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3-Rotational Hypersurface Satisfying $\Delta^{IV}\mathbf{x} = \mathcal{A}\mathbf{x}$ in $\mathbb{E}^6$

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ABSTRACT: We introduce the tri-rotational hypersurface $x(u,v,w,s,t)$ in six dimensional Euclidean space $\mathbb{E}^6$. We compute the curvatures of $\mathbf{x}$. In addition, we obtain the Laplace−Beltrami operator depends on the fourth fundamental form, and find $\Delta^{IV}\mathbf{x} = \mathcal{A}\mathbf{x}$ for a 6×6 matrix $\mathcal{A}$ in $\mathbb{E}^6$.

Keywords: Euclidean spaces, six space, 3-rotational hypersurface, Gauss map, curvature, fourth Laplace−Beltrami operator.

Contents

1. Introduction

Chen [13,14,15,16] introduced the finite type submanifolds immersed into $\mathbb{E}^m$ or $\mathbb{E}^m_\nu$ by using a finite number of eigenfunctions of their Laplacian. Then the topic has been spread until today.

Takahashi [54] gave a connected Euclidean submanifold is of 1-type, iff it is minimal or minimal in some hypersphere of $\mathbb{E}^m$. 2-type spherical closed submanifolds were studied by [9,10,14]. Garay worked [29] Takahashi's theorem in $\mathbb{E}^m$. Cheng and Yau considered the hypersurfaces having constant scalar curvature; Chen and Piccinni [18] focused the submanifolds having finite type Gauss map in $\mathbb{E}^m$. Dursun [24] introduced the hypersurfaces having the pointwise 1-type Gauss map in $\mathbb{E}^{n+1}$. Aminov [2] presented the geometry of submanifolds.

In $\mathbb{E}^3$; Takahashi [54] worked the minimal surfaces and spheres are the only surfaces having $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$; Ferrandez, Garay, and Lucas [26] gave the surfaces $\Delta H = AH$, $A \in Mat(3,3)$ are the minimal, or open part of a sphere, or of a right circular cylinder; Choi and Kim [21] considered the minimal helicoid having pointwise 1-type Gauss map of the first kind; Garay [28] obtained a class of finite type surfaces of revolution; Dillen, Pas and Verstraelen [22] worked the only surfaces having $\Delta r = Ar + B$, $A \in Mat(3,3)$, $B \in Mat(3,1)$ are the minimal surfaces, the spheres, and the circular cylinders; Stamatakis and Zoubi [53] gave the surfaces of revolution having $\Delta^{III}x = Ax$; Senoussi and Bekkar [52] introduced the helicoidal surfaces having $\Delta^J r = Ar$, $J = I, II, III$, where $A \in Mat(3,3)$; Kim, Kim, and Kim [41] focused the Cheng−Yau operator and the Gauss map of surfaces of revolution.

In $\mathbb{E}^4$; Moore [49,50] studied the general rotational surfaces; Hasanis and Vlachos [38] considered the hypersurfaces with harmonic mean curvature vector field; Cheng and Wan [19] worked the complete hypersurfaces with CMC; Kim and Turgay [42] gave the surfaces having L_1-pointwise 1-type Gauss map; Arslan et al [3] worked the Vranceanu surface having pointwise 1-type Gauss map; Arslan et al [4] considered the generalized rotational surfaces; Arslan et al [5] introduced the tensor product surfaces having pointwise 1-type Gauss map; Yoon [56] served some relations of the Clifford torus; Özkaldı and Yaylı [51] studied the tensor product surfaces in $\mathbb{R}^4$ with Lie groups; Kahraman Aksoyak and Yaylı [39] introduced the flat rotational surfaces having pointwise 1-type Gauss map; Güler, Magid, and Yaylı [33] studied the helicoidal hypersurfaces; Güler, Hacısalihoğlu, and Kim [32] studied Gauss map and

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the third Laplace-Beltrami operator of the rotational hypersurface; Güler and Turgay [34] examined the Cheng−Yau operator and Gauss map of the rotational hypersurfaces; Güler [31] worked the rotational hypersurfaces having $\Delta^I R = AR$, where $A \in Mat(4,4)$. He [30] also obtained the fundamental form IV and the curvature formulas of the hypersphere.

In Minkowski 4-space $\mathbb{E}_1^4$; Ganchev and Milousheva [27] indicated the analogue of the surfaces of [49,50]; Arvanitoyeorgos, Kaimakamais, and Magid [8] studied the mean curvature vector field of M_1^3 supplying $\Delta H = \alpha H$ (α a constant); Arslan and Milousheva [6] considered the meridian surfaces of elliptic or hyperbolic type having pointwise 1-type Gauss map; Turgay [55] worked the Lorentzian surfaces having finite type Gauss map; Dursun and Turgay [25] introduced the space-like surfaces having pointwise 1-type Gauss map. Kahraman Aksoyak and Yaylı [40] obtained the general rotational surfaces having pointwise 1-type Gauss map in $\mathbb{E}_2^4$. Bektaş, Canfes, and Dursun [11] considered the surfaces in a pseudo-sphere having 2-type pseudo-spherical Gauss map in $\mathbb{E}_2^5$. They [12] also obtained the pseudo-spherical submanifolds having 1-type pseudo-spherical Gauss map.

Arslan, Sütveren, and Bulca [7] worked the rotational λ-hypersurfaces in Euclidean spaces. Güler, Yaylı, and Hacısalihoğlu [35] introduced the bi-rotational hypersurface in $\mathbb{E}^4$. See [17,45,46,47,48] for the further works.

We consider the Laplace−Beltrami operator depends on the fourth fundamental form of the 3-rotational hypersurface in six dimensional Euclidean space $\mathbb{E}^6$. In Section 2, we give the notions of the six dimensional Euclidean geometry. In Section 3, we serve the curvature formulas of a hypersurface in $\mathbb{E}^6$. In Section 4, we present the 3-rotational hypersurface. Moreover, we obtain the 3-rotational hypersurface supplying $\Delta^{IV}\mathbf{x} = \mathcal{A}\mathbf{x}$ for some 6×6 matrix $\mathcal{A}$ of $\mathbb{E}^6$ in the last section.

2. Preliminaries

Presenting some facts and definitions, we indicate some notations used whole paper. Assume $\mathbb{E}^m$ be the Euclidean m-space with the canonical Euclidean metric tensor given by $\widetilde{g} = \langle\ ,\ \rangle = \sum_{i=1}^{m} dx_i^2$, where $(x_1, x_2, \ldots, x_m)$ is a coordinate system in $\mathbb{E}^m$. Considering an m-dimensional Riemannian submanifold of the space $\mathbb{E}^m$, we show the Levi–Civita connections [44] of the manifold $\widetilde{M}$, and its submanifold M of $\mathbb{E}^m$ by $\widetilde{\nabla}$, ∇, resp. We shall use letters X, Y, Z, W (resp., ξ, η) to denote vectors fields tangent (resp., normal) to M.

The Gauss and Weingarten formulas are presented, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \tag{2.2}$$

where h, D and A are the second fundamental form, the normal connection and the shape operator of M, respectively.

The shape operator A_ξ is a symmetric endomorphism of the tangent space T_pM at $p \in M$ for each $\xi \in T_p^\perp M$. The shape operator and the second fundamental form are related by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle .$$

The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X,Y,)Z,W \rangle = \langle h(Y,Z), h(X,W) \rangle - \langle h(X,Z), h(Y,W) \rangle, \tag{2.3}$$

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{2.4}$$

where R, R^D are the curvature tensors associated with connections ∇ and D, respectively, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Assume M be an oriented hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, $\mathbf{S}$ its shape operator, x its position vector. We consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of consisting of principal directions of M corresponding from the principal curvature k_i for $i = 1, 2, \ldots, n$. Let the dual basis of this frame field be $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Then the first structural equation of Cartan is

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i, j = 1, 2, \ldots, n, \tag{2.5}$$

where ω_{ij} denotes the connection forms corresponding to the chosen frame field. Then, from the Codazzi equation (2.3), we obtain

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j), \tag{2.6}$$

$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l) \tag{2.7}$$

for distinct $i, j, l = 1, 2, \ldots, n$.

We put $s_j = \sigma_j(k_1, k_2, \ldots, k_n)$, where σ_j is the j-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} a_{i_1} a_{i_2} \ldots a_{i_j}.$$

We use the following notation

$$r_i^j = \sigma_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_n).$$

By the definition, we get $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. We call the function s_k as the k-th mean curvature of M. Note that the functions $H = \frac{1}{n} s_1$ and $K = s_n$ are called the mean curvature and the Gauss$-$Kronecker curvature of M, respectively. Particularly, M is called the j-minimal if $s_j \equiv 0$.

In $\mathbb{E}^{n+1}$, the i-th curvature formulas $\mathfrak{C}_i$ (See [1] and [43] for details), where $i = 0, 1, \ldots, n$, are derived using the characteristic polynomial of $\mathbf{S}$:

$$\sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k} = P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda \mathcal{I}_n) = 0, \tag{2.8}$$

$\mathcal{I}_n$ indicates the identity matrix of order n. Hence, we reveal the curvature formulas $\binom{n}{i}\mathfrak{C}_i = s_i$. Here, $\binom{n}{0}\mathfrak{C}_0 = s_0 = 1$ (by definition), $\binom{n}{1}\mathfrak{C}_1 = s_1, \ldots, \binom{n}{n}\mathfrak{C}_n = s_n = K$.

For a Euclidean submanifold x: $M \longrightarrow \mathbb{E}^m$, the immersion (M, x) is called the *finite type,* if x can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of (M, x), i.e. $x = x_0 + \sum_{i=1}^{k} x_i$, where x_0 is a constant map, $x_1, \ldots, x_k$ non-constant maps, and $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. If λ_i are different, M is called the *k-type*. See [14] for details.

Let $\mathbf{x} = \mathbf{x}(u, v, w, s, t)$ be an immersion from $M^5 \subset \mathbb{E}^5$ to $\mathbb{E}^6$. A Euclidean dot product of $\overrightarrow{x} = (x_1, ..., x_6)$ and $\overrightarrow{y} = (y_1, ..., y_6)$ of $\mathbb{E}^6$ is given by $\overrightarrow{x} \cdot \overrightarrow{y} = \sum_{i=1}^{6} x_i y_i$. A quintet vector product of $\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}, \overrightarrow{p}, \overrightarrow{q}$ of $\mathbb{E}^6$ is defined by

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} \times \overrightarrow{p} \times \overrightarrow{q} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{pmatrix},$$

where e_i, $i = 1, ..., 6$, are the unit vectors of $\mathbb{E}^6$. For a hypersurface $\mathbf{x}$ in 6-space, we have $(\mathbf{g}_{ij})_{5\times5}$, $(\mathbf{h}_{ij})_{5\times5}$, $(\mathbf{t}_{ij})_{5\times5}$, $(\mathbf{f}_{ij})_{5\times5}$, where $(\mathbf{g}_{ij})$,$(\mathbf{h}_{ij})$,$(\mathbf{t}_{ij})$,$(\mathbf{f}_{ij})$ are the first, second, third and the fourth fundamental form matrices, respectively, and $\mathbf{g}_{ij} = \mathbf{x}_i \cdot \mathbf{x}_j$, $\mathbf{h}_{ij} = \mathbf{x}_{ij} \cdot G$, $\mathbf{t}_{ij} = G_i \cdot G_j$, $\mathbf{f}_{ij} = \mathbf{t}_{ij}\mathbf{S}$, $i, j = 1, 2, ..., 5$, $\mathbf{x}_i = \mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$ when $i = 1$, $\mathbf{x}_{ij} = \mathbf{x}_{uv} = \frac{\partial^2 \mathbf{x}}{\partial u \partial v}$ when $i = 1$ and $j = 2$, and

$$G = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w \times \mathbf{x}_s \times \mathbf{x}_t}{\|\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w \times \mathbf{x}_s \times \mathbf{x}_t\|} \tag{2.9}$$

is the Gauss map of the hypersurface $\mathbf{x}$.

See [32,33,34,35,36,37] for details.

3. i-th Curvatures

In this section, we reveal the curvature formulas of any hypersurface $\mathbf{x} = \mathbf{x}(u, v, w, s, t)$ *in* $\mathbb{E}^6$.

Theorem 3.1 *A hypersurface* $\mathbf{x}$ *in* $\mathbb{E}^6$ *has the following curvature formulas,* $\mathfrak{C}_0 = 1$ *by definition,*

$$5\mathfrak{C}_1 = -\frac{\mathfrak{b}}{\mathfrak{a}},\ 10\mathfrak{C}_2 = \frac{\mathfrak{c}}{\mathfrak{a}},\ 10\mathfrak{C}_3 = -\frac{\mathfrak{d}}{\mathfrak{a}},\ 5\mathfrak{C}_4 = \frac{\mathfrak{e}}{\mathfrak{a}},\ \mathfrak{C}_5 = -\frac{\mathfrak{f}}{\mathfrak{a}}, \tag{3.1}$$

where $P_{\mathbf{S}}(\lambda) = \mathfrak{a}\lambda^5 + \mathfrak{b}\lambda^4 + \mathfrak{c}\lambda^3 + \mathfrak{d}\lambda^2 + \mathfrak{e}\lambda + \mathfrak{f} = 0$ *is the characteristic polynomial of shape operator matrix* $\mathbf{S}$*,* $\mathfrak{a} = \det(\mathbf{g}_{ij})$*,* $\mathfrak{f} = \det(\mathbf{h}_{ij})$*, and* $(\mathbf{g}_{ij})$ *and* $(\mathbf{h}_{ij})$ *are the first, and the second fundamental form matrices, respectively.*

Proof: The solution matrix $(\mathbf{g}_{ij})^{-1} \cdot (\mathbf{h}_{ij})$ supplies the the shape operator matrix $\mathbf{S}$ of hypersurface $\mathbf{x}$ in 6-space. In $\mathbb{E}^6$, computing the i-th mean curvature formula $\mathfrak{C}_i$, where $i = 0, 1, 2, 3, 4, 5$, we reveal the characteristic polynomial $P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda\mathcal{I}_5) = \mathfrak{a}\lambda^5 + \mathfrak{b}\lambda^4 + \mathfrak{c}\lambda^3 + \mathfrak{d}\lambda^2 + \mathfrak{e}\lambda + \mathfrak{f} = 0$ of $\mathbf{S}$. Then, we find the i-th curvature folmulas in 6-space. We obtain the following

$$\begin{aligned}
\binom{5}{0}\mathfrak{C}_0 &= 1, \\
\binom{5}{1}\mathfrak{C}_1 &= k_1 + k_2 + k_3 + k_4 + k_5 \\
&= -\frac{\mathfrak{b}}{\mathfrak{a}}, \\
\binom{5}{2}\mathfrak{C}_2 &= k_1k_2 + k_1k_3 + k_1k_4 + k_1k_5 + k_2k_3 + k_2k_4 + k_2k_5 + k_3k_4 + k_3k_5 + k_4k_5 \\
&= \frac{\mathfrak{c}}{\mathfrak{a}}, \\
\binom{5}{3}\mathfrak{C}_3 &= k_1k_2k_3 + k_1k_2k_4 + k_1k_2k_5 + k_1k_3k_4 + k_1k_3k_5 + k_1k_4k_5 + k_2k_3k_4 \\
&\quad + k_2k_3k_5 + k_2k_4k_5 + k_3k_4k_5 \\
&= -\frac{\mathfrak{d}}{\mathfrak{a}}, \\
\binom{5}{4}\mathfrak{C}_4 &= k_1k_2k_3k_4 + k_1k_2k_4k_5 + k_1k_2k_3k_5 + k_1k_3k_4k_5 + k_2k_3k_4k_5 \\
&= \frac{\mathfrak{e}}{\mathfrak{a}}, \\
\binom{5}{5}\mathfrak{C}_5 &= k_1k_2k_3k_4k_5 \\
&= -\frac{\mathfrak{f}}{\mathfrak{a}}.
\end{aligned}$$

□

See [30] for the case $\mathbb{E}^4$.

Theorem 3.2 *A hypersurface* $\mathbf{x} = \mathbf{x}(u, v, w, s, t)$ *in* $\mathbb{E}^6$ *has the following relation*

$$\mathfrak{C}_0 VI - 5\mathfrak{C}_1 V + 10\mathfrak{C}_2 IV - 10\mathfrak{C}_3 III + 5\mathfrak{C}_4 II - \mathfrak{C}_5 I = 0,$$

where I, II, III, IV, V, VI *are the fundamental form matrices having order* 5×5 *of the hypersurface.*

Proof: Considering $n = 5$ in (2.8), it is clear. □

4. 3-Rotational Hypersurface

In this section, we define a rotational hypersurface, then find its differential geometric properties in Euclidean 6-space $\mathbb{E}^6$. Note that the definition of rotational hypersurfaces in Riemannian space forms were given in [23]. A rotational hypersurface $M \subset \mathbb{E}^{n+1}$ constructed by a surface γ around an axis γ that does not meet γ is obtained by taking the orbit of γ under those orthogonal transformations of $\mathbb{E}^{n+1}$ leaves $\mathfrak{r}$ pointwise fixed (See [23, Remark 2.3]).

We use the profile surface $\gamma(u,v) = (f(u,v), 0, g(u,v), 0, h(u,v), 0)$ with the following rotation matrix

$$\begin{pmatrix} \cos w & -\sin w & 0 & 0 & 0 & 0 \\ \sin w & \cos w & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos s & -\sin s & 0 & 0 \\ 0 & 0 & \sin s & \cos s & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos t & -\sin t \\ 0 & 0 & 0 & 0 & \sin t & \cos t \end{pmatrix},$$

and then, give the following.

Definition 4.1 *A 3-rotational hypersurface in $\mathbb{E}^6$ is defined by*

$$\mathbf{x}(u,v,w,s,t) = \begin{pmatrix} f(u,v)\cos w \\ f(u,v)\sin w \\ g(u,v)\cos s \\ g(u,v)\sin s \\ h(u,v)\cos t \\ h(u,v)\sin t \end{pmatrix}, \tag{4.1}$$

where f, g, h are the differentiable functions, $u, v \in \mathbb{R}$, and $0 \leq w, s, t < 2\pi$.

Considering the first derivatives of (4.1) depends on u, v, w, s, t, respectively, we find the first quantities of (4.1)

$$(\mathbf{g}_{ij}) = \begin{pmatrix} \mathbf{g}_{11} & \mathbf{g}_{12} & 0 & 0 & 0 \\ \mathbf{g}_{21} & \mathbf{g}_{22} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{g}_{33} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{g}_{44} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{g}_{55} \end{pmatrix}, \tag{4.2}$$

where

$$\begin{aligned} \mathbf{g}_{11} &= f_u^2 + g_u^2 + h_u^2, \\ \mathbf{g}_{12} &= f_u f_v + g_u g_v + h_u h_v = \mathbf{g}_{21}, \\ \mathbf{g}_{22} &= f_v^2 + g_v^2 + h_v^2, \\ \mathbf{g}_{33} &= f^2, \ \mathbf{g}_{44} = g^2, \ \mathbf{g}_{55} = h^2, \end{aligned}$$

and $f_u = \frac{\partial f}{\partial u}$, $f_v = \frac{\partial f}{\partial v}$. Here,

$$\mathbf{g} = \det(\mathbf{g}_{ij}) = f^2 g^2 h^2 \left(\left(f_u^2 + g_u^2 + h_u^2\right)\left(f_v^2 + g_v^2 + h_v^2\right) - \left(f_u f_v + g_u g_v + h_u h_v\right)^2 \right).$$

Using (2.9) , we obtain the Gauss map of the 3-rotational hypersurface (4.1):

$$G = \frac{1}{W^{1/2}} \begin{pmatrix} (h_u g_v - g_u h_v)\cos w \\ (h_u g_v - g_u h_v)\sin w \\ (f_u h_v - h_u f_v)\cos s \\ (f_u h_v - h_u f_v)\sin s \\ (g_u f_v - f_u g_v)\cos t \\ (g_u f_v - f_u g_v)\sin t \end{pmatrix}, \tag{4.3}$$

where

$$W = \left(f_u^2 + g_u^2 + h_u^2\right)\left(f_v^2 + g_v^2 + h_v^2\right) - \left(f_u f_v + g_u g_v + h_u h_v\right)^2 .$$

With the help of the second derivatives of the 3-rotational hypersurface(4.1) depends on u, v, w, s, t, and the Gauss map (4.3) of the (4.1), we reveal the following second quantities

$$(\mathbf{h}_{ij}) = \begin{pmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} & 0 & 0 & 0 \\ \mathbf{h}_{21} & \mathbf{h}_{22} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{h}_{33} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{h}_{44} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{h}_{55} \end{pmatrix}, \tag{4.4}$$

where

$$\begin{aligned}
\mathbf{h}_{11} &= \frac{(h_u g_v - g_u h_v)\, f_{uu} + (f_u h_v - h_u f_v) g_{uu} + (g_u f_v - f_u g_v) h_{uu}}{W^{1/2}}, \\
\mathbf{h}_{12} &= \frac{(h_u g_v - g_u h_v)\, f_{uv} + (f_u h_v - h_u f_v) g_{uv} + (g_u f_v - f_u g_v) h_{uv}}{W^{1/2}} = \mathbf{h}_{21}, \\
\mathbf{h}_{22} &= \frac{(h_u g_v - g_u h_v)\, f_{vv} + (f_u h_v - h_u f_v) g_{vv} + (g_u f_v - f_u g_v) h_{vv}}{W^{1/2}} \\
\mathbf{h}_{33} &= \frac{(g_u h_v - h_u g_v)\, f}{W^{1/2}}, \\
\mathbf{h}_{44} &= \frac{(h_u f_v - f_u h_v)\, g}{W^{1/2}}, \\
\mathbf{h}_{55} &= \frac{(f_u g_v - g_u f_v)\, h}{W^{1/2}}.
\end{aligned}$$

So, we get

$$\mathbf{h} = \det(\mathbf{h}_{ij}) = \frac{fgh\,(f_u g_v - g_u f_v)(h_u f_v - f_u h_v)(g_u h_v - h_u g_v)\left(\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{12}^2\right)}{W^{3/2}}.$$

By using (4.2) and (4.4), we compute the following shape operator matrix of (4.1):

$$\mathbf{S} = \begin{pmatrix} \frac{\mathbf{g}_{22}\mathbf{h}_{11} - \mathbf{g}_{12}\mathbf{h}_{12}}{\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2} & \frac{\mathbf{g}_{22}\mathbf{h}_{12} - \mathbf{g}_{12}\mathbf{h}_{22}}{\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2} & 0 & 0 & 0 \\ \frac{\mathbf{g}_{11}\mathbf{h}_{12} - \mathbf{g}_{12}\mathbf{h}_{11}}{\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2} & \frac{\mathbf{g}_{11}\mathbf{h}_{22} - \mathbf{g}_{12}\mathbf{h}_{12}}{\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathbf{h}_{33}}{\mathbf{g}_{33}} & 0 & 0 \\ 0 & 0 & 0 & \frac{\mathbf{h}_{44}}{\mathbf{g}_{44}} & 0 \\ 0 & 0 & 0 & 0 & \frac{\mathbf{h}_{55}}{\mathbf{g}_{55}} \end{pmatrix}.$$

Finally, using (3.1), with (4.2), (4.4), respectively, we find the curvatures of the 3-rotational hypersurface (4.1) as follows.

Theorem 4.1 *Let* $\mathbf{x} : M^5 \longrightarrow \mathbb{E}^6$ *be an immersion given by* (4.1). *Then,* $\mathbf{x}$ *has the following principal curvatures*

$$\begin{aligned}
k_1 &= \frac{-\Omega^{1/2} + \mathbf{g}_{11}\mathbf{h}_{22} - 2\mathbf{g}_{12}\mathbf{h}_{12} + \mathbf{g}_{22}\mathbf{h}_{11}}{2W}, \\
k_2 &= \frac{\Omega^{1/2} + \mathbf{g}_{11}\mathbf{h}_{22} - 2\mathbf{g}_{12}\mathbf{h}_{12} + \mathbf{g}_{22}\mathbf{h}_{11}}{2W}, \\
k_3 &= \frac{g_u h_v - h_u g_v}{f\ W^{1/2}}, \\
k_4 &= \frac{h_u f_v - f_u h_v}{g\ W^{1/2}}, \\
k_5 &= \frac{f_u g_v - g_u f_v}{h\ W^{1/2}},
\end{aligned}$$

where

$$\Omega = \left(\mathbf{g}_{11}\mathbf{h}_{22} - 2\mathbf{g}_{12}\mathbf{h}_{12} + \mathbf{g}_{22}\mathbf{h}_{11}\right)^2 - 4W\left(\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{12}^2\right),$$

$$W = \mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2,$$

$$\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{12}^2 = \frac{1}{W}\left\{\begin{array}{c} \left(g_u h_v - g_v h_u\right)^2\left(f_{uv}^2 - f_{uu}f_{vv}\right) \\ +\left(f_u h_v - f_v h_u\right)^2\left(g_{uv}^2 - g_{uu}g_{vv}\right) \\ +\left(f_u g_v - f_v g_u\right)^2\left(h_{uv}^2 - h_{uu}h_{vv}\right) \\ -\left(f_u h_v - f_v h_u\right)\left(g_u h_v - g_v h_u\right)\left(2f_{uv}g_{uv} - f_{uu}g_{vv} - f_{vv}g_{uu}\right) \\ +\left(f_u g_v - f_v g_u\right)\left(g_u h_v - g_v h_u\right)\left(2f_{uv}h_{uv} - f_{uu}h_{vv} - f_{vv}h_{uu}\right) \\ -\left(f_u g_v - f_v g_u\right)\left(f_u h_v - f_v h_u\right)\left(2g_{uv}h_{uv} - g_{uu}h_{vv} - g_{vv}h_{uu}\right) \end{array}\right\}.$$

Proof: Solving $\det(\mathbf{S} - \lambda\mathcal{I}_5) = 0$, it is clear. □

Theorem 4.2 *Let $\mathbf{x} : M^5 \longrightarrow \mathbb{E}^6$ be an immersion given by (4.1). Then, $\mathbf{x}$ has the following curvatures*

$$\begin{aligned}
\mathfrak{C}_0 &= 1, \\
5\mathfrak{C}_1 &= \frac{\left\{\begin{array}{c} \left(f^2g^2\mathbf{h}_{55} + f^2h^2\mathbf{h}_{44} + g^2h^2\mathbf{h}_{33}\right)\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right) \\ +f^2g^2h^2\left(g_{22}\mathbf{h}_{11} - 2g_{12}\mathbf{h}_{12} + g_{11}\mathbf{h}_{22}\right) \end{array}\right\}}{f^2g^2h^2\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right)}, \\
10\mathfrak{C}_2 &= \frac{\left\{\begin{array}{c} \left(f^2\mathbf{h}_{44}\mathbf{h}_{55} + g^2\mathbf{h}_{33}\mathbf{h}_{55} + h^2\mathbf{h}_{33}\mathbf{h}_{44}\right)\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right) \\ +\left(f^2g^2\mathbf{h}_{55} + f^2h^2\mathbf{h}_{44} + g^2h^2\mathbf{h}_{33}\right)\left(\mathbf{g}_{11}\mathbf{h}_{22} - 2\mathbf{g}_{12}\mathbf{h}_{12} + \mathbf{g}_{22}\mathbf{h}_{11}\right) \\ +f^2g^2h^2\left(\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{12}^2\right) \end{array}\right\}}{2f^2g^2h^2\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right)}, \\
10\mathfrak{C}_3 &= \frac{\left\{\begin{array}{c} \left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right)\mathbf{h}_{33}\mathbf{h}_{44}\mathbf{h}_{55} \\ +\left(f^2g^2\mathbf{h}_{55} + f^2h^2\mathbf{h}_{44} + g^2h^2\mathbf{h}_{33}\right)\left(\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{12}^2\right) \\ +\left(f^2\mathbf{h}_{44}\mathbf{h}_{55} + g^2\mathbf{h}_{33}\mathbf{h}_{55} + h^2\mathbf{h}_{33}\mathbf{h}_{44}\right)\left(\mathbf{g}_{11}\mathbf{h}_{22} - 2\mathbf{g}_{12}\mathbf{h}_{12} + \mathbf{g}_{22}\mathbf{h}_{11}\right) \end{array}\right\}}{f^2g^2h^2\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right)}, \\
5\mathfrak{C}_4 &= \frac{\left\{\begin{array}{c} \left(f^2\mathbf{h}_{44}\mathbf{h}_{55} + g^2\mathbf{h}_{33}\mathbf{h}_{55} + h^2\mathbf{h}_{33}\mathbf{h}_{44}\right)\left(\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{12}^2\right) \\ +\left(g_{11}\mathbf{h}_{22} - 2g_{12}\mathbf{h}_{12} + g_{22}\mathbf{h}_{11}\right)\mathbf{h}_{33}\mathbf{h}_{44}\mathbf{h}_{55} \end{array}\right\}}{f^2g^2h^2\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right)}, \\
\mathfrak{C}_5 &= \frac{\left(\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{12}^2\right)\mathbf{h}_{33}\mathbf{h}_{44}\mathbf{h}_{55}}{f^2g^2h^2\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right)}.
\end{aligned}$$

where $f^2g^2h^2\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right) \neq 0$.

Proof: Taking into account i-th curvature folmulas with k_i, and also Theorem 4.1, it is clear. □

Hence, we serve the following examples.

Example 4.1 *Let $\mathbf{x} : M^5 \longrightarrow \mathbb{E}^6$ be an immersion given by (4.1). When the profile surface γ of $\mathbf{x}$ is parametrized by the unit sphere: $f(u,v) = \cos u \cos v$, $g(u,v) = \sin u \cos v$, $h(u,v) = \sin v$, then $\mathbf{S} = \mathcal{I}_5$ and the 3-rotational hypersurface has the following curvatures $\mathfrak{C}_i = 1$, where $i = 0, 1, ..., 5$.*

Example 4.2 *Assume $\mathbf{x} : M^5 \longrightarrow \mathbb{E}^6$ be an immersion given by (4.1). While the profile surface γ of $\mathbf{x}$ is the rational parametrized by the unit sphere: $f(u,v) = \frac{1-u^2}{1+u^2}\frac{1-v^2}{1+v^2}$, $g(u,v) = \frac{2u}{1+u^2}\frac{1-v^2}{1+v^2}$, $h(u,v) = \frac{2v}{1+v^2}$, the 3-rotational hypersurface has the same results of Example 4.1.*

Example 4.3 *Let $\mathbf{x} : M^5 \longrightarrow \mathbb{E}^6$ be an immersion given by (4.1). When the generating surface γ of $\mathbf{x}$ is parametrized by the Riemann sphere: $f(u,v) = \frac{2u}{u^2+v^2+1}$, $g(u,v) = \frac{2v}{u^2+v^2+1}$, $h(u,v) = \frac{u^2+v^2-1}{u^2+v^2+1}$, the 3-rotational hypersurface has $\mathbf{S} = -\mathcal{I}_5$, and has the following curvatures $\mathfrak{C}_i = (-1)^i$, where $i = 0, 1, ..., 5$.*

Example 4.4 *Considering the hypersphere* $S^5(r) := \left\{\xi \in \mathbb{E}^6 \mid \langle \xi, \xi \rangle = r^2\right\}$ *(for radius* $r > 0$*) as*

$$\xi(u,v,w,s,t) = \begin{pmatrix} r\cos u \cos v \cos w \\ r \cos u \cos v \sin w \\ r \sin u \cos v \cos s \\ r \sin u \cos v \sin s \\ r \sin v \cos t \\ r \sin v \sin t \end{pmatrix}, \tag{4.5}$$

we get $\mathbf{S} = \frac{1}{r}\mathcal{I}_5$*, and we obtain the following curvatures* $\mathfrak{C}_i = \frac{1}{r^i}$*, where* $i = 0, 1, ..., 5$*. Here,* $H^5 = K$ *and then the hypersurface (4.5) is a* 3*-rotational umbilical hypersphere.*

5. 3-Rotational Hypersurface Satisfying $\Delta^{IV}\mathbf{x} = \mathcal{A}\mathbf{x}$ in $\mathbb{E}^6$

In this section, we give the fourth Laplace−Beltrami operator (i.e., depends on the fourth fundamental form) of a smooth function in $\mathbb{E}^6$, then calculate it by using the 3-rotational hypersurface.

By using the inverse matrix of the fourth fundamental form matrix $(\mathbf{f}_{ij})_{5\times 5}$, we get the following.

Definition 5.1 *The Laplace−Beltrami operator of a smooth function* $\phi = \phi(x^1, x^2, x^3, x^4, x^5)\mid_{\mathbf{D}}$ $(\mathbf{D} \subset \mathbb{R}^5)$ *of class* C^5 *depends on the fourth fundamental form is the operator defined by*

$$\Delta^{IV}\phi = \frac{1}{\mathbf{f}^{1/2}} \sum_{i,j=1}^{5} \frac{\partial}{\partial x^i}\left(\mathbf{f}^{1/2}\mathbf{f}^{ij}\frac{\partial \phi}{\partial x^j}\right), \tag{5.1}$$

where $\left(\mathbf{f}^{ij}\right) = (\mathbf{f}_{kl})^{-1}$ *and* $\mathbf{f} = \det(\mathbf{f}_{ij})$.

The fourth fundamental form matrix of the 3-rotational hypersurface (4.1) is given by

$$IV = (\mathbf{f}_{ij})_{5\times 5},$$

where

$$\begin{aligned}
\mathbf{f}_{11} &= \frac{\left(\mathbf{g}_{11}+\mathbf{g}_{22}+\sqrt{\mathbf{g}_{11}^2 - 2\mathbf{g}_{11}\mathbf{g}_{22} + 4\mathbf{g}_{12}^2 + \mathbf{g}_{22}^2}\right)^2 \left(\mathbf{h}_{11}+\mathbf{h}_{22}-\sqrt{\mathbf{h}_{11}^2 - 2\mathbf{h}_{11}\mathbf{h}_{22} + 4\mathbf{h}_{12}^2 + \mathbf{h}_{22}^2}\right)^3}{32\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right)^2}, \\
\mathbf{f}_{22} &= \frac{\left(\mathbf{g}_{11}+\mathbf{g}_{22}-\sqrt{\mathbf{g}_{11}^2 - 2\mathbf{g}_{11}\mathbf{g}_{22} + 4\mathbf{g}_{12}^2 + \mathbf{g}_{22}^2}\right)^2 \left(\mathbf{h}_{11}+\mathbf{h}_{22}+\sqrt{\mathbf{h}_{11}^2 - 2\mathbf{h}_{11}\mathbf{h}_{22} + 4\mathbf{h}_{12}^2 + \mathbf{h}_{22}^2}\right)^3}{32\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right)^2}, \\
\mathbf{f}_{33} &= \frac{\mathbf{h}_{33}^3}{\mathbf{g}_{33}^2}, \ \mathbf{f}_{44} = \frac{\mathbf{h}_{44}^3}{\mathbf{g}_{44}^2}, \ \mathbf{f}_{55} = \frac{\mathbf{h}_{55}^3}{\mathbf{g}_{55}^2}.
\end{aligned}$$

For the 3-rotational hypersurface (4.1), $\mathbf{f}_{ij} = 0$ when $i \neq j$. Therefore, the Laplace−Beltrami operator depends on the fourth fundamental form of the hypersurface $\mathbf{x} = \mathbf{x}(u,v,w,s,t)$ is given by

$$\Delta^{IV}\mathbf{x} = \frac{1}{|\mathbf{f}|^{1/2}} \left\{ \begin{array}{c} \frac{\partial}{\partial u}\left(|\mathbf{f}|^{1/2}\mathbf{f}^{11}\frac{\partial \mathbf{x}}{\partial u}\right) + \frac{\partial}{\partial v}\left(|\mathbf{f}|^{1/2}\mathbf{f}^{22}\frac{\partial \mathbf{x}}{\partial v}\right) \\ +\frac{\partial}{\partial w}\left(|\mathbf{f}|^{1/2}\mathbf{f}^{33}\frac{\partial \mathbf{x}}{\partial w}\right) + \frac{\partial}{\partial s}\left(|\mathbf{f}|^{1/2}\mathbf{f}^{44}\frac{\partial \mathbf{x}}{\partial s}\right) \\ +\frac{\partial}{\partial t}\left(|\mathbf{f}|^{1/2}\mathbf{f}^{55}\frac{\partial \mathbf{x}}{\partial t}\right) \end{array} \right\}. \tag{5.2}$$

By using the derivatives of the functions in (5.2), with respect to u, v, w, s, t, respectively, we obtain the following.

Theorem 5.1 *The fourth Laplace−Beltrami operator of the 3-rotational hypersurface (4.1) is given by*

$$\Delta^{IV}\mathbf{x}(u,v,w,s,t) = \begin{pmatrix} \mathbf{F}(u,v)\cos w \\ \mathbf{F}(u,v)\sin w \\ \mathbf{G}(u,v)\cos s \\ \mathbf{G}(u,v)\sin s \\ \mathbf{H}(u,v)\cos t \\ \mathbf{H}(u,v)\sin t \end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{F}(u,v) &= \left(\frac{\mathbf{f}_u}{2\mathbf{f}}\mathbf{f}^{11} + \left(\mathbf{f}^{11}\right)_u\right) f_u + \left(\frac{\mathbf{f}_v}{2\mathbf{f}}\mathbf{f}^{22} + \left(\mathbf{f}^{22}\right)_v\right) f_v + \mathbf{f}^{11} f_{uu} + \mathbf{f}^{22} f_{vv} - \frac{1}{f}, \\
\mathbf{G}(u,v) &= \left(\frac{\mathbf{f}_u}{2\mathbf{f}}\mathbf{f}^{11} + \left(\mathbf{f}^{11}\right)_u\right) g_u + \left(\frac{\mathbf{f}_v}{2\mathbf{f}}\mathbf{f}^{22} + \left(\mathbf{f}^{22}\right)_v\right) g_v + \mathbf{f}^{11} g_{uu} + \mathbf{f}^{22} g_{vv} - \frac{1}{g}, \\
\mathbf{H}(u,v) &= \left(\frac{\mathbf{f}_u}{2\mathbf{f}}\mathbf{f}^{11} + \left(\mathbf{f}^{11}\right)_u\right) h_u + \left(\frac{\mathbf{f}_v}{2\mathbf{f}}\mathbf{f}^{22} + \left(\mathbf{f}^{22}\right)_v\right) h_v + \mathbf{f}^{11} h_{uu} + \mathbf{f}^{22} h_{vv} - \frac{1}{h},
\end{aligned}$$

$$\mathbf{f} = \det\left(\mathbf{f}_{ij}\right) = \frac{\left[\mathbf{h}_{33}\mathbf{h}_{44}\mathbf{h}_{55}\left(\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{12}^2\right)\right]^3}{\left[\mathbf{g}_{33}\mathbf{g}_{44}\mathbf{g}_{55}\left(\mathbf{g}_{11}\mathbf{g}_{22} - \mathbf{g}_{12}^2\right)\right]^2},$$

$$\mathbf{f}^{11} = \frac{1}{\mathbf{f}_{11}},\ \mathbf{f}^{22} = \frac{1}{\mathbf{f}_{22}},\ \mathbf{f}^{33} = \frac{\mathbf{g}_{33}^2}{\mathbf{h}_{33}^3},\ \mathbf{f}^{44} = \frac{\mathbf{g}_{44}^2}{\mathbf{h}_{44}^3},\ \mathbf{f}^{55} = \frac{\mathbf{g}_{55}^2}{\mathbf{h}_{55}^3}.$$

Proof: By direct computing (5.2), we obtain the following functions

$$\begin{aligned}
\mathbf{F}(u,v) &= \mathbf{f}^{11} f_{uu} + \mathbf{f}^{22} f_{vv} - f\frac{\mathbf{g}_{33}^2}{\mathbf{h}_{33}^3}, \\
\mathbf{G}(u,v) &= \mathbf{f}^{11} g_{uu} + \mathbf{f}^{22} g_{vv} - g\frac{\mathbf{g}_{44}^2}{\mathbf{h}_{44}^3}, \\
\mathbf{H}(u,v) &= \mathbf{f}^{11} h_{uu} + \mathbf{f}^{22} h_{vv} - h\frac{\mathbf{g}_{55}^2}{\mathbf{h}_{55}^3}.
\end{aligned}$$

□

Therefore, we have the following.

Corollary 5.1 *Let* $\mathbf{x} : M^5 \longrightarrow \mathbb{E}^6$ *be an immersion given by (4.1). The 3-rotational hypersurface* $\mathbf{x}$ *satisfies* $\Delta^{IV}\mathbf{x} = \mathcal{A}\mathbf{x}$, *where*

$$\mathcal{A} = diag\left(\frac{\mathbf{F}}{f}\mathcal{I}_2, \frac{\mathbf{G}}{g}\mathcal{I}_2, \frac{\mathbf{H}}{h}\mathcal{I}_2\right),$$

and $\mathcal{A} \in Mat\left(6,6\right)$, $\mathcal{I}_2$ *is the identity matrix.*

Finally, we present the following examples.

Example 5.1 *Let* $\mathbf{x} : M^5 \longrightarrow \mathbb{E}^6$ *be an immersion described by (4.1). When the generating surface* γ *of* $\mathbf{x}$ *is parametrized by the unit sphere:* $f(u,v) = r\cos u\cos v$, $g(u,v) = r\sin u\cos v$, $g(u,v) = r\sin v$, *the* 3*-rotational hypersurface* $\mathbf{x}$ *supplies*

$$\Delta^{IV}\mathbf{x} = \mathcal{B}\mathbf{x},$$

where $\mathcal{B} = -5r\mathcal{I}_6$, $\mathcal{I}_6$ *is the identity matrix.*

Example 5.2 *Let* $\mathbf{x} : M^5 \longrightarrow \mathbb{E}^6$ *be an immersion determined by* (4.1). *While the generating surface* γ *of* $\mathbf{x}$ *is parametrized by the Riemann sphere:* $f(u,v) = \frac{2u}{u^2+v^2+1}$, $g(u,v) = \frac{2v}{u^2+v^2+1}$, $h(u,v) = \frac{u^2+v^2-1}{u^2+v^2+1}$, *the* 3*-rotational hypersurface* $\mathbf{x}$ *has the following*

$$\Delta^{IV}\mathbf{x} = \mathcal{C}\mathbf{x},$$

where $\mathcal{C} = 5\mathcal{I}_6$, $\mathcal{I}_6$ *is the identity matrix.*

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