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Common fixed point theorem for generalized $\theta - \phi$ contractions in E-metric spaces

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ABSTRACT: In this paper, a common fixed point theorem for generalized $\theta - \phi$ contraction map in E-metric space is proved. The result presented in this paper extend and improve the corresponding result of Ćirić [Generalized contractions and fixed point theorems, Publ. Inst. Math. 12 (1971), 19-26]. We also present an illustrated example in support of our result. As application, we give a homotopy results for operators on a set endowed with E-metric space.

Key Words: Generalized $\theta - \phi$ contraction mapping, E-metric space, E-metric, Coincidence Point, Common fixed point.

Contents

1. Introduction and Preliminaries

The study of fixed point plays central role in many disciplines. It has been implemented to deal with the solutions to problems in Fredholm integral equations, Volterra integral equations, differential equations, homotopy equations etc. Fixed point theorems are crucial tools for demonstrating the existence and originality of solutions to diverse mathematical models that depict phenomena that arise in various field such as: steady state temperature distribution, chemical equations, neutron transport theory, economics theories, epidemics facilitating existence and uniqueness theories of variational inequalities, functional equations, optimal control problems and many more. The famous Banach contraction principle proved by Stefan Banach [4] has been crucial to numerous facets of nonlinear functional analysis. In order to generalized and extended Banach contraction principle for investigating new mappings of the contraction type, many writers developed a number of fixed point theorems.([3], [6], [11]).

The generalization of Banach contraction principle in many spaces has been done by several authors. Continuing this, in 2007 Huang and Zhang [11] introduced cone metric space which is a generalization of metric space. Many authors have obtained fixed point results in this space. They obtained some fixed point theorems for different types of contractions in cone metric space. Further in 2012 Rawashdeh et al. [14] defined an ordered normed space called E-metric space. They replaced the real-valued metric with an E-valued metric and generalized the results of real-valued metric spaces to E-metric spaces. They also proved that the contractive sequence is a Cauchy sequence in E-metric space. E-metric spaces generalize metric spaces, cone metric spaces and certain other spaces. Afterwards, few mathmaticians has done work in this direction ([10]- [13]).

In this paper, we obtain the existence and uniqueness of fixed points for generalized $\theta - \phi$ contraction in the setting of E-metric spaces. We extended and improved the corresponding result of Ćirić [5] in E-metric space. We consider this to be new result, as thus far there have been no fixed point results presented for four maps in E-metric spaces. Furthermore, we have applied our result on homotopy operator and proved a fixed point theorem for this.
Throughout this paper, $\mathbb{R}$ denote the set of all real numbers, $\mathbb{R}^+$ the set of all non-negative real numbers

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respecively. Let E be a ordered normed space with norm $||.||$ which is ordered by its positive cone E^+, such that for $x, y \in E,\ x \preceq y$ if and only if $y - x \in E^+$.
Let X be a non-empty set and S, T be self-maps on X.

(i) A point $w \in X$ is said to be a fixed point of S (resp. T) if $Sw = w$(resp. $Tw = w$). The set of all fixed points of S (resp. T) is denoted by $F(S)$(resp. $F(T)$).

(ii) A point $w \in X$ is said to be a coincidence point of S and T if $u = Sw = Tw$ where $u \in X$. The set of all coincidence points of S and T is denoted by $C(S, T)$.

(iii) A point $w \in X$ is a common fixed point of S and T if $w = Sw = Tw$. The set of all common fixed points of S and T is denoted by $F(S, T)$.

(iv) S and T are weakly compatible maps if they commute at their coincidence points, i.e., if $STw = TSw$, whenever $Sw = Tw$.

Jungck and Rhoades proposed the idea of weak compatibility [7]. The basics of ordered normed spaces and E-metric spaces are following:

Definition 1.1 [14] *An ordered space E is a vector space over the real numbers, with a partial order relation $\preceq$ such that*

(i) for all x, y and $z \in E, x \preceq y$ implies $x + z \preceq y + z$;

(ii) for all $a \in \mathbb{R}^+$ and $x \in E$ with $x \succeq 0_E, ax \succeq 0_E$.

Furthermore, if E is equipped with a norm $||.||$, then E is called normed ordered space.

Definition 1.2 [14] *Let X be a non empty set and E be a real normed space. Suppose 0_E is the zero element of E. The mapping $d^E : X \times X \to E$ is said to be an $E-$metric if for all $x, y \in X$, it satisfies*

(i) $0_E \leq d^E(x, y), d^E(x, y) = 0_E \Leftrightarrow x = y$;

(ii) $d^E(x, y) = d^E(y, x)$;

(iii) $d^E(x, y) \leq d^E(x, z) + d^E(z, y)$.

The pair (X, d^E) is called an $E-$metric space.

Remark 1.1 With reference to the topology of E-metric spaces, the properties of countability, Hausdorffness, and nets, references are [[10], [11], [12], [13], [14]].
Let E be an ordered space with a norm and (X, d^E) be an E-metric space. Let a be a point in X and $e \in Int(E^+)$. Then the open ball in X centered at a of radius e is

$$B(a, e) = \{x \in X; d^E(x, a) < e\}.$$

Example 1.1 *Let $X = \mathbb{R}$, if we take $d^E(x, y) = |x - y|\ \forall\ x, y \in X$ then (X, d^E) is E-metric space as well as usual metric space.*

Definition 1.3 [14] *Let (X, d^E) be an E-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. We then say:*

(i) $\{x_n\}$ is convrgent to x if for all $e \in Int(E^+)\ \exists$, a positive integer N such that $d^E(x_n, x) < e$, for all $n > N$;

(ii) $\{x_n\}$ is a Cauchy sequence if for any $e \in Int(E^+)\ \exists$ a positive integer N such that $d^E(x_n, x_m) < e$; for all $n, m > N$;

(iii) (X, d^E) is complete E-metric space if every Cauchy sequence is convergent to some point in X.

Proposition 1.1 *Let $\{x_n\}$ be a sequence in a E-metric space (X, d^E), where E is a normed ordered space. If x_n is convergent, then it is a Cauchy sequence.*

Proposition 1.2 *Let E be a normed ordered space with $Int(E^+) \neq \emptyset$ and let (X, d^E) be an E-metric space. Then any convergent sequence $\{x_n\}$ in X has a unique limit.*

In 2014, $\theta-$contraction was defined by Jleli and Samet [9] as following:

Definition 1.4 [9] *Let Θ be the family of all functions $\theta : (0, \infty) \to [1, \infty)$ such that*

$(\theta)_1$ *θ is non-decreasing;*

$(\theta)_2$ *for each sequence $x_n \in (0, \infty)$ $\lim_{n\to\infty} x_n = 0$ iff $\lim_{n\to\infty} \theta(x_n) = 1$;*

$(\theta)_3$ *θ is continuous.*

From the above definitions we can categorise the Banach contraction as a specific type of θ contraction while vice-versa need not be true. Jleli and Samet proved a fixed point theorem for θ-contraction. According to them:

Theorem 1.1 [9] *Let (X, d) be a complete metric space and $T : X \to X$ be a θ-contraction. Then T has a unique fixed point.*

Recently, Zheng et al. [15] defined the new type of contractive mappings as follows:

Definition 1.5 [15] *Let Φ be the family of all functions $\phi : [1, \infty) \to [1, \infty)$ such that*

$(\phi)_1$ *ϕ is non-decreasing;*

$(\phi)_2$ *for each $t > 1$, $\lim_{n\to\infty} \phi^n(t) = 1$;*

$(\phi)_3$ *ϕ is continuous on $[1, \infty)$.*

Lemma 1.1 [15] *If $\phi \in \Phi$, then $\phi(1) = 1$ and $\phi(t) < t$ for each $t \in [1, \infty)$.*

Zheng et al. [15] gave the definition of $\theta - \phi$ contraction as follow:

Definition 1.6 [15] Let (X, d) be a metric space and $T : X \to X$ be a self-map, then T is said to be a $\theta - \phi$ contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,

$$\theta(d(Tx, Ty)) \leq \phi[\theta(N(x, y))]$$

where $N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

2. Main Result

Motivated and inspired by Zheng et al. [15] we define the notion of generalized $\theta - \phi$ contraction on the E-metric space and obtained the fixed point result.

Definition 2.1 *Let (X, d^E) be an E-metric space and S, T, U, V are the mappings from X to itself. The pair (S, T) is called generalized $\theta-\phi$ contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$, we have*

$$\theta(d^E(Sx, Ty)) \leq \phi[\theta(M(x, y))] \tag{2.1}$$

where

$$M(x, y) = \max\{d^E(Ux, Vy), d^E(Ux.Sx), d^E(Vy, Ty), \frac{1}{2}[d^E(Ux, Ty) + d^E(Vy, Sx)]\}$$

Theorem 2.1 *Let (X, d^E) be a complete E-metric space. If mappings S, T, U, V satisfies $\theta-\phi$ contraction and $S(X) \subset V(X)$ and $T(X) \subset U(X)$, then S, T, U, V have a coincidence point. Moreover, if the pair $\{S, U\}$ and $\{T, V\}$ are weakly compatible, then the mappings S, T, U and V have a unique common fixed point in X.*

Proof: Let x_0 be an arbitrary point in X. Let $y_0 = Sx_0$. As $S(X) \subset V(X)$, we can find a point $x_1 \in X$ such that $y_0 = Sx_0 = Vx_1$. Set $y_1 = Tx_1$. As $T(X) \subset U(X)$, there exists a point $x_2 \in X$ such that $y_1 = Tx_1 = Ux_2$. By induction, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X which satisfy for each non-negative integer n,
$y_{2n} = Sx_{2n} = Vx_{2n+1}$ and $y_{2n+1} = Tx_{2n+1} = Ux_{2n+2}$. For all nonnegative integer n we have

$$\theta(d^E(Sx_{2n+2}, Tx_{2n+1})) \leq \phi[\theta(M(x_{2n+2}, x_{2n+1}))] \tag{2.2}$$

where

$$\begin{aligned} M(x_{2n+2}, x_{2n+1}) &= \max\{d^E(Ux_{2n+2}, Vx_{2n+1}), d^E(Ux_{2n+2}, Sx_{2n+2}), d^E(Vx_{2n+1}, Tx_{2n+1}), \\ &\qquad \frac{1}{2}[d^E(Ux_{2n+2}, Tx_{2n+1}) + d^E(Vx_{2n+1}, Sx_{2n+2})]\} \\ &= \max\{d^E(Tx_{2n+1}, Sx_{2n}), d^E(Tx_{2n+1}, Sx_{2n+2}), d^E(Sx_{2n}, Tx_{2n+1}), \\ &\qquad \frac{1}{2}[d^E(Tx_{2n+1}, Tx_{2n+1}) + d^E(Sx_{2n}, Sx_{2n+2})]\} \\ &= \max\{d^E(Tx_{2n+1}, Sx_{2n}), d^E(Tx_{2n+1}, Sx_{2n+2})\}. \end{aligned}$$

If $d^E(Tx_{2n+1}, Sx_{2n}) \leq d^E(Tx_{2n+1}, Sx_{2n+2})$, then by lemma (1.1)

$$\theta(d^E(Sx_{2n+2}, Tx_{2n+1})) \leq \phi[\theta(d^E(Sx_{2n+2}, Tx_{2n+1}))] < \theta(d^E(Sx_{2n+2}, Tx_{2n+1}))$$

which is a contradiction as θ is non-decreasing. Hence,

$$M(x_{2n+2}, x_{2n+1}) = d^E(Sx_{2n+2}, Tx_{2n+1}),$$

which implies

$$\theta(d^E(Sx_{2n+2}, Tx_{2n+1})) \leq \phi[\theta(Sx_{2n}, Tx_{2n+1})],$$

and we conclude that

$$\theta(d^E(Sx_{2n+2}, Tx_{2n+1})) \leq \phi^n[\theta(Sx_0, Tx_1)].$$

This sequence is non-increasing therefore it converges to a limit. By definition of Θ and Φ_2, we have

$$\lim_{n\to\infty} \phi^n[\theta(d^E(Sx_0, Tx_1))] = 1$$

and by (Θ_2),

$$\lim_{n\to\infty} d^E(Sx_{2n+2}, Tx_{2n+1}) = 0_E. \tag{2.3}$$

Let us prove that $\{y_n\}$ is a Cauchy sequence. Since $\lim\limits_{n\to\infty} d^E(y_n, y_{n+1}) = 0_E$ therefore we need only to prove that $\{y_{2n}\}$ is a Cauchy sequence. For getting a contradiction, let there exist $\epsilon > 0$ and sequences $\{2n(k)\}, \{2m(k)\}$ with $2k \leq 2m(k) < 2n(k), (k \in \mathbb{N})$ verifying

$$d^E(y_{2n(k)}, y_{2m(k)}) > \epsilon \tag{2.4}$$

For every integer $2k$, let $2n(k)$ be the least even integer exceeding $2m(k)$. Then, we have

$$d^E(y_{2m(k)}, y_{2n(k)-2}) \leq \epsilon,$$

and by equation (2.4),

$$\begin{aligned} \epsilon &< d^E(y_{2m(k)}, y_{2n(k)}) \\ &\leq d^E(y_{2m(k)}, y_{2n(k)-2}) + d^E(y_{2m(k)-2}, y_{2n(k)-1}) + d^E(y_{2n(k)+1}, y_{2n(k)}) \end{aligned}$$

$$\epsilon < d^E(y_{2m(k)}, y_{2n(k)}) \leq \epsilon + d^E(y_{2n(k)-2}, y_{2n(k)-1}) + d^E(y_{2n(k)-1}, y_{2n(k)})$$

Letting $k\to\infty$ in the above inequality and using (2.3), we get

$$\lim_{k\to\infty} d^E(y_{2m(k)}, y_{2n(k)}) = \epsilon. \tag{2.5}$$

Again,

$$d^E(y_{2m(k)}, y_{2n(k)}) \leq d^E(y_{2m(k)}, y_{2m(k)+1}) + d^E(y_{2m(k)+1}, y_{2n(k)+1}) + d^E(y_{2n(k)+1}, y_{2n(k)})$$

and

$$d^E(y_{2m(k)+1}, y_{2n(k)+1}) \leq d^E(y_{2m(k)+1}, y_{2m(k)}) + d^E(y_{2m(k)}, y_{2n(k)}) + d^E(y_{2n(k)}, y_{2n(k)+1})$$

Letting $k\to\infty$ in the above inequality and using (2.3) and (2.5), we get

$$\lim_{k\to\infty} d^E(y_{2m(k)+1}, y_{2n(k)+1}) = \epsilon. \tag{2.6}$$

Again,

$$d^E(y_{2m(k)}, y_{2n(k)+2}) \leq d^E(y_{2m(k)}, y_{2m(k)+1}) + d^E(y_{2m(k)+1}, y_{2n(k)+1}) + d^E(y_{2n(k)+1}, y_{2n(k)+2})$$

Letting $k\to\infty$ in the above inequality and using (2.3) and (2.6), we have

$$\lim_{k\to\infty} d^E(y_{2m(k)}, y_{2n(k)+2}) = \epsilon. \tag{2.7}$$

For all non-negative integer k, we have

$$\theta(d^E(Sx_{2n(k)+2}, Tx_{2m(k)+1})) \leq \phi[\theta(M(x_{2n(k)+2}, x_{2m(k)+1}))],$$

where

$$\begin{aligned} M((x_{2n(k)+2}, x_{2m(k)+1})) &= \max\{d^E(Ux_{2n(k)+2}, Vx_{2m(k)+1}), d^E(Ux_{2n(k)+2}, Sx_{2n(k)+2}), \\ &\quad d^E(Vx_{2m(k)+1}, Tx_{2m(k)+1}), \qquad (2.8) \\ &\quad \frac{1}{2}[d^E(Ux_{2n(k)+2}, Tx_{2m(k)+1}) + d^E(Vx_{2m(k)+1}, Sx_{2n(k)+2})]\} \\ &= \max\{d^E(Tx_{2n(k)+1}, Sx_{2m(k)}), d^E(Tx_{2n(k)+1}, Sx_{2n(k)+2}), \\ &\quad d^E(Sx_{2m(k)}, Tx_{2m(k)+1}), \qquad (2.9) \\ &\quad \frac{1}{2}[d^E(Tx_{2n(k)+1}, Tx_{2m(k)+1}) + d^E(Sx_{2m(k)}, Sx_{2n(k)+2})]\} \\ &= d^E(z, Tu). \end{aligned}$$

Letting $k\to\infty$ and using (2.3), (2.5), (2.6) and (2.7) in the above inequality, we obtain

$$\theta(\epsilon) \leq \phi[\theta(\epsilon)].$$

It follows from lemma (1.1) that

$$\theta(\epsilon) \leq \phi[\theta(\epsilon)] < \theta(\epsilon),$$

which is a contradiction. Thus, $\{y_n\}$ is a Cauchy sequence. As X is complete, the sequence $\{y_n\}$ converges to a point say z. Thus we have

$$z = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Vx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Ux_{2n}.$$

Let $u\in X$ such that $z = Vu$. By inequality (2.1), we obtain

$$d^E(y_{2n}, Tu) = \theta[d^E(Sx_{2n}, Tu)] \leq \phi[\theta(M(x_{2n}, u))],$$

where

$$M(x_{2n},u) = \max\{d^E(Ux_{2n},Vu), d^E(Ux_{2n},Sx_{2n}), d^E(Tu,Vu), \frac{1}{2}[d^E(Ux_{2n},Tu)+d^E(Vu,Sx_{2n})]\}$$
$$= \max\{d^E(z,Vu), d^E(z,z), d^E(z,Tu), \frac{1}{2}[d^E(z,Tu)+d^E(Vu,z)]\}.$$

Hence, by lemma (1.1)

$$\theta(d^E(z,Tu)) \leq \phi[\theta(d^E(z,Tu))] < \theta(d^E(z,Tu)),$$

which is a contradiction. Therefore $Vu = Tu = z$. Since $T(X) \subset U(X)$, then $\exists\ w \in X$ such that $z = Tu = Uw$. Using (2.1), we have

$$\theta(d^E(Sw,z)) = \theta(d^E(Sw,Tu)) \leq \phi[\theta(M(w,u))],$$

where

$$M(w,u) = \max\{d^E(Uw,Vu), d^E(Uw,Sw), d^E(Vu,Tu), \frac{1}{2}[d^E(Uw,Tu)+d^E(Vu,Sw)]\}$$
$$= \max\{d^E(z,Sw), \frac{1}{2}[d^E(z,Sw)]\}$$
$$= d^E(z,Sw).$$

Then, we deduce that

$$\theta(d^E(Sw,z)) \leq \phi[\theta(d^E(Sw,z))] < \theta(d^E(Sw,z)),$$

again by lemma (1.1) which is a contradiction by lemma (1.1). Hence

$$Sw = Uw = z = Vu = Tu. \tag{2.10}$$

Suppose that $\{S,U\}$ and $\{T,V\}$ are weakly compatible, then by (2.8), we have $Sz = Uz$ and $Tz = Vz$. Since,

$$M(w,z) = \max\{d^E(Uw,Vz), d^E(Uw,Sw), d^E(Vz,Tz), \frac{1}{2}[d^E(z,Tz)+d^E(Vz,z)]\}$$
$$= \max\{d^E(z,Vz), \frac{1}{2}[d^E(Uw,Tz)+d^E(Vz,Sw)]\}$$
$$= d^E(z,Tz).$$

Then by (2.1), we obtain

$$\theta(d^E(z,Tz)) = \theta(d^E(Sw,Tz)) \leq \phi[\theta(d^E(z,Tz))] < \theta(d^E(z,Tz)),$$

which is a contradiction. Hence $\theta(d^E(z,Tz)) = 0 \implies d^E(z,Tz) = 0$ so we have $z = Tz = Vz$. Similarly, we can show that $z = Tz = Vz$. Thus z is a common fixed point of the mappings S, T, U and V.
Now, let u be another common fixed point of the mappings S, T, U and V. Then, using (2.1), we obtain

$$\theta(d^E(z,u)) = \theta(d^E(Sz,Tu)) \leq \phi[\theta(M(z,u))],$$

where

$$M(z,u) = \max\{d^E(Uz,Vu), d^E(Uz,Sz), d^E(Vu,Tu), \frac{1}{2}[d^E(Uz,Tu)+d^E(Vu,Sz)]\}$$
$$= \max\{d^E(z,u), d^E(z,z), d^E(u,u), \frac{1}{2}[d^E(z,u)+d^E(u,z)]\}$$
$$= d^E(z,u).$$

Then,
$\theta(d^E(z,u)) = \theta(d^E(Sz,Tu)) \leq \phi[\theta(d^E(z,u))] < \theta(d^E(z,u))$
which gives $d^E(z,u) = 0_E$ then $z = u$. Hence the obtained common fixed point is unique. □

Now, let us provide the following example:

Example 2.1 *Consider $X = [0, 1]$ endowes with E-metric d^E given by $d^E(x, y) = |x - y|$ $x, y \in X$. Here (X, d^E) is a complete E-metric space. Define the mappings $S, T, U, V : X \to X$ by*

$$S(x) = \left(\frac{x}{3}\right)^2,$$

$$T(x) = \left(\frac{x}{4}\right)^2,$$

$$U(x) = \frac{x}{3},$$

$$V(x) = \frac{x}{4}$$

Consider the functions $\theta(x) = 1 + x$, $\forall$ $x \in X$
and $\phi(t) = \begin{cases} 1 \ \ if \ \ 1 \le t \le 2 \\ t - 1 \ \ if \ \ t \ge 2 \end{cases}$
We have $S(X) \subset V(X)$ and $T(X) \subset U(X)$. Also, we can check for each $x, y \in X$,

$$\theta(d^E(Sx, Ty)) = \theta(|x - y|) \le \phi[\theta(M(x, y))]$$

where $M(x, y) = \max\{d^E(Ux, Vy), d^E(Ux, Sx), d^E(Vy, Ty), \frac{1}{2}[d^E(Ux, Ty) + d^E(Vy, Sx)]\}$
Furthermore, the pairs $\{S, U\}$ and $\{T, V\}$ are weakly compatible. Thus, S, T, U, V satisfies all the condition of theorem (2.1). Here, $x = 0$ is a common fixed point of all the four mappings.

Consequently, we have the following results:

Corollary 2.1 *Let (X, d^E) be an E-metric space and the mappings $f, S, T : X \to X$ satisfies the following $\theta - \phi$ contraction condition*

$$\theta(d^E(Tx, Sy)) \le \phi[\theta(M(x, y))],$$

where
$M(x, y) = \max\{d^E(fx, fy), d^E(Tx, fx), d^E(Sy, fy), \frac{1}{2}[d^E(fy, Tx) + d^E(fx, Sy)]\}$
then, f, S and T have a unique common fixed point. Moreover, any fixed point of f is a fixed point of S and T and conversely.

Proof: We can prove it by taking $U = V$ in theorem (2.1). □

Corollary 2.2 *Let (X, d^E) be an E-metric space and the mapping $T : X \to X$ satisfy the following $\theta - \phi$ contraction condition*

$$\theta(d^E(Tx, Ty)) \le \phi[\theta(M(x, y))],$$

where
$M(x, y) = \max\{d^E(x, y), d^E(x, Tx), d^E(y, Ty), \frac{1}{2}[d^E(x, Ty) + d^E(y, Tx)]\}$
then, T has a unique fixed point.

Proof: Taking U and V as identity map and $S = T$ in theorem (2.1) we can prove it. □

Now we give an example for the above corollary.

Example 2.2 *Let $X = \mathbb{Z}$ endowed with the E-metric $d^E(x, y) = |x - y|$ $\forall$ $x, y \in X$. Define the mapping $T : X \to X$ by*

$$Tx = \begin{cases} 0, \ \ if \ \ x = 0 \\ -(m - 1), \ \ if \ \ x = m, \ \ m = 1, 2, 3... \\ m - 1, \ \ if \ \ x = -m, \ \ m = 1, 2, 3... \end{cases}$$

Now, we consider four three cases:

(i) for all $m > n > 2$, we have

$$d^E(Tn, Tm) = m - n$$
$$d^E(m, Tm) = 2m - 1$$
$$d^E(n, Tn) = 2n - 1$$
$$d^E(m, Tn) = m + n - 1 = d^E(n, Tm)$$
$$\theta(Tm, Tn) \leq \phi[\theta(\max\{d^E(m, n), d^E(m, Tm), d^E(n, Tn), \frac{1}{2}[d^E(m, Tn) + d^E(n, Tm)]\})]$$

as θ and ϕ are any non-decreasing functions, we have

$$\theta(m - n) \leq \phi[\theta(m + n - 1)]$$

hence condition (2.9) is verified.

(ii) for $x = m \geq 1, \ \ y = 0$ or $x = -m(m \geq 1), y = 0$. We have

$$d^E(Tx, Ty) = m - 1$$
$$d^E(x, Tx) = 2m - 1$$
$$d^E(y, Ty) = 0$$
$$d^E(x, Ty) = m$$
$$d^E(y, Tx) = m - 1$$
$$\theta(Tm, Tn) \leq \phi[\theta(\max\{d^E(m, n), d^E(m, Tm), d^E(n, Tn), \frac{1}{2}[d^E(m, Tn) + d^E(n, Tm)]\})]$$

as θ and ϕ are any non-decreasing functions, we have

$$\theta(m - 1) \leq \phi[\theta(2m - 1)]$$

hence condition (2.9) is verified.

(iii) $x = m > y = n \geq 1$ or $x = -m < y = -n \leq -1$

$$d^E(Tx, Ty) = m - n \ \text{ or } \ n - m$$
$$d^E(x, Tx) = 2m - 1$$
$$d^E(y, Ty) = 2n - 1$$
$$d^E(x, Ty) = m + n - 1$$
$$d^E(y, Tx) = n + m - 1$$
$$\theta(Tm, Tn) \leq \phi[\theta(\max\{d^E(m, n), d^E(m, Tm), d^E(n, Tn), \frac{1}{2}[d^E(m, Tn) + d^E(n, Tm)]\})]$$

as θ and ϕ are any non-decreasing functions, we have

$$\theta(m - n) \leq \phi[\theta(2m - 1)]$$

hence condition (2.9) is verified.

(iv) $x = m, y = -n, m > n \geq 1$. We have

$$d^E(Tx, Ty) = m + n - 2 \ \text{ or } \ n - m$$
$$d^E(x, Tx) = 2m - 1$$
$$d^E(y, Ty) = 2n - 1$$
$$d^E(x, Ty) = 1$$
$$d^E(y, Tx) = m - n - 1$$
$$\theta(Tm, Tn) \leq \phi[\theta(\max\{d^E(m, n), d^E(m, Tm), d^E(n, Tn), \frac{1}{2}[d^E(m, Tn) + d^E(n, Tm)]\})]$$

as θ and ϕ are any non-decreasing functions, we have

$$\theta(m+n-2) \leq \phi[\theta(2m-1)]$$

hence condition (2.9) is verified. Thus T is a $\theta-\phi$ contraction. So all the hypothesis of corollary are satisfied, thus T has a fixed point. In this example $x=0$ is the fixed point.

Corollary 2.3 *Let S,T,U and V are self mappings of a complete E-metric space (X,d^E) such that $SX \subset UX, TX \subset VX$ and*

$$d^E(Sx,Ty) \leq \alpha \max\{d^E(Ux,Vy), d^E(Ux,Tx), d^E(Vy,Sy), \frac{1}{2}[d^E(Ux,Sy)+d^E(Vy,Tx)]\}$$

for all $x,y \in X$, where $\alpha \in (0,1)$ is a constant. If any of SX, TX, UX and VX is a closed subset of (X,d^E), then the pairs $\{S,V\}$ and $\{T,U\}$ have a coincidence point. Moreover, if the pairs $\{S,V\}$ and $\{T,U\}$ weakly compatible, then S,T,U and V have a unique common fixed point.

Proof: We can prove it by taking $\theta(x)=x$ and $\phi(x)=\alpha x$. □

We get a generalisation of Ćirić's contraction mapping as a result of this conclusion.

Corollary 2.4 *Let (X,d^E) be a complete E-metric space and T be a self-maps of X satisfying*

$$d^E(Tx,Ty) \leq \alpha \max\{d^E(x,y), d^E(x,Tx), d^E(y,Ty), \frac{1}{2}[d^E(x,Ty)+d^E(y,Tx)]\} \tag{2.11}$$

for all $x,y \in X$ where $\alpha \in (0,1)$ is a constant. Then T has a unique fixed point.

3. Application

In this section, we prove a fixed point theorem as an application of our main result on homotopy operator for E-metric space.
Let (X,d^E) be an E-metric space, $\mathbb{R}$ be equipped with Hausdorff topology. Let $[0,1]$ be endowed with subspace topology. Also, a topological space X is connected iff its only clopen sets are X and ϕ.

Theorem 3.1 *Let (X,d^E) be a complete E-metric space. Assume that A be an open set of X and $A \subset B$ where B be a closed subset of X. Let $H: B\times[0,1] \to X$ be an operator such that the aforementioned conditions are fulfilled:*

(i) $x \neq H(x,t)$ *for every* $x \in B\backslash A$ *and every* $t \in [0,1]$.

(ii) $\exists \phi \in \Phi, \theta \in \Theta$ *such that for all* $t \in [0,1]$ *and every* $x,y \in B$, *we have*

$$\theta[d^E(H(x,t),H(y,t))] \leq \phi[\theta(d^E(x,y))]$$

(iii) $\exists$ *a function* $\eta:[0,1] \to \mathbb{R}$ *which is continuous such that*

$$\theta[d^E(H(x,t),H(x,s))] \leq |\eta(t)-\eta(s)|$$

for all $t,s \in [0,1]$ *and each* $x \in B$.

(iv) $\psi:[1,+\infty) \to [0,+\infty)$ *is strictly non-decreasing* $(\psi(x)=x-\phi(\theta(x))$.

Then $H(.,0)$ has a fixed point iff $H(.,1)$ has a fixed point.

Proof: Suppose

$$Q := \{t \in [0,1]; x = H(x,t) \;\; for \;\; any \;\; x \in A\}$$

As (i) holds and $H(.,0)$ has a fixed point, we have that $0_E \in Q$, so Q is a non-empty set. We will demonstrate that Q is both closed and open in $[0,1]$ and since $Q = [0,1]$ we are finished as $[0,1]$ is connected.

First, let us show that Q is open in $[0,1]$. Let $t_0 \in Q$ and $x_0 \in A$ with $x_0 = H(x_0, t_0)$. As A is open in $(X, d^E), \exists\, r > 0_E$ such that $B_d^E(x_0, r) \subseteq A$. Consider $\epsilon = \psi(r + d^E(x_0, x_0)) > 0_E$. Since η and θ is continuous on t_0, there exists $\alpha(\epsilon) > 0_E$ such that $|\eta(t) - \eta(t_0)| < \epsilon \;\forall\; t \in (t_0 - \alpha(\epsilon), t_0 + \alpha(\epsilon))$.

Let $t \in (t_0 - \alpha(\epsilon), t_0 + \alpha(\epsilon))$ for $x \in B_{d^E}(x_0, r) = \{x \in X; d^E(x_0, x) \leq d^E(x_0, x_0) + r\}$, we have

$$\begin{aligned}
d^E(H(x,t), x_0) \leq \theta(d^E(H(x,t), x_0)) &= \theta(d^E(H(x,t), H(x_0, t_0)) \\
&\leq \theta(d^E(H(x,t), H(x, t_0)) + d^E(H(x, t_0), H(x_0, t_0))) \\
&\leq \theta(d^E(H(x,t), H(x, t_0))) + \theta(d^E(H(x, t_0), H(x_0, t_0))) \\
&\leq |\eta(t) - \eta(t_0)| + \phi[\theta(d^E(x, x_0))] \\
&\leq \epsilon + \phi[\theta(d^E(x, x_0))] \\
&= \psi(r + d^E(x_0, x_0)) + \phi[\theta(d^E(x_0, x_0) + r)] \\
&= r + d^E(x_0, x_0) - \phi[\theta(r + d^E(x_0, x_0))] + \phi[\theta(d^E(x_0, x_0) + r)] \\
&= r + d^E(x_0, x_0)
\end{aligned}$$

Thus, for each $t \in (t_0 - \alpha(\epsilon), t_0 + \alpha(\epsilon))$,

$$H(.,t) : \overline{B_{d^E}(x_0, r)} \to \overline{B_{d^E}(x_0, r)}$$

As also (ii) holds and θ and ϕ are non-decreasing, then all the hypothesis of theorem (2.1) are accomplished with $S = T = H(.,t)$ and $U = V = I_{\overline{B_{d^E}(x_0,r)}}$ (the identity map on $\overline{B_{d^E}(x_0, r)}$). Thus we conclude that $H(.,t)$ has a fixed point in B. However this fixed point has to be in A as (i) holds. Hence, $(t_0 - \alpha(\epsilon), t_0 + \alpha(\epsilon) \subset Q$ and therefore Q is open in $[0,1]$.

Next, we show that Q is closed in $[0,1]$. To check this, let $\{t_p\}_{p \in \mathbb{N}^*}$ be a sequence in Q with $t_p \to t^* \in [0,1]$ as $p \to \infty$. We must show that $t^* \in Q$. By the definition of Q, for all $p \in \mathbb{N}^*$, $\exists\; x_p \in A$ with $x_p = H(x_p, t_p)$. Also, for $q, p \in \mathbb{N}^*$, we have

$$\begin{aligned}
d^E(x_p, x_q) \leq \theta(d^E(x_p, x_q)) &= \theta(d^E(H(x_p, t_p), H(x_q, t_q)) \\
&\leq \theta(d^E(H(x_p, t_p), H(x_p, t_q) + d^E(H(x_p, t_q), H(x_q, t_q))) \\
&\leq \theta(d^E(H(x_p, t_p), H(x_q, t_q))) + \theta(d^E(H(x_p, t_q), H(x_q, t_q))) \\
&\leq |\eta(t_p) - \eta(t_q)| + \phi[\theta(d^E(x_p, x_q))]
\end{aligned}$$

This implies that

$$d^E(x_p, x_q) \leq |\eta(t_p) - \eta(t_q)|$$

and from (iv), we get

$$d^E(x_p, x_q) \leq \psi^{-1}[|\eta(t_p) - \eta(t_q)|]$$

Since ψ^{-1} and η are continuous and $\{t_p\}_{p \in \mathbb{N}^*}$ is convergent, letting $p, q \to \infty$ in the above inequality, we obtain $\lim\limits_{p,q \to \infty} d^E(x_p, x_q) = 0_E$, that is, $\{x_p\}_{p \in \mathbb{N}^*}$ is a Cauchy sequence in (X, d^E). Since (X, d^E) is complete, $\exists\;\, x^* \in B$ with $d^E(x^*, X^*) = \lim\limits_{p \to \infty} d^E(x^*, x_p) = \lim\limits_{p,q \to \infty} d^E(x_p, x_q) = 0_E$.

On the other hand, we have

$$\begin{aligned}
\theta(d^E(x_p, H(x^*, t^*))) &= \theta(d^E(H(x_p, t_p), H(x^*, t^*))) \\
&\leq \theta(d^E(H(x_p, t_p) H(x_p, t^*)) + d^E(H(x_p, t^*), H(x^*, t^*))) \\
&\leq \theta(d^E(H(x_p, t_p), H(x_p, t^*))) + \theta(d^E(H(x_p, t^*), H(x^*, t^*))) \\
&\leq |\eta(t_p) - \eta(t^*)| + \phi[\theta(d^E(x_p), x^*)]
\end{aligned}$$

Letting $p \to \infty$ in tha above ineqality, we get $\lim_{p\to+\infty} d^E(x_p, H(x^*, t^*)) = 0_E$ and so

$$d^E(x^*, H(x^*, t^*)) = \lim_{p\to\infty} d^E(x^*, H(x^*, t^*)) = 0_E$$

this implies that $x^*, H(x^*, t^*)$ and as (i) holds, we have $x^* \in A$. Thus, $t^* \in Q$ and Q is closed in $[0, 1]$. By employing the same method, we can demonstrate the opposite implication. □

$$y' = Ay + f, \quad y(0) = y_0 \tag{3.1}$$

$$Ae_k = \lambda_k e_k, \quad k = 1, 2, \ldots$$

4. Declarations

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No data were used to support this study.

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The authors declare that they have no competing interests.

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