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Fixed Points for Reciprocally Continuous Mappings and Variants of Compatible Mappings

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ABSTRACT: In this paper, we introduce (ψ, ϕ)-weak contraction condition involving cubic terms of distance function and prove some fixed point theorems for pairs of compatible mappings of type (E), type (K) and subcompatible mappings satisfying a newly introduced contraction condition. We also provide examples in support of our results and give an application for the mappings satisfying an integral contractive type (ψ, ϕ)-weak contraction condition.

Key Words: Weak contraction, (ψ, ϕ)-weak contraction, compatible mapping, compatible mappings of type (K) and type (E), subcompatible mapping, reciprocally continuous

Contents

1. Introduction and Preliminaries

The fixed point theory has a wider range of its application in various fields such as science, computer, numerical analysis, economics, etc. Many of the problems that arise in daily life may reduce to various types of contraction conditions. Fixed point theory concentrates mainly on the solutions of contractive type conditions which give the solutions of the problems (see [3,16,22]).

Banach contraction principle [2] which is known as basic tool of fixed point theory ensures the existence of a unique fixed point for every contraction mapping T(say) defined on a complete metric space E. The mapping T in Banach contraction principle is always continuous, but in 1968, Kannan provided an example of a discontinuous mapping satisfying contraction condition different from that of Banach contraction principle. Over the years, many authors have continuously tried to extend and generalize the Banach contraction principle in various directions.

In 1969, Boyd and Wong [6] introduced ϕ contraction condition of the form $d(Tu, Tv) \leq \phi(d(u, v))$ for all $u, v \in E$, where T is a self mapping on a complete metric space E and $\phi : [0, \infty) \to [0, \infty)$ is an upper semi continuous function from right such that $0 \leq \phi(t) < t$ for all $t > 0$. In 1997, Alber and Guerre-Delabriere [1] generalized ϕ contraction to $\phi-$weak contraction in Hilbert spaces, which was further extended and proved by Rhoades [23] in complete metric space in 2001 as follows:

A self mapping T on a complete metric space is said to be a $\phi-$weak contraction if for each $u, v \in E$, there exists a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) > 0$, for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$ such that

$$d(Tu, Tv) \leq d(u, v) - \phi(d(u, v)). \tag{1.1}$$

The function ϕ in the above inequality (1.1) is known as control function or altering distance function. The notion of control function was given by Khan *et al.* [15] as follows:

Definition 1.1 *[15] An altering distance is a function* $\phi : [0, \infty) \to [0, \infty)$ *satisfying the following:*

(i) ϕ *is an increasing and continuous function,*

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(ii) $\phi(t) = 0$ *if and only if* $t = 0$.

In 2009, Zhang and Song [29] gave the notion of generalized $\phi-$weak contraction by generalizing the concept of $\phi-$weak contraction.

Definition 1.2 *[29] Two self mappings S and T on a metric space (E, d) are said to be generalized $\phi-$weak contractions if there exists a mapping $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$ such that*

$$d(Su, Tv) \leq M(u, v) - \phi(M(u, v)) \text{ for all } u, v \in E,$$

where $M(u, v) = \max\{d(u, v), d(u, Su), d(v, Tv), \frac{d(u,Tv)+d(v,Su)}{2}\}$.

Another direction of generalization of Banach contraction principle concerns the coincidence points and common fixed points of pair of mappings satisfying contractive conditions. The use of notion of commutative mappings in fixed point theory literature became a turning moment. The first attempt to relax commutative condition of mapping to weak commutative condition was initiated by Sessa [25].

In 1986, Jungck [13] further weakened the notion of commutativity/weak commutative to compatible mappings. In 1993, Jungck, Murthy and Cho [14] further generalized the notion of compatible mappings to compatible mappings of type (A). The process of generalizing the concept of compatible mappings still goes on. Pathak and Khan [21], Pathak *et al.* [19,20], Rohen and Singh [24], Singh and Singh [27] weakened this concept of compatible mappings to compatible mappings of type (B), type (P), type (C), type (R), type (E) respectively. In 2009, Bouhadjera and Godet-Thobie [4] introduced the concept of subcompatible mappings. In 2014, Jha, Popa and Manandhar [11] generalized the concept of compatible mappings to a new type of mapping called compatible mappings of type (K).

In 2013, Murthy and Prasad [17] introduced a weak contraction that involves cubic terms of distance function. In 2018, Jain *et al.* [9] generalized this result of Murthy and Prasad [17] for the pairs of compatible mappings and Jung *et al.* [12] and Jain *et al.* [8,10] generalized the result of Murthy and Prasad [17] for variants of compatible mappings.

In this paper, we introduce a generalized (ψ, ϕ)-weak contraction condition involving cubic terms of distance function and state and prove some common fixed point theorems for pairs of compatible mappings of type (E), type (K) and subcompatible mappings satisfying that contraction condition along with reciprocal continuity.

Now we first recall some basic concepts which are useful for our work.

Definition 1.3 *Let (E, d) be a metric space. Two mappings $S, T : E \to E$ are said to be*

(i) compatible [13] if and only if

$$\lim_{n\to\infty} d(STu_n, TSu_n) = 0,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(ii) compatible of type (A) [14] if

$$\lim_{n\to\infty} d(SSu_n, TSu_n) = 0 \text{ and } \lim_{n\to\infty} d(TTu_n, STu_n) = 0,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(iii) compatible of type (B) [21] if

$$\lim_{n\to\infty} d(STu_n, TTu_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(STu_n, Sz) + \lim_{n\to\infty} d(Sz, SSu_n)]$$

and

$$\lim_{n\to\infty} d(TSu_n, SSu_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(TSu_n, Tz) + \lim_{n\to\infty} d(Tz, TTu_n)],$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(iv) compatible of type (P) *[19] if*

$$\lim_{n\to\infty} d(SSu_n, TTu_n) = 0,$$

whenever $\{u_n\}$ *is a sequence in* E *such that* $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, *for some* $z \in E$;

(v) compatible of type (C) *[20] if*

$$\lim_{n\to\infty} d(STu_n, TTu_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(STu_n, Sz) + \lim_{n\to\infty} d(Sz, SSu_n) + \lim_{n\to\infty} d(Sz, TTu_n)]$$

and

$$\lim_{n\to\infty} d(TSu_n, SSu_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(TSu_n, Tz) + \lim_{n\to\infty} d(Tz, TTu_n) + \lim_{n\to\infty} d(Tz, SSu_n)],$$

whenever $\{u_n\}$ *is a sequence in* E *such that* $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, *for some* $z \in E$;

(vi) subcompatible [4,5] if there exists a sequence $\{u_n\}$ *in* E *such that*

$$\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z,\ z \in E$$

and which satisfy

$$\lim_{n\to\infty} d(STu_n, TSu_n) = 0;$$

(vii) compatible of type (E) *[27] if*

$$\lim_{n\to\infty} SSu_n = \lim_{n\to\infty} STu_n = Tz$$

and

$$\lim_{n\to\infty} TTu_n = \lim_{n\to\infty} TSu_n = Sz,$$

whenever $\{u_n\}$ *is a sequence in* E *such that* $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, *for some* $z \in E$;

(viii) compatible of type (K) *[11] if*

$$\lim_{n\to\infty} SSu_n = Tz \ \text{ and } \ \lim_{n\to\infty} TTu_n = Sz,$$

whenever $\{u_n\}$ *is a sequence in* E *such that* $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, *for some* $z \in E$.

Remark 1.1 *If* $Sz = Tz$, *then compatibility of type* (E) *implies compatible, compatible of type* (A), *type* (B), *type* (C) *and type* (P), *however the converse may not true (see [28, Example 2.4]).*

Remark 1.2 *If* $Sz \neq Tz$, *then compatibility of type* (E) *is neither compatible nor compatible of type* (A), *type* (C), *type* (P) *(see [28, Example 2.3]).*

In 1999, Pant [18] introduced the notion of reciprocal continuity as follows.

Definition 1.4 *[18] Let* S *and* T *be self mappings on a metric space* (X, d). *Then* S *and* T *are said to be reciprocally continuous if*

$$\lim_{n\to\infty} STu_n = Sz \ \text{ and } \ \lim_{n\to\infty} TSu_n = Tz,$$

whenever $\{u_n\}$ *is a sequence in* E *such that* $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, *for some* $z \in E$.

Remark 1.3 *It is clear that a pair of continuous self mappings is reciprocally continuous, but the converse may not true (see [18]).*

Remark 1.4 *Compatibility and reciprocal continuity are independent of each other (see [26]).*

In 2011, Singh and Singh [28] split the concept of compatible mappings of type (E) to the concept of $S-$compatible mappings of type (E) and $T-$compatible mappings of type (E) and further split the notion of reciprocal continuity to the notion of $S-$reciprocally continuous and $T-$reciprocally continuous.

Definition 1.5 *[28] Let (E,d) be a metric space. Self mappings S and T on E are said to be*

(i) $S-$compatible of type (E) if

$$\lim_{n\to\infty} SSu_n = \lim_{n\to\infty} STu_n = Tz$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(ii) $T-$compatible of type (E) if

$$\lim_{n\to\infty} TTu_n = \lim_{n\to\infty} TSu_n = Sz$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(iii) $S-$reciprocally continuous if

$$\lim_{n\to\infty} STu_n = Sz$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(iv) $T-$reciprocally continuous if

$$\lim_{n\to\infty} TSu_n = Tz$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$.

Remark 1.5 *Compatibility of type (E) implies both $S-$compatible of type (E) and $T-$compatible of type (E), however the converse may not true, see the example given below.*

Example 1.1 *Let $E = [0,5]$ and d be a usual metric. Let $S,T : E \to E$ be two mappings defined by*

$$Su = \begin{cases} 5, & u \in [0, \frac{5}{2}] - \{\frac{5}{4}\}, \\ 0, & u = \frac{5}{4}, \\ \frac{5-u}{2}, & u \in (\frac{5}{2}, 5], \end{cases}$$

$$Tu = \begin{cases} 1, & u \in [0, \frac{5}{2}] - \{\frac{5}{4}\}, \\ 5, & u = \frac{5}{4}, \\ \frac{u}{2}, & u \in (\frac{5}{2}, 5]. \end{cases}$$

Clearly, S and T are not continuous at $u = \frac{5}{2}, \frac{5}{4}$. Suppose that $u_n \to \frac{5}{2}$, $u_n > \frac{5}{2}$ for all n. Then $Su_n = \frac{5-u_n}{2} \to \frac{5}{4} = t$ and $Tu_n = \frac{u_n}{2} \to \frac{5}{4} = t$. Therefore, we have $SSu_n = S(\frac{5-u_n}{2}) = 5 \to 5$, $STu_n = S(\frac{u_n}{2}) = 5 \to 5, Tt = 5$ and $TTu_n = T(\frac{u_n}{2}) = 1 \to 1,\ TSu_n = T(\frac{5-u_n}{2}) = 1 \to 1,\ St = 0$. Thus the pair (S,T) is $S-$compatible of type (E) but not compatible of type (E).

Remark 1.6 *The reciprocal continuity of the pair (S,T) implies both $S-$reciprocal continuity and $T-$reciprocal continuity, however the converse may not be true, see example given below.*

Example 1.2 *Let $E = [0,5]$ and d be a usual metric. Let $S, T : E \to E$ be two mappings defined by*

$$Su = \begin{cases} 5, & u \in [0, \frac{5}{2}), \\ 5-u, & u \in [\frac{5}{2}, 5], \end{cases}$$

$$Tu = \begin{cases} 0, & u \in [0, \frac{5}{2}), \\ u, & u \in [\frac{5}{2}, 5]. \end{cases}$$

Let $\{u_n\}$ be a sequence in E such that $u_n \to \frac{5}{2}$, $u > \frac{5}{2}$ for all n. Then $Su_n = 5 - u_n \to \frac{5}{2}$, $Tu_n = u_n \to \frac{5}{2} = t$, $STu_n = S(u_n) = 5 - u_n \to \frac{5}{2}$, $St = \frac{5}{2}$ and $TSu_n = T(5-u_n) = 0 \to 0$, $Tt = \frac{5}{2}$. It follows that $\lim_{n\to\infty} STu_n = \frac{5}{2} = St$ and $\lim_{n\to\infty} TSu_n = 0 \neq Tt = \frac{5}{2}$. Therefore, the pair (S,T) is $S-$reciprocally continuous, but it is neither $T-$reciprocally continuous nor reciprocally continuous.

Now we present a proposition which is useful for our work.

Proposition 1.1 *[28] Let S and T be two self mappings on a metric space (E,d) and $\{u_n\}$ be a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$. Assume that one of the following is satisfied:*

(i) The pair (S,T) is $S-$compatible of type (E) and $S-$reciprocally continuous;

(ii) The pair (S,T) is $T-$compatible of type (E) and $T-$reciprocally continuous.

Then (a) $Sz = Tz$ and (b) if there exists $t \in E$ such that $St = Tt = z$, then $STt = TSt$.

2. Variants of Compatible Mapping

In this paper, we will prove fixed point theorems for pairs of compatible mappings of type (E), type (K) and subcompatible mappings by using a control function $\psi \in \Psi$, where Ψ is a collection of all functions $\psi : [0,\infty)^4 \to [0,\infty)$ satisfying the following:

(ψ_1) ψ is non decreasing and upper semi continuous in each coordinate variables;

(ψ_2) $\Delta(t) = \max\{\psi(t,t,0,0), \psi(0,0,0,t), \psi(0,0,t,0), \psi(t,t,t,t)\} \leq t$, for all $t > 0$.

Let Φ be the collection of all the functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following:

(ϕ_1) ϕ is a continuous function;

(ϕ_2) $\phi(t) > t$ for all $t > 0$ and $\phi(0) = 0$.

Let (E,d) be a metric space and f, g, S and T be four self mappings on E satisfying the following:

(C_1) $S(E) \subset g(E)$ and $T(E) \subset f(E)$;

(C_2) For $\psi \in \Psi$, $\phi \in \Phi$, a real number $p > 0$ and for all $u, v \in E$,

$$\begin{aligned}[1 + pd(fu,gv)]d^2(Su,Tv) \leq p\psi\Big(& d^2(fu,Su)d(gv,Tv), d(fu,Su)d^2(gv,Tv), \\ & d(fu,Su)d(fu,Tv)d(gv,Su), d(fu,Tv)d(gv,Su)d(gv,Tv)\Big) \\ & + m(fu,gv) - \phi(m(fu,gv)),\end{aligned}$$

where

$$\begin{aligned}m(fu,gv) = \max\Big\{& d^2(fu,gv), d(fu,Su)d(gv,Tv), d(fu,Tv)d(gv,Su), \\ & \frac{1}{2}[d(fu,Su)d(fu,Tv) + d(gv,Su)d(gv,Tv)]\Big\}.\end{aligned}$$

Then for arbitrary point $u_0 \in E$, by (C_1), one can find u_1 such that $Su_0 = gu_1 = v_0$. For this u_1, one can find $u_2 \in E$ such that $Tu_1 = fu_2 = v_1$.

Continuing in this fashion, one can construct a sequence such that

$$v_{2n} = Su_{2n} = gu_{2n+1} \quad \text{and} \quad v_{2n+1} = Tu_{2n+1} = fu_{2n+2}, \tag{2.1}$$

for each $n = 0, 1, 2, 3....$

Before proving the main results, first we shall prove the following two lemmas which are useful for our work.

Lemma 2.1 $\lim_{n\to\infty} d(v_n, v_{n+1}) = 0$, *where* $\{v_n\}$ *is the sequence in* E *defined by* (2.1).

Proof: For simplicity, let us denote

$$\gamma_n = d(v_n, v_{n+1)}, n = 0, 1, 2, 3, ... \tag{2.2}$$

First, we prove that $\{\gamma_n\}$ is a non-increasing sequence, i.e., $\gamma_{n+1} \leq \gamma_n$ for $n = 1, 2, 3, ...$

Assume that n is even. Then taking $u = u_{2n}$ and $v = u_{2n+1}$ in (C_2), we get

$$\begin{aligned}[1 + pd(fu_{2n}, gu_{2n+1})]d^2(Su_{2n}, Tu_{2n+1}) \leq & \, p\,\psi\Bigg(d^2(fu_{2n}, Su_{2n})d(gu_{2n+1}, Tu_{2n+1}), \\ & d(fu_{2n}, Su_{2n})d^2(gu_{2n+1}, Tu_{2n+1}), \\ & d(fu_{2n}, Su_{2n})d(fu_{2n}, Tu_{2n+1})d(gu_{2n+1}, Su_{2n}), \\ & d(fu_{2n}, Tu_{2n+1})d(gu_{2n+1}, Su_{2n})d(gu_{2n+1}, Tu_{2n+1})\Bigg) \\ & + m(fu_{2n}, gu_{2n+1}) - \phi(m(fu_{2n}, gu_{2n+1})),\end{aligned}$$

where

$$\begin{aligned}m(fu_{2n}, gu_{2n+1}) = \max\Big\{ & d^2(fu_{2n}, gu_{2n+1}), d(fu_{2n}, Su_{2n})d(gu_{2n+1}, Tu_{2n+1}), \\ & d(fu_{2n}, Tu_{2n+1})d(gu_{2n+1}, Su_{2n}, \frac{1}{2}[d(fu_{2n}, Su_{2n})d(fu_{2n}, Tu_{2n+1}) \\ & + d(gu_{2n+1}, Su_{2n})d(gu_{2n+1}, Tu_{2n+1})]\Big\}.\end{aligned}$$

Using (2.1) in the above inequality, we have

$$\begin{aligned}[1 + pd(v_{2n-1}, v_{2n})]d^2(v_{2n}, v_{2n+1}) \leq & p\psi\Bigg(d^2(v_{2n-1}, v_{2n})d(v_{2n}, v_{2n+1}), \\ & d(v_{2n-1}, v_{2n})d^2(v_{2n}, v_{2n+1})), \\ & d(v_{2n-1}, v_{2n})d(v_{2n-1}, v_{2n+1})d(v_{2n}, v_{2n}), \\ & d(v_{2n-1}, v_{2n+1})d(v_{2n}, v_{2n})d(v_{2n}, v_{2n+1})\Bigg) \\ & + m(v_{2n-1}, v_{2n}) - \phi(m(v_{2n-1}, v_{2n})),\end{aligned}$$

where

$$\begin{aligned}m(v_{2n-1}, v_{2n}) = \max\Bigg\{ & (d^2(v_{2n-1}, v_{2n}), d(v_{2n-1}, v_{2n})d(v_{2n}, v_{2n+1}), \\ & d(v_{2n-1}, v_{2n+1})d(v_{2n}, v_{2n}), \\ & \frac{1}{2}\Big[d(v_{2n-1}, v_{2n})d(v_{2n-1}, v_{2n+1}) + d(v_{2n}, v_{2n})d(v_{2n}, v_{2n+1})\Big]\Bigg\}.\end{aligned}$$

Using (2.2) in the above inequality, we have

$$[1+p\gamma_{2n-1}]{\gamma_{2n}}^2 \le p\psi\big({\gamma_{2n-1}}^2\gamma_{2n}, \gamma_{2n-1}{\gamma_{2n}}^2, 0, 0\big) + m(v_{2n-1}, v_{2n}) - \phi\big(m(v_{2n-1}, v_{2n})\big), \tag{2.3}$$

where

$$m(v_{2n-1}, v_{2n}) = \max\big\{{\gamma_{2n-1}}^2, \gamma_{2n-1}\gamma_{2n}, 0, \frac{1}{2}\big[\gamma_{2n-1}d(v_{2n-1}, v_{2n+1}) + 0\big]\big\}.$$

Using the triangular inequality, we get

$$d(v_{2n-1}, v_{2n+1}) \le d(v_{2n-1}, v_{2n}) + d(v_{2n}, v_{2n+1}) = \gamma_{2n-1} + \gamma_{2n}.$$

Hence

$$m(v_{2n-1}, v_{2n}) \le \max\big\{{\gamma_{2n-1}}^2, \gamma_{2n-1}\gamma_{2n}, 0, \frac{1}{2}\big[\gamma_{2n-1}(\gamma_{2n-1} + \gamma_{2n})\big]\big\}. \tag{2.4}$$

Now we claim that $\{\gamma_{2n}\}$ is non-increasing.
If it is not, i.e., $\gamma_{2n-1} < \gamma_{2n}$, then by using (2.4) with the properties of ϕ and ψ, (2.3) reduces to

$$[1+p\gamma_{2n-1}]{\gamma_{2n}}^2 \le p\gamma_{2n-1}{\gamma_{2n}}^2 + \gamma_{2n-1}\gamma_{2n} - \phi(\gamma_{2n-1}\gamma_{2n}),$$

i.e., ${\gamma_{2n}}^2 < \gamma_{2n}^2$, which is a contradiction. Thus $\gamma_{2n} \le \gamma_{2n-1}$.

In a similar way, if n is odd, then we can obtain $\gamma_{2n+1} \le \gamma_{2n}$. It follows that the sequence $\{\gamma_n\}$ is non-increasing.

Now we prove that $\lim_{n\to\infty} \gamma_n = 0$.
Suppose $\lim_{n\to\infty} \gamma_n \ne 0$, i.e.,

$$\lim_{n\to\infty} \gamma_n = t, \text{ for some } t > 0. \tag{2.5}$$

Taking $n \to \infty$ in (2.3) and using the inequality (2.4) and (2.5) with the properties of ϕ, ψ, we have

$$[1+pt]t^2 \le pt^3 + t^2 - \phi(t^2).$$

This implies that $\phi(t^2) \le 0$, which is a contradiction to the definition of ϕ. Thus

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} d(v_n, v_{n-1}) = 0. \tag{2.6}$$

This completes the proof. □

Lemma 2.2 *The sequence $\{v_n\}$, defined by (2.1), is a Cauchy sequence in E.*

Proof: Let us assume that $\{v_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which one can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that

$$d(v_{m(k)}, v_{n(k)}) \ge \epsilon \quad \text{and} \quad d(v_{m(k)}, v_{n(k)-1}) < \epsilon \tag{2.7}$$

for all positive integers $k, n(k) > m(k) > k$. Then

$$\epsilon \le d(v_{m(k)}, v_{n(k)}) \le d(v_{m(k)}, v_{n(k)-1}) + d(v_{n(k)-1}, v_{n(k)}).$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} d(v_{m(k)}, v_{n(k)}) = \epsilon. \tag{2.8}$$

Now from the triangular inequality, we have

$$|d(v_{n(k)}, v_{m(k)+1}) - d(v_{m(k)}, v_{n(k)})| \le d(v_{m(k)}, v_{m(k)+1}).$$

Taking the limit as $k \to \infty$ and using (2.6) and (2.8), we have

$$\lim_{k\to\infty} d(v_{n(k)}, v_{m(k)+1}) = \epsilon. \tag{2.9}$$

Again from the triangular inequality, we have

$$|d(v_{m(k)}, v_{n(k)+1}) - d(v_{m(k)}, v_{n(k)})| \le d(v_{n(k)}, v_{n(k)+1})$$

Taking the limit as $k \to \infty$ and using (2.6) and (2.8), we have

$$\lim_{k\to\infty} d(v_{m(k)}, v_{n(k)+1}) = \epsilon. \tag{2.10}$$

Similarly, using the triangular inequality, we have

$$|d(v_{m(k)+1}, v_{n(k)+1}) - d(v_{m(k)}, v_{n(k)})| \le d(v_{m(k)}, v_{m(k)+1}) + d(v_{n(k)}, v_{n(k)+1}).$$

Taking the limit as $k \to \infty$ in the above inequality and using (2.6) and (2.8), we have

$$\lim_{k\to\infty} d(v_{n(k)+1}, v_{m(k)+1}) = \epsilon. \tag{2.11}$$

Taking $u = u_{m(k)}$ and $v = u_{n(k)}$ and using equation (2.1) in (C_2), we get

$$\begin{aligned}
&[1 + pd(v_{m(k)-1}, v_{n(k)-1})]d^2(v_{m(k)}, v_{n(k)}) \\
&\quad \le p\psi\Big(d^2(v_{m(k)-1}, v_{m(k)})d(v_{n(k)-1}, v_{n(k)}), \\
&\qquad d(v_{m(k)-1}, v_{m(k)})d^2(v_{n(k)-1}, v_{n(k)}), \\
&\qquad d(v_{m(k)-1}, v_{m(k)})d(v_{m(k)-1}, v_{n(k)})d(v_{n(k)-1}, v_{m(k)}), \\
&\qquad d(v_{m(k)-1}, v_{n(k)})d(v_{n(k)-1}, v_{m(k)})d(v_{n(k)-1}, v_{n(k)})\Big) \\
&\qquad +m(v_{m(k)-1}, v_{n(k)-1}) - \phi\big(m(v_{m(k)-1}, v_{n(k)-1})\big),
\end{aligned}$$

where

$$\begin{aligned}
m(v_{m(k)-1},v_{n(k)-1}) = \max\Big\{ & d^2(v_{m(k)-1}, v_{n(k)-1}), d(v_{m(k)-1}, v_{m(k)})d(v_{n(k)-1}, v_{n(k)}), \\
& d(v_{m(k)-1}, v_{n(k)})d(v_{n(k)-1}, v_{m(k)}), \frac{1}{2}\Big[d(v_{m(k)-1}, v_{m(k)})d(v_{m(k)-1}, v_{n(k)}) \\
& + d(v_{n(k)-1}, v_{m(k)})d(v_{n(k)-1}, v_{n(k)})\Big]\Big\}.
\end{aligned}$$

Letting $k \to \infty$ and using (2.6)–(2.11) with the properties of ϕ and ψ, we obtain

$$[1 + p\epsilon]\epsilon^2 \le p\psi(0, 0, 0, 0) + \epsilon^2 - \phi(\epsilon^2) < \epsilon^2,$$

which is a contradiction. Thus the sequence $\{v_n\}$ is a Cauchy sequence in E. This completes the proof. □

Now, we prove our main results. First, we state and prove fixed point theorem for compatible mappings of type (E) along with split reciprocal continuity as follows:

Theorem 2.1 *Four self mappings f, g, S and T on a complete metric space (E, d) satisfying conditions (C_1) and (C_2) have a unique common fixed point in E if the pairs (f, S) and (g, T) satisfy either of the following:*

(a) (f,S) *is* $f-$*compatible of type* (E) *and* $f-$*reciprocally continuous,* (g,T) *is* $g-$*compatible of type* (E) *and* $g-$*reciprocally continuous;*

(b) (f,S) *is* $S-$*compatible of type* (E) *and* $S-$*reciprocally continuous,* (g,T) *is* $T-$*compatible of type* (E) *and* $T-$*reciprocally continuous.*

Proof: By Lemma 2.2, the sequence $\{v_n\}$, defined by (2.1), is a Cauchy sequence in E. Since (E,d) is a complete metric space, $\{v_n\}$ converges to a point $w \in E$, as $n \to \infty$. Consequently, the subsequences $\{Su_{2n}\}$, $\{fu_{2n}\}$, $\{Tu_{2n+1}\}$ and $\{gu_{2n+1}\}$ also converge to the same point w.

Suppose the pair (f,S) is $f-$compatible of type (E) and $f-$reciprocally continuous. Then by Proposition 1.1, $fw = Sw$.

Since $S(E) \subset g(E)$, there exists a point $u^* \in E$ such that $Sw = gu^*$, i.e., $fw = Sw = gu^*$.

We claim that $Tu^* = gu^*$.

For this, taking $u = w$, $v = u^*$ in (C_2), we get

$$[1 + pd(fw, gu^*)]d^2(Sw, Tu^*) \leq p\psi(0,0,0,0) + m(fw, gu^*) - \phi(m(fw, gu^*)),$$

where

$$\begin{aligned} m(fw, gu^*) = \max\Big\{&d^2(fw, gu^*), d(fw, Sw)d(gu^*, Tu^*), d(fw, Tu^*)d(gu^*, Sw), \\ &\frac{1}{2}[d(fw, Sw)d(fw, Tu^*) + d(gu^*, Sw)d(gu^*, Tw)]\Big\} \\ = 0.& \end{aligned}$$

Solving the above inequality and using the value of $m(fw, gu^*)$ along with the properties of ϕ and ψ, we get $d^2(Sw, Tu^*) = 0$, which further gives $Tu^* = Sw = gu^*$.

Since the pair (g,T) is $g-$compatible of type (E) and $g-$reciprocally continuous and $gu^* = Tu^*$, by Proposition 1.1, $gw = gTu^* = Tgu^* = Tw$.

We claim that w is a fixed point of f, i.e., $fw = w$.

For this, letting $u = w$ and $v = u_{2n+1}$ in (C_2) and letting $n \to \infty$, we have

$$[1 + pd(fw, w)]d^2(Sw, w) \leq p\psi(0,0,0,0) + m(fw, w) - \phi(m(fw, w)),$$

where

$$\begin{aligned} m(fw, w) = \max\Big\{&d^2(fw, w), d(fw, Sw)d(w, w), d(fw, w)d(w, Sw), \\ &\frac{1}{2}[d(fw, Sw)d(fw, w) + d(w, Sw)d(w, w)]\Big\} \\ =&d^2(fw, w). \end{aligned}$$

So we get $d^2(fw, w) = 0$, which implies that $fw = w$. Hence we have $w = fw = Sw$.

Now, we prove that w is a fixed point of g.

For this, taking $u = v = w$ in (C_2), we get

$$[1 + pd(w, gw)]d^2(w, Tw) \leq p\psi(0,0,0,0) + m(w, gw) - \phi(m(w, gw)),$$

where

$$\begin{aligned} m(w, gw) = \max\Big\{&d^2(w, gw), d(w, w)d(gw, Tw), d(w, Tw)d(gw, w), \\ &\frac{1}{2}[d(w, w)d(w, Tw) + d(gw, w)d(gw, Tw)]\Big\} \\ = &d^2(w, gw). \end{aligned}$$

After simplification, we get $d^2(w, gw) = 0$, which implies that $w = gw$. Thus $w = fw = Sw = gw = Tw$, i.e., w is a common fixed point of f, g, S and T.

Similarly, one can complete the proof when the pairs (f, S) and (g, T) satisfy the condition (b).

For the uniqueness, suppose that w and z are two common fixed points of f, g, S and T. Taking $u = w$, $v = z$ in (C_2), we get

$$[1 + pd(w, z)]d^2(w, z) \leq p\psi(0, 0, 0, 0) + m(w, z) - \phi(m(w, z)),$$

where

$$\begin{aligned} m(w, z) = \max\Big\{&d^2(w, z), d(w, w)d(z, z), d(w, z)d(z, w),\\ &\frac{1}{2}[d(w, w)d(w, z) + d(z, w)d(z, z)]\Big\}\\ = d^2(w, z).& \end{aligned}$$

After simplification, the above inequality reduces to $d^2(w, z) = 0$, which implies that $w = z$. This proves the uniqueness of the common fixed point of f, g, S and T. This completes the proof. □

Example 2.1 *Let $E = [0, 5]$ and d be a usual metric. Let $S, T : E \to E$ be two mappings defined by*

$$Tu = Su = \begin{cases} \frac{5+u}{2}, & u \in [0, \frac{5}{2}), \\ \frac{5-u}{2}, & u \in [\frac{5}{2}, 5], \end{cases}$$

$$gu = fu = \begin{cases} \frac{5}{2} + u, & u \in [0, \frac{5}{2}), \\ \frac{5}{2}, & u = \frac{5}{4}, \\ \frac{24}{5}, & u \in (\frac{5}{2}, 5]. \end{cases}$$

Then $S(E) = [\frac{5}{2}, \frac{15}{4}) = T(E)$ and $f(E) = g(E) = [\frac{5}{2}, 5)$. The mappings are not continuous at $u = \frac{5}{2}$. Let $\{u_n\}$ be a sequence in E such that $u_n \to 0$, $u_n > 0$ for all n. Then Su_n, $fu_n \to \frac{1}{2} = t$ and $SSu_n = S(\frac{5+u_n}{2}) \to \frac{5}{2}$, $Sfu_n = S(\frac{5}{2} + u_n) \to \frac{5}{2}$, $ffu_n = f(\frac{5}{2} + u_n) \to \frac{24}{5}$, and $fSu_n = f(\frac{5+u_n}{2}) \to \frac{24}{5}$. Also, we have $ft = \frac{5}{2} = St$. Thus $SSu_n, Sfu_n \to \frac{5}{2} = ft = f\frac{5}{2}$ and $Sfu_n \to \frac{5}{2} = St = S\frac{1}{2}$. So the pair (f, S) is $S-$compatible of type (E) and $S-$reciprocally continuous and the pair (g, T) is $T-$compatible of type (E) and $T-$reciprocally continuous. In particular, if we take $\psi(t_1, t_2, t_3, t_4) = \max\{\frac{t_1+t_2}{2}, t_3, t_4\}$, $\phi(t) = \frac{3}{2}t$, and $0 < p$, then it satisfies all the conditions of Theorem 2.1 and $\frac{5}{2}$ is the unique common fixed point of f, g, S and T.

Next, we prove fixed point theorem for compatible mappings of type (K) with reciprocal continuity.

Theorem 2.2 *Let f, g, S and T be four self-mappings on a complete metric space (E, d) satisfying the conditions (C_1) and (C_2). Then S, T, f and g have a unique common fixed point in E, provided that (f, S) and (g, T) are the pairs of reciprocally continuous and compatible mappings of type (K).*

Proof: From Lemma 2.2, the sequence $\{v_n\}$, defined by (2.1), is a Cauchy sequence in E. Since (E, d) is a complete metric space, $\{v_n\}$ converges to a point $w \in E$, as $n \to \infty$. Consequently, the subsequences $\{Su_{2n}\}$, $\{fu_{2n}\}$, $\{Tu_{2n+1}\}$ and $\{gu_{2n+1}\}$ also converge to the same point w.

Since the mappings f and S are compatible of type (K), $ffu_{2n} \to Sw$, $SSu_{2n} \to fw$ as $n \to \infty$. Also the reciprocal continuity of the pair (f, S) implies that $\{fSu_{2n}\}$ converges to fw and $\{Sfu_{2n}\}$ converges to Sw as $n \to \infty$.

Similarly, the compatibility of type (K) with reciprocal continuity of the pair (g, T) implies that $ggu_{2n} \to Tw$, $TTu_{2n} \to gw$, $gTu_{2n} \to gw$ and $Tgu_{2n} \to Tw$ as $n \to \infty$.

Now we claim that $gw = fw$.

For this, taking $u = Su_{2n}$ and $v = Tu_{2n+1}$ in (C_2) and letting $n \to \infty$, we get

$$[1 + pd(fw, gw)]d^2(fw, gw) \leq p\psi(0, 0, 0, 0) + m(fw, gw) - \phi(m(fw, gw)),$$

where

$$\begin{aligned} m(fw,gw) &= \max\Big\{d^2(fw,gw), d(fw,fw)d(gw,gw), d(fw,gw)d(gw,fw),\\ &\qquad \frac{1}{2}[d(fw,fw)d(fw,gw)+d(gw,fw)d(gw,gw)]\Big\}\\ &= d^2(fw,gw). \end{aligned}$$

So we get $d^2(fw,gw)=0$, which implies that $fw=gw$.

Now we shall show that $gw=Sw$.

For this, letting $u=w$ and $v=Tu_{2n+1}$ in (C_2) and letting $n\to\infty$, we get

$$\begin{aligned} [1+pd(fw,gw)]d^2(Sw,gw) \le& p\psi\Bigg(d^2(fw,Sw)d(gw,gw), d(fw,Sw)d^2(gw,gw),\\ & d(fw,Sw)d(fw,gw)d(gw,Sw), d(fw,gw)d(gw,Sw)d(gw,gw)\Bigg)\\ &+m(fw,gw)-\phi\big(m(fw,gw)\big), \end{aligned}$$

where

$$\begin{aligned} m(fw,gw) = \max\Bigg\{& d^2(fw,gw), d(fw,Sw)d(gw,gw), d(fw,gw)d(gw,Sw),\\ & \frac{1}{2}\big[d(fw,Sw)d(fw,gw)+d(gw,Sw)d(gw,gw)\big]\Bigg\}=0. \end{aligned}$$

Solving the above inequality, we get $d^2(Sw,gw)=0$, i.e., $Sw=gw$. Hence $fw=gw=Sw$.

Next we claim that $Sw=Tw$.

For this, taking $u=v=w$ in (C_2), we have

$$\begin{aligned} [1+pd(fw,gw)]d^2(Sw,Tw) \le p\psi\Bigg(& d^2(fw,Sw)d(gw,Tw), d(fw,Sw)d^2(gw,Tw),\\ & d(fw,Sw)d(fw,Tw)d(gw,Sw),\\ & d(fw,Tw)d(gw,Sw)d(gw,Tw)\Bigg)\\ &+m(fw,gw)-\phi\big(m(fw,gw)\big), \end{aligned}$$

where

$$\begin{aligned} m(fw,gw) = \max\Bigg\{& d^2(fw,gw), d(fw,Sw)d(gw,Tw), d(fw,Tw)d(gw,Sw),\\ & \frac{1}{2}\big[d(fw,Sw)d(fw,Tw)+d(gw,Sw)d(gw,Tw)\big]\Bigg\}=0. \end{aligned}$$

i.e.,

$$[1+0]d^2(Sw,Tw)\le p\psi(0,0,0,0)+0-\phi(0),$$

That is, $d^2(Sw,Tw)=0$. This implies that $Sw=Tw$. Hence $gw=Tw=fw=Sw$, i.e., w is a coincidence point of S,T,f and g.

It remains to prove that w is a common fixed point of S,T,f and g.

For this, taking $u = u_{2n}$ and $v = w$ in (C_2) and letting $n \to \infty$, we get

$$[1 + pd(w, gw)]d^2(w, Tw) \le p\psi(0, 0, 0, 0) + m(w, gw) - \phi(m(w, gw)),$$

where

$$m(w, gw) = \max\Big\{d^2(w, gw), d(w, w)d(gw, Tw), d(w, Tw)d(gw, w), \frac{1}{2}[d(w, w)d(w, Tw) + d(gw, w)d(gw, Tw)]\Big\}.$$

Solving the above inequality with the the properties of ψ, ϕ, we obtain $d^2(w, Tw) = 0$, i.e., $w = Tw$. Hence $fw = gw = Sw = Tw = w$. Therefore, w is a common fixed point of f, g, S and T.

The uniqueness can be proved easily. This completes the proof. □

Now we prove fixed point theorem for pairs of reciprocally continuous subcompatible mappings.

Theorem 2.3 *Let f, g, S and T be four self mappings on a complete metric space (E, d) satisfying the condition (C_1) and (C_2). Then S, T, f and g have a unique common fixed point in E, provided that (f, S) and (g, T) are the pairs of subcompatible mappings and reciprocally continuous mappings.*

Proof: Since the pair (f, S) is subcompatible, there exists a sequence $\{u_n\}$ in E such that

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} Su_n = t,$$

for some $t \in E$ which satisfy

$$\lim_{n\to\infty} d(fSu_n, Sfu_n) = 0.$$

Due to the reciprocal continuity of the pair (f, S), $ft = St$, i.e., t is a coincidence point of f and S.

Next the subcompatibility of the pair (g, T) implies that there exists a sequence $\{w_n\}$ in E such that

$$\lim_{n\to\infty} gw_n = \lim_{n\to\infty} Tw_n = z$$

for some $z \in E$ which satisfy

$$\lim_{n\to\infty} d(gTw_n, Tgwu_n) = 0.$$

Due to thereciprocal continuity of the pair (g, T), $gz = Tz$, i.e., z is a coincidence point of g and T.

We claim that $t = z$.

For this, taking $u = u_n$ and $v = w_n$ in (C_2) and letting $n \to \infty$, we have

$$[1 + pd(t, z)]d^2(t, z) \le p\psi(0, 0, 0, 0) + m(t, z) - \phi(m(t, z)),$$

where

$$m(t, z) = \max\Big\{d^2(t, z), d(t, t)d(z, z), d(t, z)d(z, t), \frac{1}{2}[d(t, t)d(t, z) + d(z, t)d(z, z)]\Big\} = d^2(t, z).$$

Using the value of $m(t, z)$ along with the properties of ϕ and ψ the above inequality reduces to $d^((t, z) = 0$, which implies that $t = z$.

Now we prove that $St = t$.

For this, taking $u = t$ and $v = w_n$ in (C_2) and letting $n \to \infty$, we get

$$[1 + pd(ft, z)]d^2(St, z) \le p\psi(0, 0, 0, 0) + m(ft, z) - \phi(m(ft, z)),$$

where

$$\begin{aligned} m(ft,z) =& \max\Big\{d^2(ft,z), d(ft,St)d(z,z), d(ft,z)d(z,St), \\ & \frac{1}{2}[d(ft,St)d(ft,z)+d(z,St)d(z,z)] \\ =& d^2(St,t). \end{aligned}$$

Simplifying the above inequality, we have $d^2(St,t)=0$, which implies that $St=t$, i.e., $ft=St=t$.

Now we claim that $gt=t$.

For this, taking $u=v=t$ in (C_2), we get

$$[1+pd(ft,gt)] \leq p\psi(0,0,0,0)+m(ft,gt)-\phi(m(ft,gt)),$$

where

$$\begin{aligned} m(ft,gt) =& \max\Big\{d^2(ft,gt), d(ft,St)d(gt,Tt), d(ft,Tt)d(gt,St), \\ & \frac{1}{2}[d(ft,St)d(ft,Tt)+d(gt,St)d(gt,Tt)] \\ =& d^2(t,gt). \end{aligned}$$

The above inequality gives $d^2(t,gt)=0$, which implies that $gt=t$. Hence $ft=St=gt=Tt=t$, i.e., t is a common fixed point of f,S,g and T.

The uniqueness follows easily. This completes the proof. □

Example 2.2 *Let $E=[0,10]$ and d be a usual metric. Let $f,g,S,T:E\to E$ be four mappings defined by*

$$fu = \begin{cases} 0, & u=0, \\ 8, & u\in(0,2.5], \\ u-2.5, & u\in(2.5,10], \end{cases}$$

$$gu = \begin{cases} u, & u=0, \\ 4, & u\in(0,10], \end{cases}$$

$$Su = \begin{cases} 4, & u\in(0,2.5], \\ 0, & u\in(2.5,10]\cup\{0\}, \end{cases}$$

$$Tu = \begin{cases} u, & u=0, \\ 2, & u\in(0,10]. \end{cases}$$

Let p be a positive real number and $\phi:[0,\infty)\to[0,\infty)$ be a function defined by $\phi(t)=\frac{3}{2}t$ for $t\geq 0$ and $\psi:[0,\infty)^4\to[0,\infty)$ be a function defined by

$$\psi(w_1,w_2,w_3,w_4)=\max\{\frac{w_1+w_2}{2},\, w_3,\, w_4\},$$

where $w_i\geq 0, 1\leq i\leq 4$.

Let us consider a sequence $\{u_n\}$ with $u_n=0$. It is clear that the pair (f,S) and the pair (g,T) are the pairs of subcompatible and reciprocally continuous mappings. Also, one can easily verify that all the conditions of Theorem 2.3 are satisfied and so 0 is the unique common fixed point of f,g,S and T.

Remark 2.1 *If we consider the function $\psi:[0,\infty)^4\to[0,\infty)$ defined by*

$$\psi(t_1,t_2,t_3,t_4)=\max\{\tfrac{1}{2}[t_1+t_2],t_3,t_4\},$$

in Theorems 2.2 and 2.3, then we conclude that our results generalize the results of Jain et al. [8,10] for compatible mappings of type (E), type (K) and subcomaptible mappings, respectively.

Corollary 2.1 *Let (E,d) be a complete metric space. Suppose that $S,T:E\to E$ are two mappings satisfying the following:*

(C_1^*) $T(E)\subset S(E)$;

(C_2^*) *For all $u,v\in E$, $p>0$, $\psi\in\Psi$, $\phi\in\Phi$,*

$$\begin{aligned}[1+pd(Su,Sv)]d^2(Tu,Tv) \le p\,\psi\Big(& d^2(Su,Tu)d(Sv,Tv), d(Su,Tu)d^2(Sv,Tv),\\ & d(Su,Tu)d(Su,Tv)d(Sv,Tu), d(Su,Tv)d(Sv,Tu)d(Sv,Tv)\Big)\\ & + m(Su,Sv)-\phi(m(Su,Sv)),\end{aligned}$$

where

$$\begin{aligned}m(Su,Sv)=\max\Big\{& d^2(Su,Sv), d(Su,Tu)d(Sv,Tv), d(Su,Tv)d(Sv,Tu),\\ & \frac{1}{2}[d(Su,Tu)d(Su,Tv)+d(Sv,Tu)d(Sv,Tv)]\Big\}.\end{aligned}$$

If S and T are compatible mappings of type (K) and reciprocally continuous, then S and T have a unique common fixed point in E.

Proof: Letting $f=g=S$ and $S=T$ in Theorem 2.2 and applying Theorem 2.2, one can get the result. □

Corollary 2.2 *Let S and T be two self mappings on a metric space (E,d) satisfying the conditions (C_1^*) and (C_2^*) of Corollary 2.1. If the pair (S,T) satisfies of of the following:*

(a) (S,T) is $S-$compatible of type (E) and $S-$reciprocally continuous;

(b) (S,T) is $T-$compatible of type (E) and $T-$reciprocally continuous.

Then S and T have a unique common fixed point in E.

Proof: Letting $f=g=S$ and $S=T$ in Theorem 2.1 and applying Theorem 2.1, one can obtain the result. □

Corollary 2.3 *Let S and T be two self mappings on a metric space (E,d) satisfying the conditions (C_1^*) and (C_2^*) of Corollary 2.1. If the pair (S,T) is a pair of subcompatible and reciprocally continuous mappings, then S and T have a unique common fixed point in E.*

Proof: Letting $f=g=S$ and $S=T$ in Theorem 2.3 and applying Theorem 2.3, one can obtain the result. □

3. Application

In 2001, Branciari [7] obtained Banach contraction principle for mapping satisfying an integral type contraction condition. On the similar lines, we analyze our results for mappings satisfying a generalized $(\phi - \psi)-$weak contraction condition of integral type.

Theorem 3.1 *Let f, g, S and T be four self mappings on a complete metric space (E, d) satisfying the conditions (C_1) and*

(C_4) *for $u, v \in E$,*

$$\int_o^{M(u,v)} \gamma(t)\, dt \leq \int_o^{N(u,v)} \gamma(t)\, dt,$$

where
$M(u, v) = [1 + pd(fu, gv)]d^2(Su, Tv)$,

$$N(u, v) = p\,\psi\bigg(d^2(fu, Su)d(gv, Tv), d(fu, Su)d^2(gv, Tv), d(fu, Su)d(fu, Tv)d(gv, Su),$$
$$d(fu, Tv)d(gv, Su)d(gv, Tv)\bigg) + m(fu, gv) - \phi(m(fu, gv)),$$

where

$$m(fu, gv) = \max\Big\{d^2(fu, gv), d(fu, Su)d(gv, Tv), d(fu, Tv)d(gv, Su),$$
$$\frac{1}{2}[d(fu, Su)d(fu, Tv) + d(gv, Su)d(gv, Tv)]\Big\},$$

and $\psi \in \Psi$, $\phi \in \Phi$, $p > 0$ is a real number and $\gamma : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable function which is summable on each compact subset of $[0, \infty)$ such that for each $\epsilon > 0$, $\int_o^{\epsilon} \gamma(t)\, dt > 0$. If the pairs (f, S) and (g, T) are compatible mappings of type (K) and reciprocally continuous, then f, g, S and T have a unique common fixed point.

Proof: By putting $\gamma(t) = c$ (some nonzero constant), the result follows from Theorem 2.2. □

Theorem 3.2 *Let f, g, S and T be four self mappings on a complete metric space (E, d) satisfying the conditions (C_1) and (C_4). If the pairs (f, S) and (g, T) satisfy one of the following:*

(a) (f, S) is $f-$compatible of type (E) and $f-$reciprocally continuous, (g, T) is $g-$compatible of type (E) and $g-$reciprocally continuous;

(b) (f, S) is $S-$compatible of type (E) and $S-$reciprocally continuous, (g, T) is $T-$compatible of type (E) and $T-$reciprocally continuous.

Then f, g, S and T have a unique common fixed point.

Proof: By putting $\gamma(t) = c$ (some nonzero constant), the result follows from Theorem 2.1. □

Theorem 3.3 *Let f, g, S and T be four self mappings on a complete metric space (E, d) satisfying the conditions (C_1) and (C_4). If the pairs (f, S) and (g, T) are subcompatible mappings and reciprocally continuous, then f, g, S and T have a unique common fixed point.*

Proof: By putting $\gamma(t) = c$ (some nonzero constant), the result follows from Theorem 2.3. □

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