Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) v. 2025 (43) : 1–9.
ISSN-0037-8712
doi:10.5269/bspm.67675

Existence and uniqueness results of nonlinear hybrid Caputo-Fabrizio fractional differential equations with periodic boundary conditions

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ABSTRACT: In this manuscript, we establish the existence and uniqueness of solutions for nonlinear hybrid fractional differential equations involving Caputo-Fabrizio fractional derivatives of order $\varrho \in (0,1)$. The proofs are based on Banach's fixed point theorem and some basic concepts of Caputo-Fabrizio fractional analysis. As an application, a nontrivial example is given in the last part of this paper to illustrate our theoretical results.

Key Words: Caputo-Fabrizio fractional integral; Caputo-Fabrizio fractional derivative; Banach fixed point theorem; Gronwall theorem.

Contents

1. Introduction

Due to the development of fractional calculus in numerous scientific disciplines, the theory of fractional differential equations has recently attracted a great deal of interest. This theory is a useful tool for simulating and describing some phenomena in a variety of scientific and engineering fields, including fluid flow phenomena, earthquakes, electrochemical processes, wave propagation, signal theory, biology, and electromagnetic theory.There are many studies now that cover the fundamental theory and uses of fractional calculus and fractional differential equations(see, [2,5,7,15,19,26,35]).Though the majority of the early studies relied on the Riemann-Liouville fractional order derivative or the Caputo fractional order derivative, it has recently been noted that these derivatives have the issue that their kernels have a singularity that takes place at the endpoint of an interval of definition. As a result, the literature has recently proposed numerous new definitions of fractional derivatives such as ψ Caputo derivative [3,6],Caputo Fabrizio derivative [18]. Because of the nonlocality of fractional derivatives, fractional differential equations, such as functional hybrid fractional differential equations, have recently gained popularity for modeling and describing non-homogeneous physical phenomena that occur in their form. The significance of fractional hybrid differential equations stems from the fact that they include a variety of dynamical systems as special cases. Furthermore, hybrid differential equations arise in a variety of areas of applied mathematics and physics, such as the deflection of a curved beam with a constant or varying cross-section and electromagnetic waves or gravity-driven flows. Dhage .al [20] discussed the existence and uniqueness of solutions for the following hybrid differential equations with second-type linear perturbations :

$$\begin{cases} \dfrac{d}{dt}\left(y(t) - \mathcal{F}(t, y(t))\right) = \mathcal{G}(t, y(t)), \qquad t \in [t_0, t_0 + \xi], \\ \\ y(0) = y_0 \in \mathbb{R}. \end{cases}$$

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Submitted March 27, 2023. Published December 12, 2023
2010 *Mathematics Subject Classification:* 34A08, 34K37.

They proved the existence theorems by employing fundamental differential inequalities and comparison results. For more details of this theory, we refer the readers to the articles [8,11,12,25,23,27,33,36,37]and references therein.
The study of fractional differential equations has gained a fresh perspective thanks to the Caputo-Fabrizio fractional derivative. The nonsingular kernel of the new derivative is one of its most attractive features [17,18]. The Caputo-Fabrizio derivative shares the same additional motivating properties of heterogeneous and configuration with different scales as the Caputo and Riemann-Liouville fractional derivatives. Motivated by the above works especially [20] and [23], we develop the theory of fractional hybrid differential equations involving Caputo Fabrizio fractional differential operator of order $\varrho \in (0,1)$. To be more precise, we establish the existence and uniqueness of the solution of the following nonlinear fractional hybrid differential equation:

$$\begin{cases} {}^{CF}D^{\varrho}_{0+}\left(y(t)-\Psi(t,y(t))\right)=\Phi(t,y(t)), \qquad t\in\mathfrak{D}=[0,T], \\ \\ y(0)=y(T)=0. \end{cases} \tag{1.1}$$

Where $T>0$, ${}^{CF}D^{\varrho}_{0+}$ is the Caputo-Fabrizio derivative and Φ,Ψ in $C(\mathfrak{D}\times\mathbb{R},\mathbb{R})$.
Our paper is organized as follows. In Section 2, We provide some fundamental definitions and properties of fractional integrals and Caputo Fabrizio fractional derivatives, which will be used throughout the remainder of our manuscript. In Section 3, we establish the existence and uniqueness of solutions of the Caputo Fabrizio fractional periodic boundary value problem (1.1) by using Banach's fixed point theorem. As an application, an illustrative example is presented in Section 4 followed by a conclusion in Section 5.

2. Preliminaries

In this section, we give some notations, definitions, and results on Caputo- Fabrizio fractional derivatives and Caputo- Fabrizio fractional integrals, consult the articles [9,13,30] for more details.

Notations

• We denote by $C(\mathfrak{D},\mathbb{R})$ the space of continuous real-valued functions defined on $\mathfrak{D}$ provided with the topology of the supremum norm

$$\| x \| = \sup_{t\in\mathfrak{D}} | y(t) | .$$

• Let $H^1(a,b)=\left\{g \mid g\in L^2(a,b) \text{ and } g'\in L^2(a,b)\right\}$, where $L^2(a,b)$ is the space of square integrable functions on the interval (a,b).

Definition 2.1 ([17]) Let $g\in H^1(a,b)$ and $\varrho\in(0,1)$. Then the Caputo-Fabrizio fractional derivative [18] is defined as

$$ {}^{CF}D^{\varrho}_{at}(g(t))=\frac{\mathcal{M}(\varrho)}{1-\varrho}\int_a^t g'(x)\exp\left[-\varrho\frac{t-x}{1-\varrho}\right]dx \tag{2.1}$$

where $\mathcal{M}(\varrho)$ is a normalization function such that $\mathcal{M}(0)=\mathcal{M}(1)=1$. However, if $g\notin H^1(a,b)$, then the derivative is defined as

$$ {}^{CF}D^{\varrho}_{at}(g(t))=\frac{\varrho\mathcal{M}(\varrho)}{1-\varrho}\int_a^t (g(t)-g(x))\exp\left[-\varrho\frac{t-x}{1-\varrho}\right]dx \tag{2.2}$$

Remark 2.1 ([18]) If $\Phi=\frac{1-\varrho}{\varrho}\in(0,\infty)$, then $\varrho=\frac{1}{1+\Phi}\in(0,1)$. In consequence, Eq. (2) can be reduced to

$$ {}^{CF}D^{\varrho}_{at}(f(t))=\frac{N(\Phi)}{\Phi}\int_a^t f'(x)\exp\left[-\frac{t-x}{\Phi}\right]dx, \tag{2.3}$$

where $N(\Phi)$ is the normalization term corresponding to $\mathcal{M}(\varrho)$ such that $N(0)=N(\infty)=1$.

Remark 2.2 ([18]) We have the following property: $\left[\quad \lim_{\Phi\to 0}\frac{1}{\Phi}\exp\left[-\frac{t-x}{\Phi}\right]=\delta(x-t)\quad\right]$ where δ is the Dirac delta function.

The above Caputo-Fabrizio fractional derivative was later modified by Losada and Nieto [17] as

$$ {}^{CF}D^{\varrho}_{at}(f(t))=\frac{(2-\varrho)\mathcal{M}(\varrho)}{2(1-\varrho)}\int_a^t f'(x)\exp\left[-\varrho\frac{t-x}{1-\varrho}\right]dx \tag{2.4}$$

The fractional integral corresponding to the derivative in Eq. (5) was defined by Nieto and Losada [17] as follows.

Definition 2.2 ([17]) Let $0<\varrho<1$. The fractional integral of order ϱ of a function f is defined by

$$ {}^{CF}I^{\varrho}_{at}(f(t))=\frac{2(1-\varrho)}{(2-\varrho)\mathcal{M}(\varrho)}f(t)+\frac{2\varrho}{(2-\varrho)\mathcal{M}(\varrho)}\int_a^t f(x)dx, t\geq 0 \tag{2.5}$$

Remark 2.3 *([17]) From the definition in Eq. (6), the fractional integral of Caputo-Fabrizio type of a function f of order $0<\varrho<1$ is a mean between the function f and its integral of order one, i.e.,*

$$\frac{2(1-\varrho)}{(2-\varrho)\mathcal{M}(\varrho)}+\frac{2\varrho}{(2-\varrho)\mathcal{M}(\varrho)}=1 \tag{2.6}$$

and therefore $\mathcal{M}(\varrho)=\frac{2}{2-\varrho}, 0<\varrho<1$. Using $\mathcal{M}(\varrho)=\frac{2}{2-\varrho}$, as shown below, Losada and Nieto proposed a new Caputo derivative and its corresponding integral.

Definition 2.3 ([17]) Let $0<\varrho<1$. The fractional Caputo-Fabrizio derivative of order ϱ of a function f is given by

$$ {}^{CF}D^{\varrho}_{at}(f(t))=\frac{1}{1-\varrho}\int_a^t f'(x)\exp\left[-\varrho\frac{t-x}{1-\varrho}\right]dx, \quad t\geq 0, \tag{2.7}$$

and its fractional integral is defined as

$$ {}^{CF}I^{\varrho}_{at}(f(t))=(1-\varrho)f(t)+\varrho\int_a^t f(x)dx, \quad t\geq 0. \tag{2.8}$$

Proposition 2.1 *([10]) Let $\varrho>0$, we have*

1) ${}^{CF}D^{\varrho}_{at}({}^{CF}I^{\varrho}_{at}(u(t)))=u(t)-exp(\frac{-\varrho}{1-\varrho}(t-a))u(a)$.

2) ${}^{CF}I^{\varrho}_{at}({}^{CF}D^{\varrho}_{at}(u(t)))=u(t)-u(a)$.

3) ${}^{CF}I^{\varrho}_{at}$ is linear and bounded from $C(\mathfrak{D},\mathbb{R})$ to $C(\mathfrak{D},\mathbb{R})$.

Theorem 2.1 *([29]). Let Ω be a non-empty, closed convex subset of a Banach space X and Let $\mathcal{F}_1:\Omega\to\Omega$ be an operator such that $\mathcal{F}_1$ is a continuous,*
Then the equation $\mathcal{F}_1x=x$ has a unique solution in Ω.

Theorem 2.2 *([38]).(Gronwall). Let y,Ψ and χ be real continuous functions defined in $[a,b],\chi(t)\geq 0$ for $t\in[a,b]$.*
If we have the inequality:

$$y(t)\leq\Psi(t)+\int_a^t\chi(s)y(s)ds.$$

for all $t\in[a,b]$, Then

$$y(t)\leq\Psi(t)+\int_a^t\chi(s)\Psi(s)\exp\left[\int_a^t\chi(u)du\right]ds$$

for all $t\in[a,b]$.

Lemma 2.1 *(See [38]). If Ψ is constant, then from*

$$x(t) \leq \Psi + \int_a^t \chi(s)x(s)ds$$

it follows that

$$x(t) \leq \Psi \exp\left(\int_a^t \chi(u)du\right).$$

Throughout the rest of our paper, we will assume the following assumptions.

($\mathcal{A}_1$) $y(T) = \Psi(T,0) - \Psi(a,0) + (1-\varrho)\Phi(T,0) + \varrho\int_a^t \Phi(s,y(s))ds = 0.$

($\mathcal{A}_2$) The functions $\Psi, \Phi : \mathfrak{D} \times \mathbb{R} \to \mathfrak{D}$ are continuous.

($\mathcal{A}_3$) There exists a constant $0 < \lambda_1 < 1$ such that:

$$\mid \Psi(t,x) - \Psi(t,y) \mid \leq \lambda_1 \mid x - y \mid \quad for \quad all \quad t \in \mathfrak{D} \quad and \quad x, y \in \mathbb{R}.$$

($\mathcal{A}_4$) $\Phi(a, x(a)) = 0$,and exists a constant $0 < \lambda_2 < 1$ such that :

$$\mid \Phi(t,x) - \Phi(t,y) \mid \leq \lambda_2 \mid x - y \mid \quad for \quad all \quad t \in \mathfrak{D} \quad and \quad x, y \in \mathbb{R}.$$

3. Main results

In this section, before we give the existence result of the fractional boundary value problem (1.1), we need to prove the following fundamental lemma.

Lemma 3.1 *Suppose that hypothesis ($\mathcal{A}_1$) holds, then the function $y(t) \in C(\mathfrak{D}, \mathbb{R})$ is a solution of the periodic fractional boundary value problem (1.1) if and only if y satisfies the following fractional hybrid integral equation*

$$y(t) = \Psi(t, y(t)) - \Psi(a,0) + (1-\varrho)\Phi(t, y(t)) + \varrho \int_a^t \Phi(s, y(s))ds. \tag{3.1}$$

Proof: Let y be a solution of the problem (1.1), then we apply the Caputo Fabrizio fractional integral ${}^{CF}I_{at}^{\varrho}$ on both sides of (1.1) we obtain

$${}^{CF}I_{at}^{\varrho}({}^{CF}D_{at}^{\varrho}\,(y(t) - \Psi(t, y(t)))) = {}^{CF}\, I_{at}^{\varrho}\Phi(t, y(t)),$$

according to Proposition (2.1) we have

$$y(t) - \Psi(t, y(t)) - y(a) + \Psi(a, y(a)) = {}^{CF}\, D_{at}^{\varrho}\Phi(t, y(t)),$$

since $y(a) = y(T) = 0$ then we obtain

$$y(t) = \Psi(t, y(t)) - \Psi(a, 0) + {}^{CF}\, I_{at}^{\varrho}\Phi(t, y(t)),$$

thus

$$y(t) = \Psi(t, y(t)) - \Psi(a,0) + (1-\varrho)\Phi(t, y(t)) + \varrho \int_a^t \Phi(s, y(s))ds.$$

Hence equation (3.1) holds.
Inversely, it is clear that if $y(t)$ satisfies the equation (3.1), we apply the Caputo Fabrizio fractional derivative ${}^{CF}D_{at}^{\varrho}$; to both sides of equation (3.1) and we use Proposition 2.1, we obtain

$${}^{CF}D_{at}^{\varrho}\,(y(t) - \Psi(t, y)) = \;{}^{CF}D_{at}^{\varrho} I_{at}^{\varrho}\Phi(t, y)$$

which implies that

$$^{CF}D^{\varrho}_{at}\,(y(t)-\Psi(t,y)) = \quad \Phi(t,y) - exp(\frac{-\varrho}{1-\varrho}(t-a))\Phi(a,0)),$$

it follows that

$$^{CF}D^{\varrho}_{at}\,(y(t)-\Psi(t,y)) = \quad \Phi(t,y),$$

Finally, we need to verify that the condition $y(a) = y(T) = 0$ in the equation (1.1) also holds. For this purpose, we substitute $t = a$ and $t = T$ in (3.1), we obtain

$$y(a) = \Psi(a, y(a)) - \Psi(a, 0) + (1-\varrho)\Phi(a, y(a)) + \varrho\int_a^a \Phi(s, y(s))ds = 0,$$

and from $(\mathcal{A}_1)$, it follows that

$$y(T) = \Psi(T, y(T)) - \Psi(a, 0) + (1-\varrho)\Phi(T, y(T)) + \varrho\int_a^t \Phi(s, y(s))ds = 0,$$

thus

$$y(a) = y(T) = 0.$$

Hence, y is a solution to the problem (1.1). This completes the proof.

□

Theorem 3.1 *Assume that hypotheses* $(\mathcal{A}_1) - (\mathcal{A}_4)$ *hold,and*

$$\lambda_1 + (1-\varrho)\lambda_2 + \varrho T\lambda_2 < 1$$

then the periodic fractional boundary value problem (1.1) has a solution defined on $\mathfrak{D}$.

Proof: By using lemma 3.1, the fractional hybrid differential equation (1.1) is equivalent to the following nonlinear fractional hybrid integral equation

$$y(t) = \Psi(t, y(t)) - \Psi(0, 0) + (1-\varrho)\Phi(t, x(t)) + \varrho\int_0^t \Phi(s, x(s))ds. \tag{3.2}$$

Let $\mathcal{F}_1 : \mathcal{X} \to \mathcal{X}$ and be an operator defined by

$$\mathcal{F}_1 y(t) = \Psi(t, y(t)) - \Psi(0, 0) + (1-\varrho)\Phi(t, y(t)) + \varrho\int_0^t \Phi(s, y(s))ds,$$

we can transforme the fractional integral equation (3.2) into the operator equation as follows

$$\mathcal{F}_1 y(t) = y(t), \quad t \in \mathfrak{D}. \tag{3.3}$$

Now, we will show that the operator $\mathcal{F}_1$ satisfies the conditions of theorem 2.1.
Let us show that the operator $\mathcal{F}_1$ is a contraction on $\mathcal{X}$.

$$\| \mathcal{F}_1 x - \mathcal{F}_1 y \| = \| \Psi(.,x) - \Psi(.,y) + (1-\varrho)(\Phi(.,x) - \Phi(.,y) + \varrho\int_a^t (\Phi(s,x(s)) - \Phi(s,y(s))ds) \|,$$

$$\| \mathcal{F}_1 x - \mathcal{F}_1 y \| \leq \| \Psi(.,x) - \Psi(.,y) \| + (1-\varrho) \| (\Phi(.,x) - \Phi(.,y) \|$$
$$+ \varrho\int_a^t \| \Phi(s,x(s)) - \Phi(s,y(s)) \| ds,$$

by using assumptions $\mathcal{A}_3$ and $\mathcal{A}_4$ we get

$$\| \mathcal{F}_1 x - \mathcal{F}_1 y \| \leq \lambda_1 \| x - y) \| + (1-\varrho)\lambda_2 \| (x - y \| + \lambda_2 \varrho \int_a^t \| x - y \| ds,$$
$$\| \mathcal{F}_1 x - \mathcal{F}_1 y \| \leq (\lambda_1 + (1-\varrho)\lambda_2 + \lambda_2 \varrho T) \| x - y \|,$$

since $(\lambda_1 + (1-\varrho)\lambda_2 + \lambda_2\varrho T) < 1$, the operator $\mathcal{F}_1$ is a contraction on $\mathcal{X}$. From Theorem 2.1 we deduce that the problem (1.1) has a unique solution. □

Theorem 3.2 *Assume that hypotheses* $(\mathcal{A}_1) - (\mathcal{A}_4)$ *hold and*

$$\lambda_1 + (1-\varrho)\lambda_2 < 1$$

then the solution of fractional boundary value problem (1.1) is in

$$\mathcal{C}_{\kappa,\omega} = \{x \in \mathcal{X} : \ \| x \| \leq \kappa e^{\omega T}\}.$$

where

$$\kappa = \frac{2\Psi_0 + (1 + \varrho(T-1))\Phi_0}{1 + \varrho\lambda_2 - (\lambda_1 + \lambda_2)}, \quad \omega = \frac{\varrho\lambda_2}{1 + \varrho\lambda_2 - (\lambda_1 + \lambda_2)}$$

and

$$\Psi_0 = \sup_{t \in \mathfrak{D}} \Psi(t, 0), \quad \Phi_0 = \sup_{t \in \mathfrak{D}} \Phi(t, 0).$$

Let $\mathcal{X} = C(\mathfrak{D}, \mathbb{R})$,and y the unique solution of problem (1.1), it is easy te see that $\mathcal{C}_{\kappa,\omega}$ is a closed and convex subset of the Banach space $\mathcal{X}$. Now, we show that y is in $\mathcal{C}_{\kappa,\omega}$.
From lemma (2.1) we have

$$\begin{aligned}
&| y(t) | = |\mathcal{F}_1 y(t)| \\
&|y(t)| = | \Psi(t, y(t)) - \Psi(0,0) + (1-\varrho)\Phi(t, y(t)) + \varrho \int_0^t \Phi(s, y(s)) ds |, \\
&|y(t)| = | \Psi(t, y(t)) - \Psi(t,0) + \Psi(t,0) - \Psi(0,0) + (1-\varrho)(\Phi(t, y(t)) - \Phi(t,0)) \\
&\quad + (1-\varrho)\Phi(t,0) + \varrho \int_0^t \Phi(s, y(s)) - \Phi(s,0) ds + \varrho \int_0^t \Phi(s,0) ds |, \\
&|y(t)| \leq | \Psi(t, y(t)) - \Psi(t,0) | + | \Psi(t,0) | + | \Psi(0,0) | + (1-\varrho) | (\Phi(t, y(t)) - \Phi(t,0)) | \\
&\quad + (1-\varrho) | \Phi(t,0) | + \varrho | \int_0^t | \Phi(s, y(s)) - \Phi(s,0) | ds + \varrho \int_0^t | \Phi(s,0) | ds,
\end{aligned}$$

by using the assymtions $(\mathcal{A}_1) - (\mathcal{A}_4)$,we find

$$|y(t)| \leq \lambda_1 | y(t) | + 2\Psi_0 + (1-\varrho)\lambda_2 | y(t) | + (1-\varrho)\Phi_0 + \varrho\lambda_2 \int_0^t | y(s) | ds + \varrho T \Phi_0,$$

$$(1 - \lambda_1 - (1-\varrho)\lambda_2) |y(t)| \leq 2\Psi_0 + (1-\varrho)\Phi_0 + \varrho T \Phi_0 + \varrho\lambda_2 \int_0^t | y(s) | ds,$$

from lemma (2.1) (Gronwall's inequality), and the fact that
$\lambda_1 + (1-\varrho)\lambda_2 < 1$, we have $|y(t)| \leq \kappa e^{\omega T}$ where

$$\kappa = \frac{2\Psi_0 + (1 + \varrho(T-1))\Phi_0}{1 + \varrho\lambda_2 - (\lambda_1 + \lambda_2)}, \quad \omega = \frac{\varrho\lambda_2}{1 + \varrho\lambda_2 - (\lambda_1 + \lambda_2)}$$

as a result $y \in \mathcal{C}_{\kappa,\omega}$. This completes the proof of theorem 3.2.

4. An illustrative example

In order to illustrate the main results obtained above,we give a nontrivial example in this section. For this purpose we consider the periodic fractional boundary value problem below:

$$\begin{cases} {}^{CF}D^{\varrho}_{at}\left(y(t)-\Psi(t,y(t))\right)=\Phi(t,y(t)), \qquad t\in\mathfrak{D}=[a,T], \\ x(a)=x(T)=0. \end{cases} \tag{4.1}$$

Where $\varrho=\frac{1}{2},\;\; a=0,\;\; T=1, \quad \Phi(t,y(t))=(\dfrac{1}{4}t-\dfrac{1}{9})sin(y(t))$ and $\Psi(t,y(t))=\dfrac{1}{5}\sqrt{y^2(t)+2}$.
It is clear that Ψ,Φ are the continous functions, and the assumption $(\mathcal{A}_1)$ is satisfied:

$$\Psi(1,0)-\Psi(0,0)+(1-\varrho)\Phi(1,0)+\varrho\int_0^t\Phi(s,x(s))ds.=0$$

.

It remains to verify the assumption $(\mathcal{A}_3)$ and $(\mathcal{A}_4)$. Let $t\in\mathfrak{D}$ and $x\in\mathbb{R}$, then we have :

$$|\Psi(t,x(t))-\Psi(t,y(t))|=\left|\frac{1}{5}\sqrt{x^2(t)+2}-\frac{1}{5}\sqrt{y^2(t)+2}\right|,$$
$$|\Psi(t,x(t))-\Psi(t,y(t))|\leq\frac{1}{5}\left|x(t)-y(t)\right|\frac{|x(t)+y(t)|}{\sqrt{x^2(t)+2}+\sqrt{y^2(t)+2}},$$
$$|\Psi(t,x(t))-\Psi(t,y(t))|\leq\frac{1}{5}\left|x(t)-y(t)\right|,$$

also:

$$|\Phi(t,x(t))-\Phi(t,y(t))|=|\;(\frac{1}{4}t-\frac{1}{9})sin(x(t))-(\frac{1}{4}t-\frac{1}{9})sin(y(t))\;|,$$
$$|\Phi(t,x(t))-\Phi(t,y(t))|\leq|\;\frac{1}{4}t-\frac{1}{9}\;|\;|sin(x(t))-sin(y(t))|\,,$$
$$|\Phi(t,x(t))-\Phi(t,y(t))|\leq\frac{5}{36}\left|x(t)-y(t)\right|,$$

thus, the assumptions $(\mathcal{A}_3)$ and $(\mathcal{A}_4)$ are holds true with $\lambda_1=\frac{1}{5}$ and $\lambda_2=\frac{5}{36}$. Finally, the periodic fractional hybrid problem (4.1) has a unique solution specified on $[0,1]$ since all of the conditions of Theorem 3.1 are met.

5. Conclusion

Using the Caputo Fabrizio fractional derivative of order $\varrho\in(0,1)$, we defined solutions for the periodic fractional hybrid boundary value problem in the current paper. Furthermore, the existence and uniqueness of a solution to this problem are demonstrated using the Banach fixed point theorem. Finally, a nontrivial example is presented as an application to demonstrate our theoretical results.

Acknowledgments

The authors are thankful to the referee for her/his valuable suggestions towards the improvement of the paper.

References

1. R.P. Agarwal, Y. Zhou, J. Wang, X. Luo, *Fractional functional differential equations with causal operators in Banach spaces.* Mathematical and Compututer Modelling,1440-1452,(2011).
2. A.El Mfadel, S.Melliani, *Others Existence results for anti-periodic fractional coupled systems with p- Laplacian operator via measure of noncompactness in Banach spaces*, Journal Of Mathematical Sciences, 1-14, (2023).
3. A.El Mfadel, S.Melliani, M. Elomari, *Existence of anti-periodic solutions for ψ-Caputo-type fractional p-Laplacian problems via Leray–Schauder degree theory*,Analysis, (2023).

4. R.P. Agarwal, S.K. Ntouyas, B. Ahmad and A.K. Alzahrani,*Hadamard-type fractional functional differential equations and inclusions with retarded and advanced arguments*,Advances in Difference Equations ,1-15,(2016) .
5. F. Mainardi, *Fractals and Fractional Calculus Continuum Mechanics* Springer Verlag, (1997).
6. R.P. Agarwal, S. Hristova, D. O'Regan,*A survey of Lyapunov functions, stability and impulsive Caputo fractional differential equations*, Fract. Calc. Appl. Anal,290-318 , (2016) .
7. K. Diethelm,*The Analysis of Fractional Differential Equations* ,Springer-Verlag, Berlin, (2010).
8. M. R. Ammi, E. El Kinani, and D. F. Torres, *Existence and uniqueness of solutions to functional integro-differential fractional equations*, Electronic Journal of Differential Equations, 1-9,(2012).
9. R. Almeida,*Caputo fractional derivative of a function with respect to another function*,Commun. Nonlinear Sci. Numer. Simul,460-481, (2017).
10. R. Almeida, A.B. Malinowska, M.T.T. Monteiro. *Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications*, Mathematical Methods in the Applied Sciences, 336-352,(2018).
11. D. Baleanu, H. Jafari, H. Khan, S.J Johnston. *Results for mild solution of fractional coupled hybrid boundary value problems*, Open Math,601-608, (2015) .
12. D. Baleanu, H. Khan, H. Jafari, R.A. Alipour, *On existence results for solutions of a coupled system of hybrid boundary value problems with hybrid conditions*, Advances in Difference Equations ,1-14,(2015).
13. A. Belarbi, M. Benchohra, A. Ouahab. *Uniqueness results for fractional functional differential equations with infinite delay in Frechet spaces*, Appl. Anal, 1459-1470 ,(2006).
14. T.A. Burton, *A fixed point theorem of Krasnoselskii*, Appl. Math. Lett,85-88,(1998) .
15. D.Kumar, J. Singh, D. Baleanu,*Numerical computation of a fractional model of differential-difference equation*, Journal Of Computational And Nonlinear Dynamics, (2016).
16. M. Caputo,*Linear models of dissipation whose Q is almost frequency independent*, International Journal of Geographical InformationScience, 529-539 ,(1967).
17. J.Losada, J. Nieto,*Properties of a new fractional derivative without singular kernel*,Progr.Fract. Differ. Appl, 87-92, (2015)
18. M.Caputo, M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progress In Fractional Differentiation and Applications, 73-85,(2015).
19. M.Ma, D.Baleanu, Y. Gasimov,X.Yang, *New results for multidimensional diffusion equations in fractal dimensional space*, Rom. J. Phys, 784-794 ,(2016).
20. B.C. Dhage,*Basic results in the theory of hybrid differential equations with mixed perturbations of second type, Funct*, Differ. Equ, 87-106, (2012).
21. B.C. Dhage, V. Lakshmikantham,*Basic results on hybrid differential equations*, Nonlinear Anal. Hybrid, 414-424,(2010).
22. A. El Mfadel, S. Melliani, and M. Elomari,*Existence and uniqueness results for Caputo fractional boundary value problems involving the p-Laplacian operator* U.P.B. Sci. Bull. Series A, 37-46,(2022).
23. A. El Mfadel, S. Melliani, M. Elomari,*New existence results for nonlinear functional hybrid differential equations involving the $\psi-$ Caputo fractional derivative*, Results in Nonlinear Analysis, 78-86,(2022).
24. A. El Mfadel, S. Melliani, M. Elomari,*Existence results for nonlocal Cauchy problem of nonlinear $\psi-$Caputo type fractional differential equations via topological degree methods*, Advances in the Theory of Nonlinear Analysis and its Application, (2022).
25. A. El Mfadel, S. Melliani, M. Elomari, *On the Existence and Uniqueness Results for Fuzzy Linear and Semilinear Fractional Evolution Equations Involving Caputo Fractional Derivative*, Journal of Function Spaces, (2021).
26. A. El Mfadel, S. Melliani, M. Elomari,*A Note on the Stability Analysis of Fuzzy Nonlinear Fractional Differential Equations Involving the Caputo Fractional Derivative*, International Journal of Mathematics and Mathematical Sciences,(2021).
27. A. El Mfadel, S. Melliani, M. Elomari,*Notes on Local and Nonlocal Intuitionistic Fuzzy Fractional Boundary Value Problems with Caputo Fractional Derivatives*, Journal of Mathematics, (2021).
28. J. W. Green, F. A. Valentine,*On the arzela-ascoli theorem*, Mathematics Magazine,199-202, (1961) .
29. N.Derdar,*Nonlinear implicit Caputo-Fabrizio fractional hybrid differential equations*, International Journal Of Nonlinear Analysis And Applications, (2023).
30. T.L. Guo, M. Jiang,*Impulsive fractional functional differential equations*, Comput.Math.Appl,3414-3424, (2012).
31. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical studies ,204,(2006) .
32. R. Hilfer, *Applications of Fractional Calculus in Physics* ,Singapore, (2000).

33. H. Lu, S. Sun, D. Yang, H. Teng, *Theory of fractional hybrid differential equations with linear perturbations of second type*, Boundary Value Problems, 1-16,(2013).

34. Y. Luchko, J.J. Trujillo, *Caputo-type modification of the Erdelyi-Kober fractional derivative*, Fract. Calc. Appl. Anal, 249-267,(2007).

35. I. Podlubny, *Fractional Differential Equations* ,Mathematics in Science and Engineering, Academic Press, (1999).

36. H.A. Wahash, S.K. Panchal, M.S. Abdo, *Existence and stability of a nonlinear fractional differential equation involving a ψ-Caputo operator*, Advances in the Theory of Nonlinear Analysis and its Application,266-278,(2020).

37. Y. Zhao, S. Sun, Z. Han, Q. Li, *Theory of fractional hybrid differential equations*, Computers and Mathematics with Applications,1312-1324,(2011) .

38. G.Nchama, *Properties of Caputo-Fabrizio fractional operators*, New Trends In Mathematical Sciences, 1-25,(2020).

39. S.Dragomir,M. City, *Some Gronwall type inequalities and applications*,(2002).

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