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Characterization of Pairwise C- Lindelöf Spaces

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ABSTRACT: This article presents representation of pairwise C-Lindelöf spaces, some of which make use of filter-base. Sufficient conditions for a space to be pairwise C-Lindelöf spaces are studied

Key Words: A bitopological space, pairwise C-Lindelöf, B-P-space, adherent convergent.

Contents

1. Introduction

In 1963, Kelly [5] established the idea of bitopology. Certain spaces provided its two (arbitrary) topologies. Many concepts in topological spaces have been generalised to bitopological spaces such as separation axioms, covering properties, mappings and others (see [2] [3] and [6]).

The Lindelófness notion was also introduced and various generalizations of this concept have been studied and investigated separately for distinct basis and aims (see [6] and [7]).

Similarly, the notion of C-compact space was introduced in bitopology by many authors (see [1] [4] and [9]). Further, the properties of C-compactness has been extended to bitopology (see the details in [9] and [10]).

In the present study we are concern with the pairwise C-Lindelöf spaces and provide different characterizations of these spaces. Throughout this paper, all spaces (X,τ) and (X,τ_1,τ_2) (or simply X) always mean topological spaces and bitopological spaces, respectively. In this work, we use the notation (τ_i,τ_j)- to denote certain properties with respect to topology τ_i and τ_j as bitopological spaces, where $i,j=1,2$. By τ_i-open set, we shall mean the open set with respect to topology τ_i in X. By τ_i-open cover of X, we mean that the cover of X by τ_i-open sets in X. The reader may consult Kelly 1963 for the detail notations and discussions.

2. Preliminaries

Definition 2.1 *[6]. A bitopological space X is called τ_i-P-space if any countable intersection of τ_i-α-open sets is τ_i-α-open. X is said P-space if it is τ_i-P-space for $i=1,2$.*

Definition 2.2 *[7]. (X,τ_1,τ_2) is pairwise Hausdorff iff for each pair of distinct points x and y of X there are a τ_1-open set U and a τ_2-open set V such that $x\in U, y\in V$ and $U\cap V=\emptyset$.*

Definition 2.3 *[7]. Let (X,τ_1,τ_2) be a bitopological space, then a subset U is called (τ_i,τ_j)-regular open set if $U=\tau_i\text{–}int(\tau_j\text{–}cl(U))$. Similarly, U is said pairwise regular open if it is both (τ_i,τ_j)-regular open and (τ_j,τ_i)-regular open.*

Definition 2.4 *[6]. A bitopological space (X,τ_1,τ_2) is said to be (τ_i,τ_j)-P-space if countable intersection of τ_i-open sets in X is τ_j-open. X is said B-P-space if it is (τ_1,τ_2)-P- space and (τ_2,τ_1)-P-space.*

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Definition 2.5 *[9] A bitopological space (X, τ_1, τ_2) is said to be (τ_i, τ_j)-C-compact if given τ_i-closed set A of X and τ_i-open covering $\{U_\alpha : \alpha \in I\}$ of A, there is a finite number of elements of I, with $A \subset \bigcup_{\alpha=1}^{n} \tau_j - cl(U_\alpha)$. X is called pairwise-C-compact if it is (τ_1, τ_2)-C-compact and (τ_2, τ_1)-C-compact.*

3. Pairwise C–Lindelöf Spaces

According to definition 2.5, we generalize pairwise C-compact spaces to pairwise C-Lindelöf as the following.

Definition 3.1 *A bitopological space (X, τ_1, τ_2) is said to be (τ_i, τ_j)-C-Lindelö f if given τ_i-closed set A of X and τ_i-open covering $U_\alpha : \alpha \in I$ of A, there is a countable number of elements of I, say I_0, with $A \subset \cup_{\alpha \in I_0} \tau_j - cl(U_\alpha)$. X is called pairwise C- Lindelöf if it is (τ_1, τ_2)-C-Lindelöf and (τ_2, τ_1)-C-Lindelöf.*

Definition 3.2 *In a bitopological space (X, τ_1, τ_2), the family $U_\alpha : \alpha \in I$ is called (τ_i, τ_j)-regular-open cover if each U_α is (τ_i, τ_j)-regular-open set for all $\alpha \in I$. $U_\alpha : \alpha \in I$ is said to be pairwise regular open cover if it is both (τ_i, τ_j)-regular-open and (τ_j, τ_i)-regular-open.*

Proposition 3.1 *The bitopological space (X, τ_1, τ_2) is (τ_i, τ_j)-C-Lindelöf if and only if for any τ_i-closed set A of X and (τ_i, τ_j)-regular-open cover $U_\alpha : \alpha \in I$ of A, there is a countable number of elements of I, say I_0, such that*

$$A \subset \cup_(\alpha \in I_0)\tau_j - cl(U_\alpha).$$

Proof: $\Longleftarrow$ Suppose the condition holds. Now, let A be τ_i-closed subset of X and $U_\alpha : \alpha \in I$ be τ_i-open cover of A. Thus $\{\tau_i\text{–}int(\tau_j\text{–}cl(U_\alpha)) : \alpha \in I\}$ is (τ_i, τ_j)-regular-open cover of A, so there is a countable number of elements of I, say I_0, such that

$$\begin{aligned} A &\subset \textstyle\bigcup_{\alpha \in I_0} \tau_j - cl(\tau_i\text{–}int(\tau_j\text{–}cl(U_\alpha))) \\ &\subset \textstyle\bigcup_{\alpha \in I_0} \tau_j - cl(U_\alpha). \end{aligned}$$

Therefore, (X, τ_1, τ_2) is (τ_i, τ_j)-C-Lindelöf.

$\Longrightarrow$ Consider that X is (τ_i, τ_j)-C-Lindelöf. So the condition follows from the definition of (τ_i, τ_j)-C-Lindelöfness.

□

Theorem 3.1 *For a bitopological $\tau_i - P$-space (X, τ_1, τ_2), if X is (τ_i, τ_j)-C-Lindelöf, then if A is a τ_i-closed set of (X, τ_1, τ_2) and F is a collection of τ_i-closed sets of X with $(\cap F) \bigcap A = \emptyset$, then there is a countable number of elements of $\mathcal{F}$, say F_n and $n \in N$, with $\cap_{n \in N}(\tau_j - int(F_n)) \bigcap A = \emptyset$.*

Proof: Consider A be a τ_i-closed subset of a (τ_i, τ_j)-C-Lindelöf space (X, τ_1, τ_2) and F be a collection of τ_i-closed sets of X such that

$$\begin{aligned} &(\cap F) \textstyle\bigcap A = \emptyset. \\ &\Longrightarrow A \subset X - (\cap F) \\ &= \cup\{X - F : F \in F\} \end{aligned}$$

Now, $\mathcal{U} = \{U : U = X - F, F \in F\}$ is a collection of τ_i-open sets of X covering A. Since A is τ_i-closed subset of a (τ_i, τ_j)-C-Lindelöf space, there is a countable number of elements of U such that

$$\begin{aligned} A &\subset \cup_{n \in N} \tau_j - cl(U_n) \\ A &\subset \cup_{n \in N} \tau_j - cl(X - F_n) \\ A &\subset X - \cap_{n \in N}(\tau_j - int(F_n)) \end{aligned}$$

Then, $\cap_{n \in N}(\tau_j - int(F_n)) \bigcap A = \emptyset$. □

Example 3.1 *Let X be a set and $card(X) = 2^c$ where $c = card(\Re)$. Let τ_1 be a co-countable topology on X consisting of $\emptyset$ and all subsets of X whose complements have cardinality at most c and let τ_2 be a co-finite topology on X. So $(X, \tau_{cof}, \tau_{coc})$ is (τ_{cof}, τ_{coc})-C-Lindelöf but is not p-Lindelöf (see [6]).*

Remark 3.1 *From definition 2.5 and definition 3.1, it is obviously that every pairwise C-compact is pairwise C-Lindelöf but not the converse by the following example.*

Example 3.2 *Let Ω denotes the set of ordinals which are less than or equal to the first uncountable ordinal number ω_1, i.e., $\Omega = [0, \omega_1]$. This Ω is an uncountable well-ordered set with a largest element ω_1, having the property that if $\alpha \in \Omega$ with $\alpha < \omega_1$, then $\{\beta \in \Omega : \beta \leq \alpha\}$ is countable. Since Ω is a totally ordered space, it can be provided with its order topology. Let $\tau_1 = \tau_2$ be an order topology on Ω. Thus (Ω, τ_1, τ_2) is a bitopological space. Since (Ω, τ_1, τ_2) is p-Lindelöf (see [7]), it is pairwise C-Lindelöf . But (Ω, τ_1, τ_2) is not pairwise C-compact (see [1] and [4]).*

Definition 3.3 *In a space (X, τ_1, τ_2), X is said to be $\tau_i\tau_j$-regular if, for each point if, for any point $x \in X$ and each τ_i–closed set P such that $x \notin P$, there exist τ_i-open set U and a τ_i-open set V such that $x \in U$, $P \subseteq V$, and τ_j–$(U) \bigcap \tau_j$–$(V) = \emptyset$. X is p^*–regular if it is $\tau_1\tau_2$-regular and $\tau_2\tau_1$-regular.*

Definition 3.4 *A bitopological space (X, τ_1, τ_2) is called $\tau_i\tau_j$-normal if, given a τ_i–closed set A and a τ_i-closed set B with $A \bigcap B = \emptyset$, there exist a τ_j-open set U and a τ_j–open set V such that $A \subseteq U, B \subseteq V$, and $U \bigcap V = \emptyset$. X is p^*–normal if it is $\tau_1\tau_2$- normal and $\tau_2\tau_1$- normal.*

Theorem 3.2 *If (τ_i, τ_j)-P-space (X, τ_1, τ_2) is $\tau_i\tau_j$-regular and (τ_i, τ_j)-C-Lindelöf, then it is $\tau_i\tau_j$-normal.*

Proof: Let A be τ_i–closed and B be τ_i-closed subsets of X such that $A \bigcap B = \emptyset$. Since X is $\tau_i\tau_j$-regular, then, for any $a \in A$, there is a τ_i-open set U_a and a τ_i-open set V_a such that $a \in U_a, B \subseteq V$, and τ_j–$(U_a) \bigcap \tau_j$–$(V_a) = \emptyset$. The family $\{U_a : a \in A\}$ is τ_i-open cover of A. Because X is (τ_i, τ_j)-C-Lindelöf, there is a countable subfamily $\{U_(a_n) : n \in \aleph\}$ such that

$$A \subset \bigcup_{n\in\aleph} \tau_j - cl(U_{a_n}).$$

Consider that $W = \bigcap_{n\in\aleph} V_{a_n}$. Since X is (τ_i, τ_j)–P–space, W is τ_j-open set containing B. Let $Q = X - \bigcap_{n\in\aleph} \tau_j$–$cl(V_{a_n})$.. Thus, Q is τ_j-open set containing A and $W \bigcap Q = \emptyset$. Thus, X is $\tau_i\tau_j$-normal. □

Definition 3.5 *[3]. A function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be i-continuous if it is continuous. f is called continuous if it is i-continuous for each $i = 1, 2$.*

Proposition 3.2 *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a continuous function from (τ_i, τ_j)-C-Lindelöf space to a pairwise Hausdorff and (σ_j, σ_i)-P–space. Then for each τ_i-closed (resp. τ_j-closed) subset $A \subset X$, $f(A)$ is σ_i-closed (resp. σ_j-closed).*

Proof: Consider A be τ_i-closed subset of X such that $a \in Y$ and $a \notin f(A)$. For each $y \in f(A)$, pick a σ_i-open set V_y such that $a \notin \sigma_j - cl(V_y)$. The family $\{f^{-1}(V_y) : y \in f(A)\}$ is a τ_i-open cover of A and because of (τ_i, τ_j)-C-Lindelöfness of X, there is $n \in \aleph$ such that

$$A \subset \bigcup_{n\in\aleph} \tau_j - cl(f^{-1}(V_y)).$$

Because f is i-continuous, then we get

$$\begin{aligned} f(A) &\subset f(\bigcup_{n\in\aleph} \tau_j - cl(f^{-1}(V_y))) \\ &\subset f(\bigcup_{n\in\aleph} f^{-1}(\tau_j - cl(V_y))) \\ &= f(f^{-1}(\bigcup_{n\in\aleph} \sigma_j - cl(V_y))). \end{aligned}$$

Then,

$$f(A) \subset \bigcup_{n\in\aleph} \sigma_j - cl(V_y).$$

Since Y is (σ_j, σ_i)-P–space, $W = Y - \bigcup_{n\in\aleph} \sigma_j - cl(V_y)$ is σ_i-open set such that $a \in W$ and $W \bigcap f(A) = \emptyset$. Therefore, $f(A)$ is σ_i-closed. □

Theorem 3.3 *Let* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *be a continuous function. If* X *is* (τ_i, τ_j)-*C-Lindelöf space, then* Y *is* (σ_i, σ_j)-*C-Lindelöf.*

Proof: consider F be σ_j-closed set and $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be σ_i-open cover of F. Because f is i-continuous, $\{f^{-1}(U_\alpha)\}$ is τ_i-open cover of τ_i-closed set $f^{-1}(F)$. Since X is (τ_i, τ_j)-C- Lindelöf space, there is a countable collection $\Delta_0 \in \Delta$ such that

$$f^{-1}(F) \subset \bigcup_{\alpha \in \Delta_0} \tau_j - cl(f^{-1}(U_\alpha))$$
$$\subset \bigcup_{\alpha \in \Delta_0} f^{-1}(\sigma_j - cl(U_\sigma))$$

Then, $F \subset \bigcup_{\alpha \in \Delta_0} (\sigma_j - cl(U_\alpha))$. □

In [4] and [8], the ideas of adherent point and filter-base have been introduced and studied in topological spaces. We extend these notions to bitopological spaces as follows.

Definition 3.6 *Let* (X, τ_1, τ_2) *be a bitopological space and* $A \subset X$, *then an element* $x \in X$ *is said to be* τ_i-*adherent point of a set* A *if any* τ_i-*open set* G *containing* x *such that* $G \bigcap A \neq \emptyset$. *The element* $x \in X$ *is said to be adherent point of* A *if it is both* τ_i-*adherent point for* $i = 1, 2$.

Definition 3.7 *let* (X, τ_1, τ_2) *be a bitopological space.* τ_i-*open filter-base* $\mathcal{F}$ *on* (X, τ_1, τ_2) *(i.e. a filter-base composed exclusively of* τ_i-*open sets of* X*) is said to be* τ_i-*adherent convergent if any* τ_i-*open neighborhood of the* τ_i-*adherent set of* $\mathcal{F}$ *includes a member of* $\mathcal{F}$, $i = 1, 2$.

Theorem 3.4 *A bitopological* (τ_j, τ_i)-*P–space* (X, τ_1, τ_2) *is* (τ_i, τ_j)-*C-Lindelöf if and only if each* τ_i-*open filter base is* $\tau_i j$-*adherent convergent.*

Proof: $\Longrightarrow$ Consider $\mathcal{F}$ be τ_i-open filter base of (τ_i, τ_j)-C-Lindelöf space X. Let W be the τ_i-adherent set of $\mathcal{F}$. So we have $W = \bigcap\{\tau_i - cl(U) : U \in \mathcal{F}\}$. If D is τ_i-open neighborhood of W, we get $X - D$ is τ_i- closed set.
Let $x \in X - D$. So

$$x \in X - W \Longrightarrow x \in \bigcap\{\tau_i - cl(U) : U \in \mathcal{F}\}.$$
$$\Longrightarrow x \in X - \bigcap\{\tau_i - cl(U) : U \in \mathcal{F}\}.$$

Then,

$$X - D \subset X - \bigcap\{\tau_i - cl(U) : U \in \mathcal{F}\}$$
$$= \bigcup\{X - \tau_i - cl(U) : U \in \mathcal{F}\}.$$

Thus $\{X - \tau_i - cl(U) : U \in \mathcal{F}\}$ is τ_i-open cover of $X - D$. Since X is (τ_i, τ_j)-C-Lindelöf space, we get

$$X - D \subset \bigcup_{n \in \aleph} \tau_j - cl(X - \tau_i - cl(U_n)).$$
$$= \bigcup_{n \in \aleph} (X - \tau_j - int(\tau_i - cl(U_n)))$$
$$= X - \bigcap_{n \in \aleph} \tau_j - int(\tau_i - cl(U_n)).$$

Thus, $\bigcap_{n \in \aleph} \tau_j - int(\tau_i - cl(U_n)) \subset D$. Because $U_n \subset \tau_j - int(\tau_i - cl(U_n))$ for $n \in \aleph$, so

$$\bigcap_{n \in \aleph} U_n \subset \bigcap_{n \in \aleph} \tau_j - int(\tau_i - cl(U_n)) \subset D.$$

This implies that D contains a point of $\mathcal{F}$. Then $D \in \mathcal{F}$.

$\Longleftarrow$ Suppose that X is not (τ_i, τ_j)-C-Lindelöf space, so there exists τ_i-closed set $C \subset X$ and $\mathcal{V}$ covering of C consisting of τ_i-open sets of X such that, for every countable subcollection $\{V_n : n \in \aleph\}$, $C \subset X - \bigcup_{n \in \aleph} \tau_j - cl(V_n)$.

Now, let $\mathcal{P} = \{X - \bigcup_{n \in \aleph} \tau_j - cl(V_n) : V_n \in \mathcal{V}\}$. Since X is (τ_j, τ_i)-P–space, $\mathcal{P}$ is τ_i-open filter base on X and $(\cap \mathcal{P}) \bigcap C \neq$. Let x be τ_i-adherent point of $\mathcal{P}$ such that

$$x \in \{\tau_i - cl(X - \bigcup_{n\in\aleph} \tau_j - cl(U_n)) : U_n \in \mathcal{V}\} = \bigcap_{n\in\aleph}\{X - \tau_i - int(\tau_j - cl(U_n)) : U_n\mathcal{V}\}$$
$$= \{X - \bigcup_{n\in\aleph} \tau_i - int(\tau_j - cl(U_n)) : U_n \in \mathcal{V}\} \Longrightarrow x \in \bigcup_{n\in\aleph} \tau_i - int(\tau_j - cl(U_n)) \Longrightarrow x \in C.$$

Then, $C \subset \bigcup_{n\in\aleph} \tau_i - int(\tau_j - cl(U_n))$. Thus the τ_i-adherent set of $\mathcal{P}$ is hold in $X - C$ that is τ_i-open neighborhood but there is not any element of $\mathcal{P}$ is contained in $X - C$ which leads to a contradiction. Therefore (X, τ_1, τ_2) is (τ_i, τ_j)-C-Lindelöf space. □

4. Conclusion

In current paper, we introduced the c-Lindelöfness in bitopological spaces namely; pairwise c-Lindelöf space as a generalization of C-compactness. Also, some necessary notions such as adherent point and filter-base have been defined and studied in bitopology in sense of convergence concept. Hopefully, these results which obtained in this work will encourage the researchers for further studies the C-paracompactenss and C-paralindelöfness in bitopological settings.

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