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Weak Solutions for Obstacle Problems Via Young Measure in Variable-exponent Sobolev Space

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ABSTRACT: We first discuss the existence and uniqueness of weak solution for the obstacle problem $\int_\Omega \phi_1\big(y, Dw - \mathcal{T}(w)\big) \colon D(\nu - w) + \phi_2(y, Dw) : (\nu - w) + \Big\langle w|w|^{r(y)-2}, \nu - w \Big\rangle \ \mathrm{d}y \geq 0$ with variable exponent, where $w : \Omega \to \mathbb{R}^m$ is a vector-valued function and Ω is a bounded open domain in $\mathbb{R}^n (n \geq 2)$, the existence is proved by means of the Young measure and under assumptions on ϕ_1, ϕ_2 and a theorem of Kinderlehrer and Stampacchia.

Keywords: Obstacle problem, Kinderlehrer and Stampacchia, Young measures, Sobolev spaces.

Contents

1. Introduction and main result

This article will look at the role of Young measure theory in tackling the obstacle problem in partial differential equations (PDE) and the significance of the Kinderlehrer and Stampacchia theorem which proves the existence of a weak solution for the obstacle problem. The obstacle problem is a boundary value problem in which the solution must stay above a given function, the obstacle.
In [8] the author proved existence and uniqueness of weak solution for the obstacle problem of the nonhomogeneous A-harmonic equation for differential forms $d^\star A(x, d\omega) = B(x, d\omega)$ in the ev space $W_d^{p(x)}\left(\Omega, \Lambda^l, \mu\right)$.

$$\int_\Omega (A(x, du) \cdot d(v-u) + B(x, du) \cdot (v-u))dx \geq 0,$$

for v belonging to

$$\Omega_{\psi,\theta} = \left\{ v \in W_d^{p(x)}\left(\Omega, \Lambda^l, \mu\right) : v \geq \psi \ \text{ a.e. } \ x \in \Omega, v - \theta \in W_{0d}^{p(x)}\left(\Omega, \Lambda^l, \mu\right) \right\}.$$

In this paper, we investigate the conditions under which a weak solution for an obstacle problem can be established as uniquely existing.

$$\begin{cases} \displaystyle\int_\Omega \phi_1\big(y, Dw - \mathcal{T}(w)\big) \colon D(\nu - w) + \phi_2(y, Dw) : (\nu - w) + \Big\langle w|w|^{r(y)-2}, \nu - w \Big\rangle \ \mathrm{d}y & \geq 0 \\ \\ \nu \in \mathcal{G}_{\varphi,\Lambda} \end{cases} \tag{1.1}$$

where

$$\mathcal{G}_{\varphi,\Lambda} = \left\{ \nu \in W^{1,r(y)}(\Omega; \mathbb{R}^m) : \ \nu - \Lambda \in W_0^{1,r(y)}(\Omega; \mathbb{R}^m), \ \nu \geq \varphi \text{ a.e in } \Omega \right\}. \tag{1.2}$$

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Here Ω is a bounded open domain in $\mathbb{R}^n (n \geq 2)$ and $u : \Omega \to \mathbb{R}^m$ is a vector-valued function and $\mathcal{T} : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a Lipschitz continuous function and variable exponent $r(y)$ with locally log-Hölder continuity in Ω satisfies

$$1 < r^- \leq r(y) \leq r^+ < \infty \quad \text{for a.e. } \ y \in \Omega. \tag{1.3}$$

Let $\mathbb{M}^{m\times n}$ the set of real m by n matrices equipped with the usual inner product $\mathcal{A} : \mathcal{B} = \sum_{i,j} \mathcal{A}_{ij}\mathcal{B}_{ij}$. The obstacle function $\varphi : \Omega \to \mathbb{R}^m$ defined in (1.2) and $\Lambda \in W^{1,r(y)}(\Omega; \mathbb{R}^m)$ is a function which gives the boundary values. We will study the solution $w \in \mathcal{G}_{\varphi,\Lambda}$ for (1.1) under the following hypotheses:

($\mathcal{I}_0$) **(continuity)** $\phi_1 : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ and $\phi_2 : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ are measurable functions, for all $\mathcal{A} \in \mathbb{M}^{m\times n}$ with respect to y and continuous for a.e. $y \in \Omega$ with respect to $\mathcal{A}$.

($\mathcal{I}_1$) **(coercivity and growth)** There exist $d_1, d_3 \in L^{r'}(\Omega)$, $d_2, d_4 \in L^1(\Omega)$ and $c_i > 0$, $i = 1, 2, 3, 4$ such that

$$\begin{gathered}
\big|\phi_1\big(y, \mathcal{A} - \mathcal{T}(w)\big)\big| \leq d_1(y) + c_1|\mathcal{A} - \mathcal{T}(w)|^{r(y)-1}, \\
\phi_1\big(y, \mathcal{A} - \mathcal{T}(w)\big) \colon \mathcal{A} \geq -d_2(y) + c_2\big|\mathcal{A} - \mathcal{T}(w)\big|^{r(y)}, \\
\big|\phi_2(y, \mathcal{A})\big| \leq d_3(y) + c_3\big|\mathcal{A}\big|^{r(y)-1}, \\
\phi_2(y, D\mathcal{A}) : \mathcal{A} \geq -d_4(y) + c_4\big|\mathcal{A}\big|^{r(y)}.
\end{gathered}$$

The function $\mathcal{T} : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is continuous such that

$$\mathcal{T}(0) = 0 \text{ and } \Big|\mathcal{T}(w_1) - \mathcal{T}(w_2)\Big| \leq C_{\mathcal{T}}\Big|w_1 - w_2\Big| \quad \forall w_1, w_2 \in \mathbb{R}^m,$$

where $C_{\mathcal{T}}$ is a positive constant related to the exponent $r(y)$ and the diameter of Ω (diam (Ω)) by the following $C_{\mathcal{T}} < \frac{1}{\text{diam}(\Omega)}\left(\frac{1}{2}\right)^{\frac{1}{r^+}}$.

($\mathcal{I}_2$) ϕ_1 satisfies one of the following conditions:

(a) The map $\mathcal{A} \mapsto \phi_1\big(y, \mathcal{A} - \mathcal{T}(w)\big)$ is strictly quasimonotone, i.e., there exists constants $c_5 > 0$ such that

$$\int_\Omega \Big(\phi_1\big(y, \mathcal{A} - \mathcal{T}(w)\big) - \phi_1\big(y, \mathcal{B} - \mathcal{T}(w)\big)\Big) \colon (\mathcal{A} - \mathcal{B}) \, \mathrm{d}y \geq c_5 \int_\Omega \Big|\mathcal{A} - \mathcal{B}\Big|^{r(y)} \, \mathrm{d}y,$$

for all $y \in \Omega$ and $\mathcal{A}, \mathcal{B} \in \mathbb{M}^{m\times n}$.

(b) There exists a function $Q : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\phi_1\big(y, \mathcal{A} - \mathcal{T}(w)\big) = \frac{\partial Q}{\partial \mathcal{A}}\big(y, \mathcal{A} - \mathcal{T}(w)\big)$, and $\mathcal{A} \to \phi_1\big(y, \mathcal{A} - \mathcal{T}(w)\big)$ is convex and C^1.

(c) **(monotonicity)** For all $y \in \Omega$, the map $\mathcal{A} \mapsto \phi_1(y, \mathcal{A})$ is a C^1-function and is monotone, i.e.

$$\Big(\phi_1\big(y, \mathcal{A} - \mathcal{T}(w)\big) - \phi_1\big(y, \mathcal{B} - \mathcal{T}(w)\big)\Big) \colon (\mathcal{A} - \mathcal{B}) \geq 0,$$

for all $y \in \Omega$ and $\mathcal{A}, \mathcal{B} \in \mathbb{M}^{m\times n}$.

($\mathcal{I}_3$) ϕ_2 satisfies one of the following conditions:

(a) The map $\mathcal{A} \mapsto \phi_2(y, \mathcal{A})$ is strictly quasimonotone, i.e., there exists constants $c_6 > 0$ such that

$$\int_\Omega \big(\phi_2(y, D\mathcal{A}) - \phi_2(y, D\mathcal{B})\big) \colon (\mathcal{A} - \mathcal{B}) \, \mathrm{d}y \geq c_6 \int_\Omega \Big|\mathcal{A} - \mathcal{B}\Big|^{r(x)} \, \mathrm{d}y,$$

for all $y \in \Omega$ and $\mathcal{A}, \mathcal{B} \in \mathbb{M}^{m\times n}$.

(b) There exists a function $\hbar : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\phi_2(y,\mathcal{A}) = \frac{\partial \hbar}{\partial \mathcal{A}}(y,\mathcal{A})$, and $\mathcal{A} \to \phi_2(y,\mathcal{A})$ is convex and C^1.

(c) (monotonicity) For all $y \in \Omega$, the map $\mathcal{A} \mapsto \phi_2(y,\mathcal{A})$ is a C^1-function and is monotone, i.e.

$$\big(\phi_2(y, D\mathcal{A}) - \phi_2(y, D\mathcal{B})\big) \colon (\mathcal{A} - \mathcal{B}) \geq 0,$$

for all $y \in \Omega$ and $\mathcal{A}, \mathcal{B} \in \mathbb{M}^{m\times n}$.

We will prove the following existence result for the obstacle problem (1.1) - (1.2):

Theorem 1.1 *Suppose $\mathcal{G}_{\varphi,\Lambda} \neq \emptyset$ and a satisfies the conditions $(\mathcal{I}_0)$, $(\mathcal{I}_2)$ and $(\mathcal{I}_3)$. Then there exists a weak solution $w \in \Im_{\psi,\theta}$ to the obstacle problem (1.1)-(1.2). In other words, there exists a function $w \in \mathcal{G}_{\varphi,\Lambda}$ satisfying*

$$\int_\Omega \phi_1\big(y, Dw - \mathcal{T}(w)\big)\colon D(\nu - w) + \phi_2(y, Dw) : (\nu - w) + \Big\langle w|w|^{r(x)-2}, \nu - w\Big\rangle \ \mathrm{d}y \geq 0,$$

for each $\nu \in \mathcal{G}_{\varphi,\Lambda}$.

This document is composed of four parts. Part two begins with the demonstration of a theorem from Kinderlehrer and Stampacchia, followed by a short description of Young measures, which will be utilized to verify the primary outcomes. In part three, the demonstration of the existence of solutions to obstacle problems is presented. Lastly, section four confirms the singularity of a weak solution to obstacle problems.

2. Preliminary

This section presents some important preliminary results and notation that will be used later.

2.1. Spaces of Lebesgue and Sobolev featuring variable exponents

We recall some necessary notations, definitions and properties for our function spaces (see [12,7] and an overview about Young measures (see [4,13,3]). For each open bounded subset Ω of $\mathbb{R}^n (n \geq 2)$, we denote $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}), r(y) > 1 \text{ for any } y \in \Omega\}$. We define for every $r \in C^+(\bar{\Omega})$,

$$r^- = \inf_{y\in\Omega} r(y) \text{ and } r^+ = \sup_{y\in\Omega} r(y).$$

The Sobolev space $W^{1,r(y)}(\Omega;\mathbb{R}^m)$ consists of all functions w in the Lebesgue space

$$L^{r(y)}(\Omega;\mathbb{R}^m) = \left\{ w : \Omega \to \mathbb{R}^m \text{ measurable} : \int_\Omega |w(y)|^{r(y)} \ \mathrm{d}y < \infty \right\},$$

such that $Dw \in L^{r(y)}(\Omega;\mathbb{M}^{m\times n})$. The space $L^{r(y)}(\Omega;\mathbb{R}^m)$ is endowed with the norm

$$\|w\|_{r(y)} = \inf \left\{ \zeta > 0, \int_\Omega \left| \frac{w(y)}{\zeta} \right|^{r(y)} \ \mathrm{d}y \leq 1 \right\}$$

It is a Banach space. Moreover, it is reflexive if and only if $1 < r^- \leq r^+ < \infty$. Its dual is defined by $L^{r'(x)}(\Omega;\mathbb{R}^m)$ where $\frac{1}{r(y)} + \frac{1}{r'(y)} = 1$. For any $w \in L^{r(y)}(\Omega;\mathbb{R}^m)$ and $v \in L^{r'(y)}(\Omega;\mathbb{R}^m)$, the generalized Hölder inequality

$$\left| \int_\Omega w\nu \ \mathrm{d}y \right| \leq \left(\frac{1}{r^-} + \frac{1}{r^+} \right) \|w\|_{r(y)} \|\nu\|_{r'(y)}$$

holds true. The space $W^{1,r(y)}(\Omega;\mathbb{R}^m)$ is endowed with the norm

$$\|w\|_{1,r(y)} = \|w\|_{r(y)} + \|Dw\|_{r(y)}$$

Proposition 2.1 *([14]). We denote* $\rho(w)=\int_{\Omega}|w|^{r(y)}dy, \forall u \in L^{r(y)}(\Omega;\mathbb{R}^m)$*. If* $w_k, w \in L^{r(y)}(\Omega;\mathbb{R}^m)$ *and* $r^+<\infty$*, then:*

i) $\|w\|_{r(y)}<1(=1;>1) \Leftrightarrow \rho(w)<1(=1;>1)$.

ii) $\|w\|_{r(y)}>1 \Rightarrow \|w|^{r^-}_{r(y)} \leq \rho(w) \leq \|w\|^{r^+}_{r(y)}; |w\|_{r(y)}<1 \Rightarrow \|w\|^{r^+}_{r(y)} \leq \rho(w) \leq \|w\|^{r^-}_{r(y)}$.

iii) $\|w_k\|_{r(y)} \rightarrow 0 \Leftrightarrow \rho(w_k) \rightarrow 0; \|w_k\|_{r(y)} \rightarrow +\infty \Leftrightarrow \rho(w_k) \rightarrow +\infty$

We denote by $W_0^{1,r(y)}(\Omega;\mathbb{R}^m)$ the closure of $C_0^{\infty}(\Omega;\mathbb{R}^m)$ in $W^{1,r(y)}(\Omega;\mathbb{R}^m)$ and $W^{-1,r'(y)}(\Omega;\mathbb{R}^m)$ is its dual space. We denote $r^*(y)=\frac{nr(y)}{n-r(y)}$ for $r(y)<n;=\infty$ for $r(y)>n$.

Theorem 2.1 *(see [5,15]) If* $r(y)$ *satisfies* (1.3)*, then the inequality*

$$\int_{\Omega}\langle w(y),\nu(y)\rangle \text{ dy} \leq C\|w(y)\|_{L^{r(y)}(\Omega,\mathbb{R}^m)}\|\nu(y)\|_{L^{r'(y)}(\Omega,\mathbb{R}^m)}$$

holds for every $w(y) \in L^{r(y)}(\Omega,\mathbb{R}^m), \nu(y) \in L^{r'(y)}(\Omega,\mathbb{R}^m)$ *with the constant* C *dependent on* $r(y)$ *only.*

Theorem 2.2 *(see [5,15]) If* $r(y)$ *satisfies* (1.3)*, then the spaces* $L^{r(y)}(\Omega,\mathbb{R}^m)$ *and* $W^{1,r(y)}(\Omega,\mathbb{R}^m)$ *are a reflexive Banach space.*

2.2. Important knowledge on Young Measures

Evans [6] demonstrates that the weak convergence of nonlinear functionals and operators can be addressed through the use of Young measure. This technique provides a way to prove the desired result by preserving the compactness properties of the convergence in finite dimensional spaces. To assist those who are unfamiliar with the concept, numerous resources are available, including [2,6,9,11] and the references therein.

By $C_0(\mathbb{R}^m)$ we denote the set of functions $h \in C(\mathbb{R}^m)$ satisfying $\lim_{|\eta|\rightarrow\infty} h(\eta)=0$. Its dual is the well known space of signed Radon measures with finite mass and denoted by $\mathcal{M}(\mathbb{R}^m)$. The duality pairing of these spaces is defined for $\nu:\Omega \longrightarrow \mathcal{M}(\mathbb{R}^m)$ as

$$\langle\nu, h\rangle=\int_{\mathbb{R}^m} h(\eta) d\nu(\eta).$$

Definition 2.1 *Assume that the sequence* $\{f_j\}_{j\geq 1}$ *is bounded in* $L^{\infty}(\Omega;\mathbb{R}^m)$*. Then there exist a subsequence* $\{f_k\}_{k\geq 1} \subset \{f_j\}_{j\geq 1}$ *and a Borel probability measure* ν_y *on* $\mathbb{R}^m$ *for a.e.* $y \in \Omega$*, so that for each* $\varphi \in C(\mathbb{R}^m)$ *we have*

$$\varphi(f_k) \rightarrow^* \bar{\varphi} \text{ weakly }^* \text{ in } L^{\infty}(\Omega),$$

where $\bar{\varphi}(y):=\int_{\mathbb{R}^m}\varphi(\lambda)d\nu_y(\lambda)$ *for a.e.* $y \in \Omega$*. We call* $\{\nu_y\}_{y\in\Omega}$ *the family of Young measure associated with* $\{f_k\}_{k\geq 1}$*.*

Lemma 2.1 *([4]). Let* $\Omega \subset \mathbb{R}^n$ *be Lebesgue measurable and* $w_j:\Omega \rightarrow \mathbb{R}^m$*,* $j=1,2,\ldots$ *be a sequence of Lebesgue measurable functions. Then there exist a subsequence* w_k *and a collection of non-negative Radon measures* $\{\nu_y\}$ *in* $\mathbb{R}^n$ *can be found such that*

(ℓ_1*)* $\|\nu_y\|_{\mathcal{M}}:=\int_{\mathbb{R}^m} d\nu_y(\lambda) \leq 1$ *for almost every* $y \in \Omega$*.*

(ℓ_2*)* $\varphi(w_k) \rightarrow^* \bar{\varphi}$ *weakly* * *in* $L^{\infty}(\Omega)$ *for any* $\varphi \in C_0(\mathbb{R}^m)$*, where* $\bar{\varphi}=\langle\nu_y,\varphi\rangle$ *and* $C_0(\mathbb{R}^m)=\left\{\varphi \in C(\mathbb{R}^m): \lim_{|w|\rightarrow\infty}|\varphi(w)|=0\right\}$.

(ℓ_3) *If for any* $R > 0$

$$\lim_{L\to\infty}\sup_{k\in\mathbb{N}}\left|\{y\in\Omega\cap B_R(0):|w_k(y)|\geq L\}\right|=0,$$

then $\|\nu_y\|_{\mathcal{M}}=1$ *for almost every* $y\in\Omega$, *and for any measurable* $\Omega'\subset\Omega$ *we have* $\varphi(w_k)\to\bar{\varphi}=\langle\nu_y,\varphi\rangle$ *weakly in* $L^1(\Omega')$ *for continuous* φ *provided the sequence* $\varphi(w_k)$ *is weakly precompact in* $L^1(\Omega')$.

The Young measure is demonstrated by Lemma 2.1, and it is supplemented by a Fatou-type lemma that is invaluable in our work.

Lemma 2.2 *([4]). Let* $\mathcal{Q}:\Omega\times\mathbb{M}^{m\times n}\to\mathbb{R}$ *be a Carathéodory function and* $w_k:\Omega\to\mathbb{R}^m$ *a sequence of measurable functions such that* Dw_k *generates the Young measure* ν_y. *Then*

$$\liminf_{k\to\infty}\int_\Omega\mathcal{Q}\left(y,Dw_k(y)\right)\,\mathrm{d}y\geq\int_\Omega\int_{\mathbb{M}^{m\times n}}\mathcal{Q}(y,\eta)d\nu_y(\eta)\,\mathrm{d}y,$$

provided that the negative part $\mathcal{Q}^-\left(y,Dw_k(y)\right)$ *is equiintegrable.*

We first invoke a theorem of Kinderlehrer and Stampacchia in order to prove the existence and singularity of a weak resolution to the obstacle problem with variable growth.

Theorem 2.3 ***(Kinderlehrer and Stampacchia [10])*** *Let* $\mathcal{K}$ *be a nonempty closed convex subset of* X *and let* $\mathcal{S}:\mathcal{K}\to X'$ *be monotone, coercive and strong-weakly continuous on* $\mathcal{K}$. *Then there exists an element* w *such that*

$$\left\langle\mathcal{S}(w),\nu-w\right\rangle\geq 0\ \text{ for all }\ \nu\in\mathcal{K}.$$

Young measure is a tool for comprehending and managing the issues that arise when weak convergence does not perform as desired with respect to nonlinear operators and functions.

3. Existence of weak solutions for the obstacle problem (1.1)

In this section, we can use the concept of Young measure and the theorem of Kinderlehrer and Stampacchia to demonstrate the existence of weak solutions for the obstacle problem stated in (1.1)-(1.2), by defining a mapping $\mathcal{S}:\mathcal{G}_{\varphi,\Lambda}\to W^{-1,r'(y)}(\Omega;\mathbb{R}^m)$ as

$$\left\langle\mathcal{S}(w),v\right\rangle=\int_\Omega\phi_1\big(y,Du-\mathcal{T}(w)\big):D\nu+\phi_2(y,Dw):\nu+\left\langle w|w|^{r(y)-2},v\right\rangle\,\mathrm{d}y,$$

which fulfills the criteria of Theorem 1.1. We will demonstrate the proof in a series of assertion which we will present here.

Assertion 4

$\imath_1$) *$\mathcal{G}_{\varphi,\Lambda}$ is a closed convex set.*

$\imath_2$) *For each $\nu\in\mathcal{G}_{\varphi,\Lambda}$, $\mathcal{S}w\in W^{-1,r'(y)}(\Omega;\mathbb{R}^m)$.*

Proof.

$\imath_1$) is immediate that $\mathcal{G}_{\varphi,\Lambda}$ is a closed convex set.

$\imath_2$) Now,

$$\begin{aligned}\left|\left\langle\mathcal{S}w,\nu\right\rangle\right|&=\left|\int_\Omega\phi_1\big(y,Dw-\mathcal{T}(w)\big):D\nu+\phi_2(y,Dw):\nu+\left\langle w|w|^{r(y)-2},\nu\right\rangle\,\mathrm{d}y\right|\\&\leq\underbrace{\left|\int_\Omega\phi_1\big(y,Dw-\mathcal{T}(w)\big):D\nu\,\mathrm{d}y\right|}_{=:\alpha_1}+\underbrace{\left|\int_\Omega\phi_2(y,Dw):\nu\,\mathrm{d}y\right|}_{=:\alpha_2}+\\&\quad\underbrace{\left|\int_\Omega\left\langle w|w|^{r(y)-2},\nu\right\rangle\,\mathrm{d}y\right|}_{=:\alpha_3}.\end{aligned}$$

On the other hand, we deduce from $(\mathcal{I}_1)$ that

$$\begin{aligned}
|\alpha_1| &\le \int_\Omega |\phi_1(y, Dw-\mathcal{T}(w))||D\nu|dy\\
&\le \int_\Omega d_1(y)|D\nu|dy + c_1\int_\Omega |Dw-\mathcal{T}(w)|^{r(y)-1}|Dv|dy\\
&\le \|d_1(y)\|_{-1,r'(y)}\|D\nu\|_{r(y)} + c_1\left(\int_\Omega |Dw-\mathcal{T}(w)|^{r(y)}dy\right)^{\frac{1}{r'(y)}}\|D\nu\|_{r(y)}\\
&\le \left(\int_\Omega 2^{r(y)-1}c_1\left(|Dw|^{r(y)}+|\mathcal{T}(w)|^{p(y)}\right)dy\right)^{\frac{r(y)-1}{r(y)}}\|D\nu\|_{r(y)} + \|d_1(y)\|_{-1,r'(y)}\|D\nu\|_{r(y)}\\
&\le 2^{\frac{\left(r^+-1\right)^2}{r^-}}c_1\left(\|Dw\|_{r(y)}^{r(y)}+\|\mathcal{T}(w)\|_{r(y)}^{r(y)}\right)^{\frac{r(y)-1}{r(y)}}\|D\nu\|_{r(y)} + \|d_1\|_{-1,r'(y)}\|D\nu\|_{r(y)}\\
&\le C\|\nu\|_{1,r(y)}.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
|\alpha_2| &= \left|\int_\Omega \left\langle w|w|^{r(y)-2}, \nu\right\rangle \mathrm{d}y\right|\\
&\le C'\|w\|_{r(y)}\|\nu\|_{r(y)}\\
&\le C\|\nu\|_{1,r(y)}\,.
\end{aligned}$$

It is deduced from $(\mathcal{I}_1)$ that

$$\begin{aligned}
|\alpha_3| &\le \int_\Omega |\phi_2(y, Dw)||\nu|\ \mathrm{d}y\\
&\le \left(\|d_3\|_{r'(y)} + c_3\|Dw\|_{r(y)}^{r(y)-1}\right)\|\nu\|_{r(y)}\\
&\le \left(\|d_3\|_{r'(y)} + c_3\|Dw\|_{r(y)}^{r(y)-1}\right)\|\nu\|_{1,r(y)}\\
&\le\ C\|\nu\|_{1,r(y)}\,.
\end{aligned}$$

At last, we have arrived at

$$|\langle \mathcal{S}(w), \nu\rangle| \le |\alpha_1| + |\alpha_2| + |\alpha_3| \le C\|\nu\|_{1,r(y)}\,,$$

whence $\mathcal{S}w \in W^{-1,r'(y)}(\Omega;\mathbb{R}^m)$.

Assertion 5 $\mathcal{S}$ *is monotone and coercive on* $\mathcal{G}_{\varphi,\Lambda}$.

Proof. ▷ Monotonicity
For fixed $\nu \in \mathcal{G}_{\varphi,\Lambda}$, by the strict quasimonotone of $(\mathcal{I}_1)$ and $(\mathcal{I}_2)$, we have

$$\begin{aligned}
\langle \mathcal{S}w - \mathcal{S}\nu, w-\nu\rangle &= \int_\Omega (\phi_1(y, Dw-\mathcal{T}(w) - \phi_1(y, D\nu-\mathcal{T}(w))) : (Dw - D\nu)dy\\
&+ \int_\Omega \left\langle w|w|^{r(y)-2} - \nu|v|^{r(y)-2}, w-\nu\right\rangle \mathrm{d}y + \int_\Omega (\phi_2(y, Dw) - \phi_2(y, D\nu)) : (w-\nu)\ \mathrm{d}y\\
&\ge \int_\Omega (\phi_1(y, Dw-\mathcal{T}(w) - \phi_1(y, D\nu-\mathcal{T}(w))) : (Dw - D\nu)dy\\
&+ \int_\Omega (\phi_2(y, Dw) - \phi_2(y, D\nu)) : (w-\nu)\ \mathrm{d}y\\
&\ge C_5\int_\Omega |Dw - D\nu|^{r(y)}\ \mathrm{d}y + C_6\int_\Omega |w-\nu|^{r(y)}\ \mathrm{d}y\\
&\ge 0.
\end{aligned}$$

▷ **Coercivity**

For fixed $\nu \in \mathcal{G}_{\varphi,\Lambda}$, and using the condition $(\mathcal{I}_1)$ then

$$\begin{aligned}\langle \mathcal{S}w - \mathcal{S}v, w-\nu\rangle &= \int_\Omega (\phi_1(y, Dw - \mathcal{T}(w)) - \phi_1(y, D\nu - \mathcal{T}(w))) : (Dw - D\nu)dy \\ &+ \int_\Omega \left\langle w|w|^{r(y)-2} - \nu|\nu|^{r(y)-2}, w-\nu \right\rangle \mathrm{d}y + \int_\Omega (\phi_2(y, Dw) - \phi_2(y, D\nu)) : (w-\nu)\ \mathrm{d}y \\ &\geq C_5 \int_\Omega |Dw - D\nu|^{r(y)}\ \mathrm{d}y + C_6 \int_\Omega |w-\nu|^{r(y)}\ \mathrm{d}y.\end{aligned}$$

For a sufficiently small constant δ, we have

$$\begin{aligned}\frac{\displaystyle\int_\Omega |Dw - D\nu|^{r(y)}\ \mathrm{d}y}{\|Dw - D\nu\|_{L^{r(y)}(\Omega,\mathbb{R}^m)}} &= \int_\Omega \left(\frac{|Dw - D\nu|}{\|Dw - D\nu\|_{L^{r(y)}(\Omega,\mathbb{R}^m)} - \delta}\right)^{r(y)} \frac{\left(\|Dw - D\nu\|_{L^{r(y)}(\Omega,\mathbb{R}^m)} - \delta\right)^{r(y)}}{\|Dw - D\nu\|_{L^{r(y)}(\Omega,\mathbb{R}^m)}}\ \mathrm{d}y \\ &\geq \frac{\left(|Dw - D\nu|_{L^{r(y)}(\Omega,\mathbb{R}^m)} - \delta\right)^{r(y)}}{\|Dw - D\nu\|_{L^{r(y)}(\Omega,\mathbb{R}^m)}}.\end{aligned}$$

Taking $\delta = \frac{1}{2}\|Dw - D\nu\|_{L^{r(y)}(\Omega,\mathbb{R}^m)}$, we have

$$\frac{\displaystyle\int_\Omega |Dw - D\nu|^{r(y)}\ \mathrm{d}y}{\|Dw - D\nu\|_{L^{r(y)}(\Omega,\mathbb{R}^m)}} \to \infty$$

as $\|Dw - D\nu\|_{L^{r(y)}(\Omega,\mathbb{R}^m)} \to \infty$. In the same way, we also obtain

$$\frac{\displaystyle\int_\Omega |w - v|^{r(y)}\ \mathrm{d}y}{\|w - \nu\|_{L^{r(y)}(\Omega,\mathbb{R}^m)}} \to \infty$$

as $\|w - \nu\|_{L^{r(y)}(\Omega,\mathbb{R}^m)} \to \infty$. From there, it is simple to get

$$\frac{(\mathcal{S}w - \mathcal{S}v, w - \nu)}{\|w - \nu\|_{W^{1,r(y)}(\Omega,\mathbb{R}^m)}} \to \infty$$

as $\|w - \nu\|_{W^{1,r(y)}(\Omega,\mathbb{R}^m)} \to \infty$. That is to say, $\mathcal{S}$ is coercive on $\mathcal{G}_{\varphi,\Lambda}$.

Assertion 6 *$\mathcal{S}$ is strongly-weakly continuous.*

Proof.

- We choose a sequence $w_k \in \mathcal{G}_{\varphi,\Lambda}$ such that $w_k \to w \in \mathcal{G}_{\varphi,\Lambda}$ in $W^{1,r(y)}(\Omega; \mathbb{R}^m)$. Then $\|w_k\|_{1,r(y)} \leq C$ for some constant C. In view of Lemma 2.1, there exist a Young measure v_y generated by $\{Dw_k\}$ such that $\|v_y\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ and

$$Dw_k \to \langle \nu_y, id\rangle = \int_{\mathbb{M}^{m\times n}} \eta d\nu_y(\eta) \quad \text{in } L^1(\Omega). \tag{6.1}$$

  Since $L^{r(y)}(\Omega; \mathbb{M}^{m\times n})$ is reflexive, then $Dw_k \to Dw$ in $L^{r(y)}(\Omega; \mathbb{M}^{m\times n}) \subset L^1(\Omega; \mathbb{M}^{m\times n})$, thus $Dw(y) = \langle v_y, \text{ id } \rangle$ for a.e. $y \in \Omega$ (by uniqueness of limit, see also [1, Lemma 4.1].)

The following lemmas allow us to prove the Lemma Assertion 6.

Lemma 6.1 ***(div-curl inequality)****. Suppose ϕ_1 satisfies $(\mathcal{I}_0)$-$(\mathcal{I}_2)$ and $\{Dw_k\}$ generates the Young measure ν_y, then*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} (\phi_1(y,\eta-\mathcal{T}(w)) - \phi_1(y,Dw-\mathcal{T}(w))) : (\eta - Dw) d\nu_y(\eta)\ \mathrm{d}y \le 0.$$

Proof.
Let consider the sequence

$$\begin{aligned} I_k &:= \left(\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) - \phi_1(y, Dw-\mathcal{T}(w))\right) : (Dw_k - Du) \\ &= \phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) : (Dw_k - Dw) - \phi_1(y, Dw - \mathcal{T}(w)) : (Dw_k - Dw)\,. \end{aligned}$$

By the growth condition in $(\mathcal{I}_1)$, we have

$$\begin{aligned} \int_\Omega |\phi_1(y, Dw - \mathcal{T}(w))|^{r'(y)}\ \mathrm{d}y &\le C \int_\Omega |Dw|^{r(y)}\ \mathrm{d}y \\ &< \infty. \end{aligned}$$

Since $w \in W^{1,r(y)}\left(\Omega;\mathbb{R}^m\right)$. Then $\phi_1 \in L^{r'(y)}\left(\Omega, \mathbb{M}^{m\times n}\right)$. According to the weak convergence in (6.1), we obtain

$$\begin{aligned} \liminf_{k\to\infty} \int_\Omega \phi_1(y, Dw - \mathcal{T}(w)) : (Dw_k - Dw)\ \mathrm{d}y &= \int_\Omega \int_{\mathbb{M}^{m\times n}} \phi_1(y, Dw - \mathcal{T}(w)) : (\eta - Dw) d\nu_y(\eta)\ \mathrm{d}y \\ &= \int_\Omega \phi_1(y, Dw - \mathcal{T}(u)) : (\underbrace{\int_{\mathbb{M}^{m\times n}} \eta d\nu_y(\eta)}_{=:Dw(y)} - Dw)\ \mathrm{d}y = 0. \end{aligned}$$

Therefore

$$I := \liminf_{k\to\infty} \int_\Omega I_k\ \mathrm{d}y = \liminf_{k\to\infty} \int_\Omega \phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) : (Dw_k - Dw)\ \mathrm{d}y.$$

Let $\Omega' \subset \Omega$ be an arbitrary measurable subset. By the growth condition in $(\mathcal{I}_1)$ together with Holder's inequality, we have

$$\int_{\Omega'} \left|\phi_1\Big(y, Dw_k - \mathcal{T}(w_k)\Big) \colon Dw\right|\ \mathrm{d}y \le \left(\|\, d_1(y)\|_{r'(y)} + 2^{\frac{\left(r^+-1\right)^2}{r^-}} c_1 \underbrace{\left(\|Dw_k\|_{r(y)}^{r(y)} + \|\mathcal{T}(w_k)\|_{r(y)}^{r(y)}\right)^{\frac{r(y)-1}{r(y)}}}_{\le C}\right) \times$$

$$\left(\int_{\Omega'} \mid Dw^{r(y)}\ \mathrm{d}y\right)^{\frac{1}{r(y)}}.$$

Since $\int_{\Omega'} |Dw|^{r(y)}\ \mathrm{d}y$ is arbitrary small if we choose the measure of Ω' small enough, it follows that the negative part $\Big(\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) : Dw\Big)^-$ is equiintegrable. On the other hand, by the coercivity condition in (ϕ_1), we have

$$\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) : Dw_k \ge -d_2(y) + c_2 \left|Dw_k - \mathcal{T}(w_k)\right|^{r(y)} \ge -d_2(y).$$

Thus

$$\int_{\Omega'} \left(\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) : Dw_k\right)^-\ \mathrm{d}y \le \int_{\Omega'} |d_2(y)|\ \mathrm{d}y.$$

Hence $\left(\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) : Dw_k\right)^-$is equiintegrable. We infer from Lemma 2.2 that

$$I = \liminf_{k\to\infty} \int_\Omega \phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) : (Dw_k - Dw)\ \mathrm{d}y \quad \ge \int_\Omega \int_{\mathbb{M}^{m\times n}} \phi_1(y, \eta - \mathcal{T}(w)) : (\eta - Dw) d\nu_y(\eta)\ \mathrm{d}y.$$

We prove that $I \leq 0$. According to Mazur's theorem (see, e.g., [16, Theorem 2, page 120]) there exists a sequence $(\nu_k) \in W^{1,r(y)}(\Omega;\mathbb{R}^m)$ where each ν_k is a convex linear combination of $\{w_1,\dots,w_k\}$ such that $\nu_k \to w_k$ in $W^{1,r(y)}(\Omega;\mathbb{R}^m)$. This implies that ν_k belongs to the same space as w_k. Hence

$$\begin{aligned}
I &= \liminf_{k\to\infty} \int_\Omega \phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) : (Dw_k - Dw)\ \mathrm{d}y \\
&= \liminf_{k\to\infty} \left(\int_\Omega \phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) : (Dw_k - D\nu_k)\ \mathrm{d}y + \int_\Omega \phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) : (D\nu_k - Dw)\ \mathrm{d}y \right) \\
&\leq \liminf_{k\to\infty} \|\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right)\|_{r'(y)} \|Dw_k - D\nu_k\|_{r(y)} + \|\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right)\|_{r'(y)} \|D\nu_k - Dw\|_{r(y)} = 0,
\end{aligned}$$

by the boundedness of $\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right)$ in $L^{r'(y)}(\Omega;\mathbb{M}^{m\times n})$, the construction of the sequence v_k and the following fact

$$\|w_k - v_k\|_{1,r(y)} \leq \|w_k - w\|_{1,r(y)} + \|\nu_k - w\|_{1,r(y)} \to 0 \quad \text{as } k \to \infty.$$

Since

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \phi_1(y, Dw - \mathcal{T}(w)) : (\eta - Dw) d\nu_y(\eta)\ \mathrm{d}y = \int_\Omega \phi_1(y, Dw - \mathcal{T}(w)) : \left(\int_{\mathbb{M}^{m\times n}} \eta d\nu_y(\eta) - Dw \right)\ \mathrm{d}y = 0,$$

together with $I \leq 0$, the inequality of Lemma 6.1 follows .

Remark 6.1 An intermidiary result is the following inequality:

$$\liminf_{k\to\infty} \int_\Omega \left(\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) - \phi_1(y, Dw - \mathcal{T}(w)) \right) : (Dw_k - Dw)\ \mathrm{d}y \leq 0.$$

To see this it is sufficient to repeat the proof of Lemma 6.1.

Lemma 6.2
For almost every $y \in \Omega$, we have

$$(\phi_1(y, \eta - \mathcal{T}(w)) - \phi_1(y, Dw - \mathcal{T}(w))) : (\eta - Dw) = 0 \quad \textit{on } \operatorname{supp} \nu_y.$$

Proof. By Lemma 6.1, we have

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} (\phi_1(y, \eta - \mathcal{T}(w)) - \phi_1(y, Dw - \mathcal{T}(w))) : (\eta - Dw) d\nu_y(\eta)\ \mathrm{d}y \leq 0.$$

By the monotonicity of ϕ_1, the above integrand is nonnegative, thus must vanish with respect to the product measure $\mathrm{d}v_y(\eta) \otimes \mathrm{d}y$. Therefore

$$(\phi_1(y, \eta - \mathcal{T}(w)) - \phi_1(y, Dw - \mathcal{T}(w))) : (\eta - Dw) = 0 \quad \text{on } \operatorname{supp} \nu_y.$$

Now we prove the Assertion 6 for each claim listed in (I_2).
Step 1. Suppose that ϕ_1 satisfy the condition (I_2)(a). We have

$$\int_\Omega |Dw_k - Dw|^{r(y)}\ \mathrm{d}y \leq c \int_\Omega \left(\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) - \phi_1(y, Dw - \mathcal{T}(w)) \right) : (Dw_k - Dw)\ \mathrm{d}y.$$

We remark that the limit inferior of the right hand side of the above inequality is less than or equal to zero by Remark 6.1. It follows that

$$\liminf_{k\to\infty} \int_\Omega |Dw_k - Dw|^{r(y)} dy = 0.$$

Let $E_{k,\epsilon} = \{y : |Dw_k - Dw| \geq \epsilon\}$. We have

$$\int_\Omega |Dw_k - Dw|^{r(y)}\ \mathrm{d}y \geq \int_{E_{k,e}} |Dw_k - Dw|^{r(y)}\ \mathrm{d}y \geq \epsilon^{r(y)} |E_{k,\epsilon}|,$$

which gives

$$|E_{k,\epsilon}| \leq \frac{1}{\epsilon^{r(y)}} \int_\Omega |Dw_k - Dw|^{r(y)} \ \mathrm{d}y \to 0 \quad \text{ as } k \to \infty.$$

that is to say, $Dw_k \to Du$ in $L^{r(y)}(\Omega, \mathbb{R}^m)$.
Hence

$$Dw_k \to Dw \quad \text{ in measure on } \Omega \text{ (for a subsequence).}$$

After extracting a suitable subsequence if necessary, we can infer that $Dw_k \to Dw$ for almost every $y \in \Omega$. Then $\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) \to \phi_1(y, Dw - \mathcal{T}(w))$ for almost every $y \in \Omega$, and in the measure. By the equiintegrability of $\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) : D\nu$, the Vitali theorem implies

$$\int_\Omega \phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right) : D\nu \ \mathrm{d}y \to \int_\Omega \phi_1(y, Dw - \mathcal{T}(w)) : D\nu \ \mathrm{d}y \quad \text{ as } k \to \infty.$$

Suppose that ϕ_2 satisfy the condition (I_3)(a).In the same way we find:

$$\int_\Omega \phi_2\left(y, Dw_k\right) : v \ \mathrm{d}y \to \int_\Omega \phi_2(y, Dw) : v \ \mathrm{d}y \quad \text{ as } k \to \infty.$$

Step 2. For the case (I_2)(b), we argue as follows: We start by proving that for almost every $y \in \Omega$,

$$\operatorname{supp} \nu_y \subset E_y = \left\{\eta \in \mathbb{M}^{m \times n} : Q(y, \eta) = Q(y, Dw - \mathcal{T}(w)) + \phi_1(y, Dw - \mathcal{T}(w)) : (\eta - Dw)\right\}.$$

Let $\eta \in \operatorname{supp} \nu_y$, then by Lemma 6.2, we get

$$(1-\theta)(\phi_1(y, \eta - \mathcal{T}(w)) - \phi_1(y, Dw - \mathcal{T}(w))) : (\eta - Dw) = 0, \quad \forall \theta \in [0,1]. \tag{6.2}$$

By monotonicity, we have on the other hand, for $\theta \in [0,1]$

$$(1-\theta)(\phi_1(y, Dw - \mathcal{T}(w) + \theta(\eta - Dw)) - \phi_1(y, \eta - \mathcal{T}(w))) : (Dw - \eta) \geq 0. \tag{6.3}$$

Subtracting (6.2) from (6.3), we get

$$(1-\theta)(\phi_1(y, Dw - \mathcal{T}(w) + \theta(\eta - Dw)) - \phi_1(y, Dw - \mathcal{T}(w))) : (Dw - \eta) \geq 0 \tag{6.4}$$

for $\theta \in [0,1]$. By monotonicity,

$$(\phi_1(y, Dw - \mathcal{T}(w) + \theta(\lambda - Dw)) - \phi_1(y, Dw - \mathcal{T}(w))) : \theta(\eta - Dw) \geq 0,$$

and since $\theta \in [0,1]$, we have

$$(\phi_1(y, Dw - \mathcal{T}(w) + \theta(\eta - Dw)) - \phi_1(y, Dw - \mathcal{T}(w)) : (1-\theta)(\eta - Dw) \geq 0.$$

The above inequality together with (6.4) implies

$$(\phi_1(y, Dw - \mathcal{T}(w) + \theta(\eta - Dw)) - \phi_1(y, Dw - \mathcal{T}(w))) : (\eta - Dw) = 0 \quad \forall \theta \in [0,1].$$

Integrating this equality over $[0,1]$ and using the fact that

$$\phi_1(y, Dw - \mathcal{T}(w) + \theta(\eta - Dw)) : (\eta - Dw) = \frac{\partial Q}{\partial \theta}(y, Dw - \mathcal{T}(w) + \theta(\eta - Dw)) : (\eta - Dw),$$

we conclude that

$$\begin{aligned} Q(y, \eta - \mathcal{T}(w)) &= Q(y, Dw - \mathcal{T}(w)) + \int_0^1 \phi_1(y, Dw - \mathcal{T}(w) + \theta(\eta - Dw)) : (\eta - Dw) d\theta \\ &= Q(y, Dw - \mathcal{T}(w)) + \phi_1(y, Dw - \mathcal{T}(w)) : (\eta - Dw). \end{aligned}$$

Hence $\eta \in E_y$, i.e. $\operatorname{supp} \nu_y \subset E_y$. In view of the convexity of Q, we have

$$Q(y,\eta) \geq Q(y, Dw - \mathcal{T}(w)) + \phi_1(y, Dw - \mathcal{T}(w)) : (\eta - Dw).$$

For all $\eta \in E_y$, put $A(\eta) = Q(y, \eta - \mathcal{T}(w))$ and $B(\eta - \mathcal{T}(w)) = Q(y, Dw - \mathcal{T}(w)) + \phi_1(y, Dw - \mathcal{T}(w)) :$ $(\eta - Dw)$. Since $\eta \mapsto A(\eta)$ is continuous and differentiable, we obtain for all $\mathcal{F} \in \mathbf{M}^{m\times n}$ and $\theta \in \mathbb{R}$

$$\begin{aligned}
\frac{A(\lambda + \theta\mathcal{F}) - A(\eta)}{\theta} &\geq \frac{B(\eta + \theta\mathcal{F}) - B(\lambda)}{\theta} \quad \text{if } \theta > 0,\\
\frac{A(\eta + \theta\mathcal{F}) - A(\eta)}{\theta} &\leq \frac{B(\eta + \theta\mathcal{F}) - B(\eta)}{\theta} \quad \text{if } \theta < 0.
\end{aligned}$$

Thus $DA = DB$ and then

$$\phi_1(y, \eta - \mathcal{T}(w)) = \phi_1(y, Dw - \mathcal{T}(w)) \quad \forall \eta \in E_y \supset \operatorname{supp} \nu_y. \tag{6.5}$$

The equiintegrability of $\phi_1\left(y, Dw_k - \mathcal{T}(w_k)\right)$ implies that its weak L^1-limit is given by

$$\begin{aligned}
\bar{\phi_1}(y) := \int_{\mathbb{M}^{m\times n}} \phi_1(y, \eta - \mathcal{T}(w))\, \mathrm{d}v_y(\eta) &= \int_{\operatorname{supp} \nu_y} \phi_1(y, \eta - \mathcal{T}(w))\, \mathrm{d}\nu_y(\eta)\\
&= \int_{\operatorname{supp} v_y} \phi_1(y, Dw - \mathcal{T}(w))\, \mathrm{d}\nu_y(\eta) = \phi_1(y, Dw - \mathcal{T}(w)),
\end{aligned} \tag{6.6}$$

where we have used (6.5) and $\|\nu_y\|_{\mathcal{M}} = 1$. Now consider the Carathodory function

$$g(y, \eta - \mathcal{T}) = |\phi_1(y, \eta - \mathcal{T}(w)) - \bar{\phi_1}(y)|, \quad \eta \in \mathbb{M}^{m\times n}.$$

The sequence $g_k(y) := g\left(y, Dw_k(y)\right)$ is equiintegrable by that of $\phi_1\big(y, Dw_k(y)\big)$, hence its weak L^1-limit is given by

$$g_k \to \bar{g} \text{ in } L^1(\Omega),$$

where

$$\begin{aligned}
\bar{g}(y) &= \int_{\mathbb{M}^{m\times n}} |\phi_1\big(y, \eta - \mathcal{T}(w)\big) - \bar{\phi_1}(y)|\, \mathrm{d}\nu_y(\eta)\\
&= \int_{\operatorname{supp} \nu_y} |\phi_1\big(y, \eta - \mathcal{T}(w)\big) - \bar{\phi_1}(y)| d\nu_y(\eta) = 0. \quad \text{(by (6.6) and (6.5))}
\end{aligned}$$

As g_k is non-negative for all k, we can conclude that g_k converges to zero in the $L^1(\Omega)$ norm as k approaches infinity.

Hence

$$\int_\Omega \phi_1\big(y, Dw_k - \mathcal{T}(w_k)\big) \colon D\nu \,\mathrm{d}y \to \int_\Omega \phi_1\big(y, Dw - \mathcal{T}(w)\big) \colon D\nu \,\mathrm{d}y \quad \text{as} \quad k \to \infty.$$

For the case (I_3)(b).In the same way we find:

$$\int_\Omega \phi_2\left(y, Dw_k\right) : v \,\mathrm{d}y \to \int_\Omega \phi_2(y, Dw) : \nu \,\mathrm{d}y \quad \text{as} \quad k \to \infty.$$

Step 3. The last case $(\mathcal{I}_2)$(c), It is asserted that for almost every $y \in \Omega$ and every $\mathcal{A} \in \mathbb{M}^{m\times n}$

$$\phi_1\big(y, \eta - \mathcal{T}(w)\big) \colon \mathcal{A} = \phi_1\big(y, Dw - \mathcal{T}(w)\big) \colon \mathcal{A} + \big(\nabla\phi_1\big(y, Dw - \mathcal{T}(w)\big)\big) \colon \big(Dw - \mathcal{A}\big),$$

holds on $\operatorname{supp} \nu_y$, where ∇ is the derivative with respect to the second variable of ϕ_1. The monotonicity of ϕ_1 implies that for $\tau \in \mathbb{R}$

$$\left(\phi_1\big(y, \eta - \mathcal{T}(w)\big) - \phi_1\big(y, Dw - \mathcal{T}(w) + \theta\mathcal{A}\big) \colon \big(\eta - Dw - \theta\mathcal{A}\big) \geq 0,\right.$$

which implies

$$-\phi_1\big(y, \eta - \mathcal{T}(w)\big) \colon \tau\mathcal{A} \geq -\phi_1\big(y, \eta - \mathcal{T}(w)\big) \colon \big(\eta - Dw\big) + \phi_1\big(y, Dw - \mathcal{T}(w) + \theta\mathcal{A}\big) \colon \big(\eta - Dw - \theta\mathcal{A}\big).$$

By virtue of Lemma 6.2, we get

$$-\phi_1\big(y,\eta-\mathcal{T}(w)\big)\colon\theta\mathcal{A}\geq-\phi_1\big(y,Dw-\mathcal{T}(w)\big)\colon(\eta-Dw)+\phi_1\big(y,Dw-\mathcal{T}(w)+\theta\mathcal{A}\big)\colon\big(\eta-Dw-\theta\mathcal{A}\big).$$

Note that
$\phi_1\big(y,Dw-\mathcal{T}(w)+\theta\mathcal{A}\big)=\phi_1\big(y,Dw-\mathcal{T}(w)\big)+\nabla\phi_1\big(y,Dw-\mathcal{T}(w)\big)\theta\mathcal{A}+o(\theta)$,
thus

$$\begin{aligned}\phi_1\big(y,Dw-\mathcal{T}(w)+\theta\mathcal{A}\big)&\colon\big(\eta-Dw-\theta\mathcal{A}\big)=\\ &\phi_1\big(y,Dw-\mathcal{T}(w)+\theta\mathcal{A}\big)\colon(\eta-Dw)-\phi_1\big(y,Dw-\mathcal{T}(w)+\theta\mathcal{A}\big)\colon\theta\mathcal{A}\\ &=\phi_1\big(y,Dw-\mathcal{T}(w)\big)\colon(\eta-Dw)+\nabla\phi_1\big(y,Dw-\mathcal{T}(w)\big)\theta\mathcal{A}:\big(\eta-Dw-\mathcal{T}(w)\big)\\ &\quad-\phi_1\big(y,Dw-\mathcal{T}(w)\big)\colon\theta\mathcal{A}\quad-\nabla\phi_1\big(y,Dw-\mathcal{T}(w)\big)\theta\mathcal{A}:\theta\mathcal{A}+o(\theta)\\ &=\phi_1\big(y,Dw-\mathcal{T}(w)\big)\colon(\eta-Dw)+\theta\Big[\nabla\phi_1\big(y,Dw-\mathcal{T}(w)\big)\mathcal{A}:(\eta-Dw)\\ &\quad-\phi_1\big(y,Dw-\mathcal{T}(w)\big)\Big]+o(\theta).\end{aligned}$$

Therefore

$$-\phi_1\big(y,\eta-\mathcal{T}(w)\big)\colon\theta\mathcal{A}\geq\theta\Big[\big(\nabla\phi_1\big(y,Dw-\mathcal{T}(w)\big)\mathcal{A}\big)\colon(\eta-Dw)-\phi_1\big(y,Dw-\mathcal{T}(w)\big)\colon\mathcal{A}\Big]+o(\theta).$$

Since θ is arbitrary in $\mathbb{R}$, then our claim follows. By the equiintegrability of $\phi_1\left(y,Dw_k-\mathcal{T}(w_k)\right)$, its weak L^1-limit is then given by

$$\begin{aligned}\bar{\phi_1}(y)&=\int_{\operatorname{supp}\nu_y}\phi_1\big(y,\eta-\mathcal{T}(w)\big)\ \mathrm{d}\nu_x(\eta)\\ &=\int_{\operatorname{supp}\nu_y}\phi_1\big(y,Dw-\mathcal{T}(w)\big)\ \mathrm{d}v_y(\eta)+\big(\nabla\phi_1\big(y,Dw-\mathcal{T}(w)\big)\big)^t\int_{\operatorname{supp}\nu_y}(Dw-\eta)\ \mathrm{d}\nu_y(\eta)\\ &=\phi_1\big(y,Dw-\mathcal{T}(w)\big),\end{aligned}$$

where we have used our claim and $Dw(y)=\langle\nu_y,id\rangle$. On the other hand, since $L^{p'(y)}\left(\Omega;\mathbb{M}^{m\times n}\right)$ is reflexive, the sequence $\big\{\phi_1\left(y,Dw_k-\mathcal{T}(w_k)\right)\big\}$ converges weakly in $L^{p'(y)}\left(\Omega;\mathbb{M}^{m\times n}\right)$ and its weak $L^{p'(y)}$-limit is also $\phi_1(y,Dw-\mathcal{T}(w))$. Then we conclude that

$$\int_\Omega\phi_1\big(y,Dw_k-\mathcal{T}(w_k)\big)\colon D\nu\ \mathrm{d}y\to\int_\Omega\phi_1\big(y,Dw-\mathcal{T}(w)\big)\colon D\nu\ \mathrm{d}y\quad\text{ as } k\to\infty.$$

The last case $(\mathcal{H}_3)$(c). In the same way we find:

$$\int_\Omega\phi_2(y,Dw_k):\nu\ \mathrm{d}y\to\int_\Omega\phi_2(y,Dw):\nu\ \mathrm{d}y\quad\text{ as } k\to\infty.$$

Hence
$$\int_\Omega\phi_1\big(y,Dw_k-\mathcal{T}(w_k)\big)\colon D\nu+\phi_2(y,Dw_k):\nu\ \mathrm{d}y\to\int_\Omega\phi_1\big(y,Dw-\mathcal{T}(w)\big)\colon D\nu+\phi_2(y,Dw):\nu\ \mathrm{d}y \text{ as } k\to\infty$$
in the case (a), (b) and (c).

- It is clear that
$$\Big\langle w_k\left|w_k\right|^{r(y)-2},v\Big\rangle\to\Big\langle w\left|w\right|^{r(y)-2},\nu\Big\rangle\quad\text{ as } k\to\infty.$$

Next, we pass to the limit , we have

$$\begin{aligned}(\mathcal{S}w_k,\nu)&=\int_\Omega\phi_1\big(y,Dw_k-\mathcal{T}(w_k)\big)\colon D\nu+\phi_2(y,Dw_k):\nu+\Big\langle w_k\left|w_k\right|^{r(y)-2},\nu\Big\rangle\ \mathrm{d}y\\ &\to\int_\Omega\phi_1\big(y,Du-\mathcal{T}(w)\big)\colon D\nu+\phi_2(y,Dw):\nu+\Big\langle w|w|^{r(y)-2},\nu\Big\rangle\ \mathrm{d}y\\ &=(\mathcal{S}w,\nu).\end{aligned}$$

This is the strong-weakly continuous of $\mathcal{S}$ on $\mathcal{G}_{\varphi,\Lambda}$. This ends the proof of Assertion 6.
Now we can apply Theorem 2.3 and the above lemmas to obtain the existence. For this we conclude the existence of an element $w \in \mathcal{G}_{\varphi,\Lambda}$ such that $\langle \mathcal{S}(w), \nu - w\rangle \geq 0$, i.e.

$$\int_\Omega \phi_1\big(y, Dw - \mathcal{T}(w)\big)\colon (D\nu - Dw) + \phi_2(y, Dw) : (v - w) + \Big\langle w\,|w|^{r(y)-2}, \nu - w\Big\rangle \ \mathrm{d}y \geq 0 \text{ for all } \nu \in \mathcal{G}_{\varphi,\Lambda}.$$

4. Uniqueness of Weak Solutions to a Problem

In order to obtain the uniqueness result, we need to prove the theorem. :

Theorem 4.1 *Suppose $\mathcal{G}_{\varphi,\Lambda} \neq \phi$ and $r(y)$ satisfies (1). Under conditions $(\mathcal{I}_1) - (\mathcal{I}_2)$ (c), there exists a unique solution $w \in \mathcal{G}_{\varphi,\Lambda}$ to the obstacle problem (1.1). That is to say, there exists a unique $w \in \mathcal{G}_{\varphi,\Lambda}$ such that*

$$\int_\Omega \phi_1\big(y, Dw - \mathcal{T}(w)\big)\colon (D\nu - Dw) + \phi_2(y, Dw) : (\nu - w) + \Big\langle w\,|w|^{r(y)-2}, \nu - w\Big\rangle \ \mathrm{d}y \geq 0 \textit{ for all } \nu \in \mathcal{G}_{\varphi,\Lambda}.$$

Proof. Suppose $\mathcal{G}_{\varphi,\Lambda} \neq \phi$ and $r(y)$ fulfills (1). Under the assumptions $(I_1) - (I_2)$ (c), there exists a unique $w \in \mathcal{G}_{\varphi,\Lambda}$ such that

$$\int_\Omega \phi_1\big(y, Dw - \mathcal{T}(w)\big)\colon (D\nu - Dw) + \phi_2(y, Dw) : (\nu - w) + \Big\langle w\,|w|^{r(y)-2}, \nu - w\Big\rangle \ \mathrm{d}y \geq 0 \text{ for all } \nu \in \mathcal{G}_{\varphi,\Lambda}.$$

It is obvious to acquire the existence from the above lemmas. If there are two weak solutions $w_1, w_2 \in \mathcal{G}_{\varphi,\Lambda}$ to the obstacle problem (1.1)-(1.2) then

$$\int_\Omega \phi_1\big(y, Dw_1 - \mathcal{T}(w_1)\big)\colon (Dw_2 - Dw_1)\ \mathrm{d}y + \phi_2(y, Dw_1) : (w_2 - w_1) + \Big\langle w_1\,|w_1|^{r(y)-2}, w_2 - w_1\Big\rangle \ \mathrm{d}y \geq 0.$$

and

$$\begin{aligned}
&-\int_\Omega \phi_1\big(y, Dw_2 - \mathcal{T}(w_2)\big)\colon (Dw_2 - Dw_1)\ \mathrm{d}y + \phi_2(y, Dw_2) : (w_2 - w_1) + \Big\langle w_2\,|w_2|^{r(y)-2}, w_2 - w_1\Big\rangle \ \mathrm{d}y \\
=&\int_\Omega \phi_1(y, Dw_2 - \mathcal{T}(w_2)) : (Dw_1 - Dw_2)\ \mathrm{d}y + \phi_2(y, Dw_2) : (w_1 - w_2) + \Big\langle w_2\,|w_2|^{r(y)-2}, w_1 - w_2\Big\rangle \ \mathrm{d}y \\
&\geq 0
\end{aligned}$$

furthermore

$$\begin{aligned}
&\int_\Omega \phi_1\big(y, Dw_1 - \mathcal{T}(w_1)\big) - \phi_1\big(y, Dw_2 - \mathcal{T}(w_2)\big)\colon (Dw_1 - Dw_2) + \big(\phi_2(y, Dw_1) - \phi_2(y, Dw_2)\big)\colon (w_1 - w_2) \\
&+ \Big\langle w_1\,|w_1|^{r(y)-2} - w_2\,|w_2|^{r(y)-2}, w_1 - w_2\Big\rangle \ \mathrm{d}y \leq 0.
\end{aligned}$$

In consideration of $(\mathcal{I}_2)$(c), we can further deduce that

$$\int_\Omega \phi_1\big(y, Dw_1 - \mathcal{T}(w_1)\big) - \phi_1\big(y, Dw_2 - \mathcal{T}(w_2)\big)\colon (Dw_1 - Dw_2)\ \mathrm{d}y = 0 \text{ on } \Omega.$$

In view of (I_3)(c), we can further infer that

$$\int_\Omega \big(\phi_2(y, Dw_1) - \phi_2(y, Dw_2)\big)\colon (w_1 - w_2)\ \mathrm{d}y = 0 \text{ on } \Omega,$$

and

$$\int_\Omega \Big\langle w_1\,|w_1|^{r(y)-2} - w_2\,|w_2|^{r(y)-2}, w_1 - w_2\Big\rangle \ \mathrm{d}x = 0$$

that is to say, $w_1 = w_2$ a.e. on Ω.
We can infer the uniqueness of the solution.

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