.3}$$

where $f_1, f_2, \ldots, f_k : I^{3k} \to I$ has continuous partial derivatives. Here I is an interval (bounded or unbounded).

Define the function F by

$$F(x_1, x_2, \ldots, x_{3k}) = \begin{pmatrix} f_1(x_1, \ldots, x_{3k}) \\ x_1 \\ x_2 \\ f_2(x_1, \ldots, x_{3k}) \\ x_4 \\ x_5 \\ f_3(x_1, \ldots, x_{3k}) \\ \vdots \\ f_k(x_1, \ldots, x_{3k}) \\ x_{3k-2} \\ x_{3k-1} \end{pmatrix} \tag{1.4}$$

The Jacobian matrix J(F) of F is given by

$$J(F) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \frac{\partial f_1}{\partial x_{3k}} \\ 1 & 0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \frac{\partial f_2}{\partial x_{3k}} \\ 0 & 0 & 0 & 1 & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \ldots & \ldots & \ldots & 0 \\ & \ddots & & \ddots & & \ddots & & \ddots & & \ddots \\ \frac{\partial f_k}{\partial x_1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \frac{\partial f_k}{\partial x_{3k}} \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 & 0 & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & 1 & 0 \end{pmatrix}. \tag{1.5}$$

A point $\bar{X} = (\bar{x}_1, \ldots, \bar{x}_k) \in I^k$ is called an equilibrium point of System (1.3) if $\bar{x}_1 = f_1(\bar{x}_1, \bar{x}_1, \bar{x}_1, \bar{x}_2, \bar{x}_2, \bar{x}_2, \ldots, \bar{x}_k, \bar{x}_k, \bar{x}_k), \ldots, \bar{x}_k = f_k(\bar{x}_1, \bar{x}_1, \bar{x}_1, \bar{x}_2, \bar{x}_2, \bar{x}_2, \ldots, \bar{x}_k, \bar{x}_k, \bar{x}_k)$. We recall the concept of global asymptotic stability of equilibrium points. See [21] and [22].

Definition 1.1 *An equilibrium point* $\bar{X} = (\bar{x}_1, \ldots, \bar{x}_k) \in I^k$ *of System* (1.3) *is said to be*

(i) *stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for any initial conditions $x_{-i}^{(j)} \in I, i = 0,1,2, j = 1,2,\dots,k$, we have*

$$\max_{i=0}^{2} |x_{-i}^{(j)} - \bar{x}_j| < \delta, j = 1,\dots,k \ \Rightarrow \max_{j=1}^{k} |x_n^{(j)} - \bar{x}_j| < \epsilon, n \in \mathbb{N}.$$

(ii) *attractor if for any initial conditions $x_{-i}^{(j)} \in I, i = 0,1,2, j = 1,2,\dots,k$, $\lim_{n\to\infty} x_n^{(j)} = \bar{x}_j, j = 1,\dots,k$.*

(iii) *globally asymptotically stable if it is stable and attractor.*

(iv) *unstable if it is not stable.*

Theorem 1.2 *[11], [22]. Assume that $\bar{X} = (\bar{x}_1,\dots,\bar{x}_k) \in I^k$ is an equilibrium point of System (1.3). The following statement are true:*

(i) *If all eigenvalues of $J(F)$, evaluated at $(x_1, x_2, \dots, x_{3k}) = (\bar{x}_1, \bar{x}_1, \bar{x}_1, \dots, \bar{x}_k, \bar{x}_k, \bar{x}_k)$ lie inside the open unit disk, then $\bar{X}$ is stable.*

(ii) *If at least one eigenvalue lies outside the open unit disk, then $\bar{X}$ is unstable.*

Our main objective of this paper is to study asymptotic behavior of solutions of the following system

$$x_{n+1} = \frac{x_n}{A + y_n y_{n-1} y_{n-2}}, y_{n+1} = \frac{y_n}{B + x_n x_{n-1} x_{n-2}}, \ n = 0,1,\dots. \tag{1.6}$$

We apply Theorem 1.2 for the function F defined by

$$F(x_1,x_2,x_3,x_4,x_5,x_6) = \begin{pmatrix} f_1(x_1,x_2,x_3,x_4,x_5,x_6) \\ x_1 \\ x_2 \\ f_2(x_1,x_2,x_3,x_4,x_5,x_6) \\ x_4 \\ x_5 \end{pmatrix}, \tag{1.7}$$

where

$$\begin{aligned} f_1(x_1,x_2,x_3,x_4,x_5,x_6) &= \frac{x_1}{A + x_4 x_5 x_6}, \\ f_2(x_1,x_2,x_3,x_4,x_5,x_6) &= \frac{x_4}{B + x_1 x_2 x_3}. \end{aligned}$$

So,

$$J(F) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} & \frac{\partial f_1}{\partial x_5} & \frac{\partial f_1}{\partial x_6} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} & \frac{\partial f_2}{\partial x_5} & \frac{\partial f_2}{\partial x_6} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.8}$$

The Jacobian $J = J(F)$ of F evaluated at the equilibrium point $(\bar{x}_1, \bar{x}_2)$ is given by

$$J = \begin{pmatrix} \frac{1}{A+\bar{x}_2^3} & 0 & 0 & \frac{-\bar{x}_1\bar{x}_2^2}{(A+\bar{x}_2^3)^2} & \frac{-\bar{x}_1\bar{x}_2^2}{(A+\bar{x}_2^3)^2} & \frac{-\bar{x}_1\bar{x}_2^2}{(A+\bar{x}_2^3)^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-\bar{x}_2\bar{x}_1^2}{(B+\bar{x}_1^3)^2} & \frac{-\bar{x}_2\bar{x}_1^2}{(B+\bar{x}_1^3)^2} & \frac{-\bar{x}_2\bar{x}_1^2}{(B+\bar{x}_1^3)^2} & \frac{1}{B+\bar{x}_1^3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.9}$$

Our paper is organized as follows. In Section 2, we find the equilibrium points of System (1.6). Section 3 is devoted to studying the stability of the zero equilibrium point. In Section 4, we invetigate the convergence of solutions. We establish two important asymptotic relations of solutions in Section 5. Finally, many illustrative examples in Section 6, are presented.

2. Equilibrium points of the system

In this section, we will find the equilibrium points of (1.6). For an equilibrium point $\bar{X} = (\bar{x}_1, \bar{x}_2)$, one can readily see that $\bar{x}_1$ and $\bar{x}_2$ satisfies

$$\begin{cases} \bar{x}_1 = \frac{\bar{x}_1}{A+\bar{x}_2^3} \\ \\ \bar{x}_2 = \frac{\bar{x}_2}{B+\bar{x}_1^3}. \end{cases} \tag{2.1}$$

We summerize the results of this section in the following Lemma. The proof is straightforward and will be omitted.

Lemma 2.1 *The following statements are true:*

(i) If $A = B = 1$, then $\bar{X}$ has the two forms $(a, 0)$ and $(0, a)$, where $a \geq 0$.

(ii) If $A \neq 1$ and $B \neq 1$, then (1.6) has two equilibrium points $(0, 0)$ and $(\sqrt[3]{1-B}, \sqrt[3]{1-A})$.

(iii) If $A \neq 1$ and $B = 1$, then (1.6) has two equilibrium points $(0, 0)$ and $(0, \sqrt[3]{1-A})$.

(iv) If $A = 1$ and $B \neq 1$, then (1.6) has two equilibrium points $(0, 0)$ and $(\sqrt[3]{1-B}, 0)$.

3. Stability of the zero solution

In this section, we investigate the stability of the zero solution.

Theorem 3.1 *The following statements are true :*

(i) If either A or B is less than 1, then the equilibrium point $(0, 0)$ of System (1.6) is unstable.

(ii) If both A and B are greater than 1, then the equilibrium point $(0, 0)$ of System (1.6) is stable.

(iii) If one of them is equal to 1 and the other is less than 1, then $(0, 0)$ of System (1.6) is unstable.

Proof: The Jacobian J of F at $\bar{X} = (0, 0)$ is given by

$$J = \begin{pmatrix} \frac{1}{A} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{B} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.1}$$

The characteristic equation of J is

$$|J - \lambda I| = \begin{vmatrix} \frac{1}{A} - \lambda & 0 & 0 & 0 & 0 & 0 \\ 1 & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 1 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{B} - \lambda & 0 & 0 \\ 0 & 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & -\lambda \end{vmatrix} = 0, \tag{3.2}$$

that is, $(\frac{1}{A} - \lambda)(-\lambda)(-\lambda)(\frac{1}{B} - \lambda)(-\lambda)(-\lambda) = 0$. It follows that

$$\lambda = 0, \lambda = 0, \lambda = 0, \lambda = \frac{1}{A}, \lambda = \frac{1}{B}. \tag{3.3}$$

(i) If either A or B is less than 1, then one root of the characteristic equation lie outside the unit disk. Therefore, the zero equilibrium point is unstable.

(ii) If both of A and B are greater than 1, all roots of the characteristic equation lie inside the unit disk. Therefore, the zero equilibrium point is stable.

(iii) In this case, one root lies outside the unit disk. Therefore, the zero equilibrium point is unstable.

□

Theorem 3.2 *Suppose that $A > 1$ and $B > 1$. Then the zero equilibrium point is globally asymptotically stable.*

Proof: From (ii) of Theorem 3.1, the zero equilibrium point is stable. Let (x_n, y_n) be a solution of (1.6). Then

$$x_{n+1} = \frac{x_n}{A + y_n y_{n-1} y_{n-2}} \leq \frac{x_n}{A}, \quad y_{n+1} = \frac{y_n}{B + x_n x_{n-1} x_{n-2}} \leq \frac{y_n}{B}.$$

Then $x_n \leq \dfrac{x_0}{A^n}$ and $y_n \leq \dfrac{y_0}{B^n}$. This implies that the sequences $\{x_n\}$ and $\{y_n\}$ are decreasing to zero. Therefore, the zero equilibrium point is globally asymptotically stable. □

4. Convergence of solutions

We investigate in this section the convergence of solutions.

Theorem 4.1 *If $A = 1$ and $B < 1$, then every solution (x_n, y_n) of System (1.6), with $y_n \neq 0, n \in \mathbb{N}$, satisfies*

(i) $\lim_{n\to\infty} x_n = a \in [0, \infty)$.

(ii)

$$\lim_{n\to\infty} y_n = \begin{cases} 0 & \text{if } a > 0 \\ \infty & \text{if } a = 0 \end{cases}.$$

Proof: Let (x_n, y_n) be a solution of System (1.6), with $y_n \neq 0, n \in \mathbb{N}$. Then

$$x_{n+1} = \frac{x_n}{1 + y_n y_{n-1} y_{n-2}} \tag{4.1}$$

$$\leq x_n, n \in N_0.$$

We deduce that

$$\lim_{n\to\infty} x_n = a \in [0, \infty).$$

On the other hand, from the relation

$$y_{n+1} = \frac{y_n}{B + x_n x_{n-1} x_{n-2}},$$

we get

$$\frac{y_{n+1}}{y_n} = \frac{1}{B + x_n x_{n-1} x_{n-2}}, \tag{4.2}$$

from which we obtain

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \frac{1}{B + a^3}. \tag{4.3}$$

We have the following two cases

(1) $a > 0$. In this case $B + a^3 \geq 1$. Otherwise, if $B + a^3 < 1$, then $\lim_{n\to\infty} \frac{y_{n+1}}{y_n} > 1$. Then there is $\alpha > 1$ such that $\frac{y_{n+1}}{y_n} > \alpha$ for sufficiently large n. It follows that $\lim_{n\to\infty} y_n = \infty$. Taking $n \to \infty$ in (4.1), we get $\lim_{n\to\infty} x_n = 0$ which is a contradiction. So $B + a^3 \geq 1$. First, assume $B + a^3 > 1$. Then

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \frac{1}{B + a^3} < 1.$$

Hence there is $\beta < 1$ such that $\frac{y_{n+1}}{y_n} < \beta$. This implies that $\lim_{n\to\infty} y_n = 0$. Second, assume $B + a^3 = 1$. Since $\{x_n\}$ is decreasing, then from (4.2), the sequence $\{\frac{y_{n+1}}{y_n}\}$ is increasing to 1. This means that the sequence $\{y_n\}$ is decreasing to a non-negative number b. From relation (4.1), and taking the limit as $n \to \infty, b = 0$.

(2) $a = 0$. This case implies $B + a^3 = B < 1$ which in turn ensures $\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \frac{1}{B} > 1$. Therefore, $\lim_{n\to\infty} y_n = \infty$.

□

Theorem 4.2 *If $A = 1$ and $B = 1$, then every solution (x_n, y_n) of (1.6), we have*

(i) $\lim_{n\to\infty} x_n = a$,

(ii) $\lim_{n\to\infty} y_n = b$,

for some $a, b \in [0, \infty)$ such that $ab = 0$.

Proof: Let (x_n, y_n) be a solution of (1.6). Then

$$\begin{aligned} x_{n+1} &= \frac{x_n}{1 + y_n y_{n-1} y_{n-2}} \\ &\leq x_n, n \in N_0. \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} y_{n+1} &= \frac{y_n}{1 + x_n x_{n-1} x_{n-2}} \\ &\leq y_n, n \in N_0. \end{aligned} \tag{4.5}$$

Hence $\{x_n\}$ and $\{y_n\}$ are decreasing to some non-negative numbers a and b respectively. Taking the limit of both of equations (4.4) and (4.5), we get the equations

$$a = \frac{a}{1 + b^3},$$

and

$$b = \frac{b}{1 + a^3}.$$

This implies that $ab = 0$. □

Theorem 4.3 *If $A > 1$ and $B = 1$, then every solution (x_n, y_n) of (1.6), satisfies*

(i) $\lim_{n\to\infty} x_n = 0$,

(ii) $\lim_{n\to\infty} y_n = a \in [0, \infty)$.

Proof: Let (x_n, y_n) be a solution of (1.6). Then

$$x_{n+1} = \frac{x_n}{A + y_n y_{n-1} y_{n-2}}, n \in \mathbb{N}_0. \tag{4.6}$$

By induction, we deduce

$$x_n \leq \frac{x_0}{A^n}, n \in \mathbb{N}_0, \tag{4.7}$$

and consequently $\{x_n\}$ is decreasing to 0. This prove (i). We have

$$\begin{aligned} y_{n+1} &= \frac{y_n}{1 + x_n x_{n-1} x_{n-2}} \\ &\leq y_n, n \in \mathbb{N}_0. \end{aligned}$$

This implies that the sequence $\{y_n\}$ is decreasing. Therefore, $\lim_{n\to\infty} y_n = a \in [0, \infty)$.

□

Theorem 4.4 *Assume* $A < B < 1$. *Let* $\alpha \in (\sqrt[3]{1-B}, \sqrt[3]{1-A})$. *If the initial conditions* $x_{-i} \in [\alpha, \infty), i = 0, 1, 2$ *and* $y_{-i} \in [0, \alpha], i = 0, 1, 2$, *then*

(i) $\lim_{n\to\infty} x_n = \infty$,

(ii) $\lim_{n\to\infty} y_n = 0$.

Proof: Assume that $x_{-i} \in [\alpha, \infty), i = 0, 1, 2$ and $y_{-i} \in [0, \alpha], i = 0, 1, 2$. We can prove by induction that

$$x_n \geq \frac{\alpha}{(A + \alpha^3)^n}, n = 0, 1, \ldots$$

and

$$y_n \leq \frac{\alpha}{(B + \alpha^3)^n}, n = 0, 1, \ldots.$$

This implies (i) and (ii). □

Theorem 4.5 *Assume* $B < A < 1$. *Let* $\alpha \in (\sqrt[3]{1-A}, \sqrt[3]{1-B})$. *If the initial conditions* $x_{-i} \in [0, \alpha], i = 0, 1, 2$ *and* $y_{-i} \in [\alpha, \infty), i = 0, 1, 2$, *then*

(i) $\lim_{n\to\infty} x_n = 0$,

(ii) $\lim_{n\to\infty} y_n = \infty$.

Proof: Assume that $x_{-i} \in [0, \alpha], i = 0, 1, 2$ and $y_{-i} \in [\alpha, \infty), i = 0, 1, 2$. We can prove by induction that

$$x_n \leq \frac{\alpha}{(A + \alpha^3)^n}, n = 0, 1, \ldots$$

and

$$y_n \geq \frac{\alpha}{(B + \alpha^3)^n}, n = 0, 1, \ldots.$$

This implies (i) and (ii). □

Theorem 4.6 *If* $A > 1$ *and* $B < 1$, *then any solution* (x_n, y_n) *of* (1.6) *with* $y_n \neq 0, n \in \mathbb{N}$ *satisfies*

(i) $\lim_{n\to\infty} x_n = 0$,

(ii) $\lim_{n\to\infty} y_n = \infty$.

Proof: Assume (x_n, y_n) is a solution of (1.6) with $y_n \neq 0, n \in \mathbb{N}$. By (4.7), $\lim_{n\to\infty} x_n = 0$. From the following relation

$$\frac{y_{n+1}}{y_n} = \frac{1}{B + x_n x_{n-1} x_{n-2}}, \tag{4.8}$$

we obtain

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \frac{1}{B} > 1. \tag{4.9}$$

This implies that

$$\lim_{n\to\infty} y_n = \infty.$$

□

In a similar way, one can see the following result

Theorem 4.7 *If $A < 1$ and $B > 1$, then any solution (x_n, y_n) of (1.6) with $x_n \neq 0, n \in \mathbb{N}$ satisfies*

(i) $\lim_{n\to\infty} x_n = \infty$,

(ii) $\lim_{n\to\infty} y_n = 0$.

5. Asymptotic relations for solutions

For $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, we denote by

$$\|x\| = \max_{1\leq i\leq k} |x_i|.$$

Also, for a $k \times k$-matrix C, with eigenvalues $\lambda_i, i = 1, \ldots, k$, the spectral radius $\rho(C)$ of C is defined to be

$$\rho(C) = \max_{1\leq i\leq k} |\lambda_i|.$$

Let J be the Jacobian matrix of F, evaluated at the zero equilibrium point of (1.6), which is defined in (3.1). In the following theorem we establish an asymptotic inequality. In the following theorem we establish an asymptotic inequality.

Theorem 5.1 *Assume that $A > 1$ and $B > 1$. If $z_n = (x_n, y_n) \in (0,\infty)^2$ is a solution of (1.6), then the following asymptotic inequality holds*

$$\min(\frac{1}{A}, \frac{1}{B}) \leq \liminf \frac{\|z_{n+1}\|}{\|z_n\|} \leq \limsup \frac{\|z_{n+1}\|}{\|z_n\|} \leq \max(\frac{1}{A}, \frac{1}{B}).$$

Proof: Assume $z_n = (x_n, y_n) \in (0,\infty)^2$ is a solution of (1.6). We have

$$\frac{x_{n+1}}{x_n} = \frac{1}{A + y_n y_{n-1} y_{n-2}},$$

and

$$\frac{y_{n+1}}{y_n} = \frac{1}{B + x_n x_{n-1} x_{n-2}}.$$

Since $\{x_n\}$ and $\{y_n\}$ are decreasing to 0, then the sequence $\{\frac{x_{n+1}}{x_n}\}$ and $\{\frac{y_{n+1}}{y_n}\}$ are increasing to $\frac{1}{A}$ and $\frac{1}{B}$ respectively. Set

$$C_{1,n} = \frac{1}{A} - \frac{x_{n+1}}{x_n}, \quad C_{2,n} = \frac{1}{B} - \frac{y_{n+1}}{y_n} \tag{5.1}$$

Hence, $\lim_{n\to\infty} C_{1,n} = 0$ and $\lim_{n\to\infty} C_{2,n} = 0$. From relation (5.1), we obtain

$$x_{n+1} = x_n \left(\frac{1}{A} - C_{1,n}\right), \quad y_{n+1} = y_n \left(\frac{1}{B} - C_{2,n}\right).$$

Consequently,

$$\begin{aligned}
\|z_{n+1}\| &= \max(x_{n+1}, y_{n+1}) \\
&= \max\left(x_n(\frac{1}{A} - C_{1,n}), y_n(\frac{1}{B} - C_{2,n})\right) \\
&\leq \max\left(\|z_n\|(\frac{1}{A} - C_{1,n}), \|z_n\|(\frac{1}{B} - C_{2,n})\right) \\
&= \|z_n\| \max\left(\frac{1}{A} - C_{1,n}, \frac{1}{B} - C_{2,n}\right) \\
&\leq \|z_n\| \max(\frac{1}{A}, \frac{1}{B}).
\end{aligned}$$

This implies

$$\limsup \frac{\|z_{n+1}\|}{\|z_n\|} \leq \max(\frac{1}{A}, \frac{1}{B}). \tag{5.2}$$

Again by relation (5.1), we have

$$x_n = \frac{x_{n+1}}{\frac{1}{A} - C_{1,n}}, \quad y_n = \frac{y_{n+1}}{\frac{1}{B} - C_{2,n}}.$$

From which we deduce

$$\begin{aligned}
\|z_n\| &= \max(x_n, y_n) \\
&= \max\left(\frac{x_{n+1}}{\frac{1}{A} - C_{1,n}}, \frac{y_{n+1}}{\frac{1}{B} - C_{2,n}}\right) \\
&\leq \max\left(\frac{\|z_{n+1}\|}{\frac{1}{A} - C_{1,n}}, \frac{\|z_{n+1}\|}{\frac{1}{B} - C_{2,n}}\right) \\
&= \|z_{n+1}\| \max\left(\frac{1}{\frac{1}{A} - C_{1,n}}, \frac{1}{\frac{1}{B} - C_{2,n}}\right).
\end{aligned}$$

So,

$$\limsup \frac{\|z_n\|}{\|z_{n+1}\|} \leq \max(A, B).$$

As a direct consequence we get

$$\liminf \frac{\|z_{n+1}\|}{\|z_n\|} = \frac{1}{\limsup \frac{\|z_n\|}{\|z_{n+1}\|}} \geq \frac{1}{\max(A, B)} = \min(\frac{1}{A}, \frac{1}{B}). \tag{5.3}$$

Relations (5.2) and (5.3) imply the required result. □

Corollary 5.1 *Assume that $A = B > 1$, If $z_n = (x_n, y_n) \in (0, \infty)^2$ is a solution of (1.6), then the following asymptotic relations*

$$\lim_{n\to\infty} \frac{\|z_{n+1}\|}{\|z_n\|} = \rho(J), \tag{5.4}$$

and

$$\lim_{n\to\infty} \sqrt[n]{\|z_n\|} = \rho(J), \tag{5.5}$$

hold.

Proof: The statement Relation (5.4) is a direct consequence of Theorem 5.1. By (5.4), we obtain (5.5). See [23]. □

In the following Theorem, we prove that relations (5.4) and (5.5) hold, for any coefficients $A, B > 1$.

Theorem 5.2 *Assume that $A > 1$ and $B > 1$. If $z_n = (x_n, y_n) \in (0,\infty)^2$ is a solution of (1.6), then relations (5.4) and (5.5) hold.*

Proof: If $A = B$, then by Corollary 5.1, relations (5.4) and (5.5) are true. Assume now $1 < A < B$, and assume $z_n = (x_n, y_n) \in (0,\infty)^2$ is a solution of (1.6). We have

$$x_{n+1} = \frac{x_n}{A + y_n y_{n-1} y_{n-2}},$$

and

$$y_{n+1} = \frac{y_n}{B + x_n x_{n-1} x_{n-2}}.$$

Consequently,

$$\begin{aligned}\frac{x_{n+1}}{y_{n+1}} &= \frac{x_n}{A + y_n y_{n-1} y_{n-2}} \quad \frac{B + x_n x_{n-1} x_{n-2}}{y_n} \\ &= \frac{x_n}{y_n} \quad \frac{B + x_n x_{n-1} x_{n-2}}{A + y_n y_{n-1} y_{n-2}}.\end{aligned}$$

Letting $L_n = \dfrac{x_n}{y_n}$, we get

$$\frac{L_{n+1}}{L_n} = \frac{B + x_n x_{n-1} x_{n-2}}{A + y_n y_{n-1} y_{n-2}}.$$

Then $\lim_{n\to\infty} \dfrac{L_{n+1}}{L_n} = \dfrac{B}{A} > 1$, since $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$. Hence $\lim_{n\to\infty} L_n = \infty$. It follows that

$$\max(x_n, y_n) = x_n,$$

for sufficiently large n. So,

$$\lim_{n\to\infty} \frac{\|z_{n+1}\|}{\|z_n\|} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{A} = \rho(J).$$

The case, where $1 < B < A$, can be treated similarly. As a direct consequence we get (5.5). See [23]. □

6. Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider some interesting numerical examples in this section.

Example 1

Consider the System (1.6) with the initial conditions $x_{-2} = 0.6, x_{-1} = 1.4, x_0 = 1.7, y_{-2} = 1.6, y_{-1} = 2.4, y_0 = 2.7$. Moreover, choosing the parameters $A = 1.8 > 1$ and $B = 1.5 > 1$. Then System (1.6) can be written as

$$x_{n+1} = \frac{x_n}{1.8 + y_n y_{n-1} y_{n-2}}, \;\; y_{n+1} = \frac{y_n}{1.5 + x_n x_{n-1} x_{n-2}}. \tag{6.1}$$

The plot of System (6.1), which is shown in Fig. 1, conforms with Theorem 3.2.

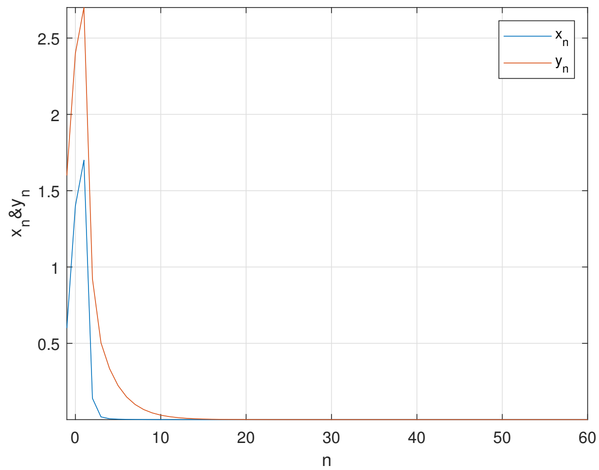

Figure 1. The plot of System(6.1).

Example 2

Consider the System (1.6) with initial conditions $x_{-2} = 30, x_{-1} = 50, x_0 = 60, y_{-2} = 5, y_{-1} = 10, y_0 = 4$. Moreover, choosing the parameters $A = 1$ and $B = 0.4 < 1$. Then System (1.6) can be written as

$$x_{n+1} = \frac{x_n}{1 + y_n y_{n-1} y_{n-2}}, \ y_{n+1} = \frac{y_n}{0.4 + x_n x_{n-1} x_{n-2}}. \tag{6.2}$$

The plot of System (6.2), which is shown in Fig. 2, conforms with Theorem 4.1.

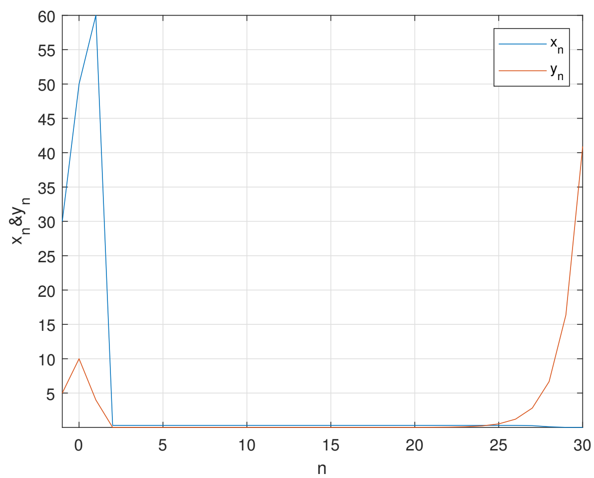

Figure 3. The plot of System(6.2).

Example 3

Consider the System (1.6) with initial conditions $x_{-2} = 0.8, x_{-1} = 0.6, x_0 = 0.5, y_{-2} = 0.9, y_{-1} = 0.7, y_0 = 0.5$. Moreover, choosing the parameters $A = 1, B = 1$. Then System (1.6) can be written as

$$x_{n+1} = \frac{x_n}{1 + y_n y_{n-1} y_{n-2}}, \ y_{n+1} = \frac{y_n}{1 + x_n x_{n-1} x_{n-2}}. \tag{6.3}$$

The plot of System (6.3), which is shown in Fig. 3, conforms with Theorem 4.2.

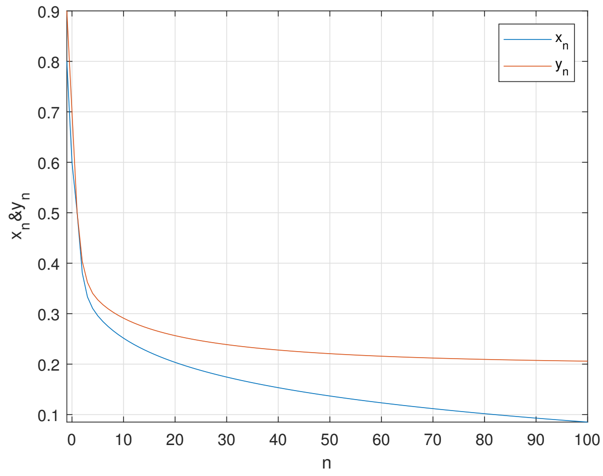

Figure 3. The plot of System(6.3).

Example 4

Consider the System (1.6) with initial conditions $x_{-2} = 0.8, x_{-1} = 0.5, x_0 = 2, y_{-2} = 0.4, y_{-1} = 0.1, y_0 = 1$. Moreover, choosing the parameters $A = 2.5 > 1$ and $B = 1$. Then System (1.6) can be written as

$$x_{n+1} = \frac{x_n}{2.5 + y_n y_{n-1} y_{n-2}}, \quad y_{n+1} = \frac{y_n}{1 + x_n x_{n-1} x_{n-2}}. \tag{6.4}$$

The plot of System (6.4), which is shown in Fig. 4, conforms with Theorem 4.3.

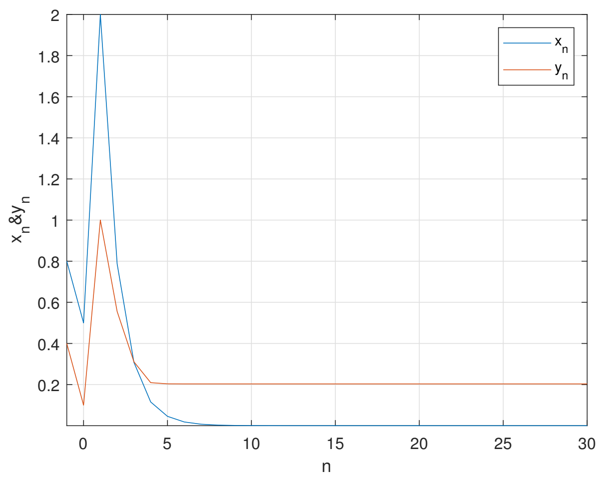

Figure 4. The plot of System(6.4).

References

1. A. S. Kurbanlı, C. Çinar and İ. Yalçinkaya, “On the behavior of positive solutions of the system of rational difference equation, $x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1}+1}, y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1}+1}$, $n = 0, 1, \ldots$,” Mathematical and Computer Modelling, vol. 53, no. 5-6, pp. 1261–1267, 2011.

2. Qi Wang, Gengrong Zhang, and Linlin Fu. "On the behavior of the positive solutions of the system of two higher-order rational difference equation, $x_{n+1} = \frac{x_{n-2k+1}}{Ay_{n-k+1}x_{n-2k+1}+\alpha}, y_{n+1} = \frac{y_{n-2k+1}}{Bx_{n-k+1}y_{n-2k+1}+\beta}, n \geqslant 0$," Applied Mathematics, vol. 4, no. 8, pp. 1220–1225, 2013, Article ID:35588, 6 pages, 2013. http://dx.doi.org/10.4236/am.2013.48164 Published Online August 2013 (http://www.scirp.org/journal/am)

3. Cinar C, and Yalcinkaya I. "On the positive solutions of difference equation system $x_{n+1} = \frac{1}{z_n}, y_{n+1} = \frac{x_n}{x_{n-1}}, z_{n+1} = \frac{1}{z_{n-1}}$," International Mathematical Journal, vol. 5, no. 5, pp.525–527, 2004.

4. A. Y. Ozban, "On the system of rational difference equations $x_n = \frac{a}{y_{n-3}}, y_n = \frac{by_{n-3}}{x_{n-q}y_{n-q}}$," Applied Mathematics and Computation, vol. 188, no. 1, pp. 833–837, 2007. doi:10.1016/j.amc.2006.10.034.

5. A. Y. Ozban. "On the positive solutions of the system of rational difference equations $x_{n+1} = \frac{1}{y_{n-k}}, y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m+k}}$," Journal of Mathematical Analysis and Applications, vol. 323, no.1, pp. 26–32, 2006.

6. Papaschinopoulos G, and Schinas CJ. "On a system of two nonlinear difference equations," Journal of Mathematical Analysis and Applications. vol. 219, no. 2, pp. 415–426, 1998.

7. Mehmet Gumus. "On a competitive system of rational difference equations $x^{(1)}_{n+1} = \frac{\alpha x^{(1)}_{n-2}}{\beta+\gamma\prod_{i=0}^{2} x^{(2)}_{n-i}}, x^{(2)}_{n+1} = \frac{\alpha_1 x^{(2)}_{n-2}}{\beta_1+\gamma_1\prod_{i=0}^{2} x^{(1)}_{n-i}}, n = 0, 1, \ldots$," Universal Journal of Mathematics and Applications, vol. 2, no. 4, pp. 224–228, 2019. Journal Homepage: www.dergipark.gov.tr/ujma ISSN 2619-9653 https://doi.org/10.32323/ujma.649122.

8. Hongmei Bao. "Asymptotic behavior of the system of second order nonlinear difference equations," Journal of Advances in Mathematics and Computer Science, vol. 24, no. 2, pp. 1–8, 2017.

9. Clark D, Kulenovic MRS, and Selgrade JF. "Global asymptotic behavior of a two-dimensional difference equation modelling competition," Nonlinear Analysis. Theory, Methods & Applications, vol. 52, no. 7, pp. 1765–1776, 2003.

10. Cinar C, Yalcinkaya I, and Karatas R. "On the positive solutions of the difference equation system $x_{n+1} = \frac{m}{y_n}, y_{n+1} = \frac{py_n}{x_{n-1}y_{n-1}}$," Journal of Institute of Mathematics and Computer Sciences, vol. 18, pp. 135–136, 2005.

11. Kocic VL, Ladas G. "Global behavior of nonlinear difference equations of higher order with application," vol. 256, Kluwer Academic, Dordrecht, The Netherlands, 1993.

12. Liu K, Zhao Z, Li. X, and Li. P. "More on three-dimensional systems of rational difference equations," Discrete Dynamics in Nature and Society. An International Multidisciplinary Research and Review Journal, vol. 2011, Article ID 178483, 9 pages, 2011.

13. Ibrahim TF and Zhang Q. "Stability of an anti-competitive system of rational difference equations," Archives des Sciences, vol. 66, no. 5, pp. 44–58, 2013.

14. Zayed EME and El-Moneam MA. "On the global attractivity of two nonlinear difference equations," Journal of Mathematical Sciences, vol. 177, no. 3, pp. 487–499, 2011.

15. Touafek N and Elsayed EM. "On the periodicity of some systems of nonlinear difference equations," Bulletin Mathematique ´de la Societe des Sciences Math ´ ematiques de Roumanie ´ , vol. 55, no. 2, pp. 217–224, 2012.

16. Touafek N and Elsayed EM. "On the solutions of systems of rational difference equations," Mathematical and Computer Modelling,vol. 55, no. 7-8, pp. 1987–1997, 2012.

17. Kalabusic S, Kulenovic MRS, and Pilav E. "Dynamics of a two-dimensional system of rational difference equations of Leslie-Gower type," Advances in Difference Equations, vol. 2011, article 29, 2011.

18. Ibrahim TF. "Boundedness and stability of a rational difference equation with delay," Romanian Journal of Pure and Applied Mathematics, vol. 57, no. 3, pp. 215–224, 2012.

19. Ibrahim TF and Touafek N. "On a third order rational difference equation with variable coefficients," Dynamics of Continuous, Discrete & Impulsive Systems. Series B. Applications & Algorithms, vol. 20, no. 2, pp. 251–264, 2013.

20. Ibrahim TF. "Oscillation, non-oscillation, and asymptotic behavior for third order nonlinear difference equations," Dynamics of Continuous, Discrete & Impulsive Systems A. Mathematical Analysis, vol. 20, no. 4, pp. 523–532, 2013.

21. M. R. S. Kulenovic and G. Ladas. "Dynamics of second order rational difference equations, with open problems and conjectures," Chapman and Hall/CRC, 2001.

22. Saber Elaydi. "An introduction to difference equations," Third Edition, Trinity University, 2005.

23. Walter Rudin. "Principles of mathematical analysis," Third Edition.

Alaa E. Hamza,
Department of Mathematics,
Cairo University, Faculty of Science, Giza,
Egypt.
E-mail address: hamzaaeg2003@yahoo.com

and

Abdullah Alshamasi,
Department of Mathematics,
University of Jeddah, College of Science, Jeddah, 21589,
Saudi Arabia.
E-mail address: a.shamasi1992@hotmail.com

and

Abdeljabbar Ghanmi,
Department of Mathematics,
Tunis Elmanar University, Faculty of Sciences, Tunis 2092,
Tunisia.
E-mail address: abdeljabbar.ghanmi@lamsin.rnu.tn