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A new bound for the zeros of Quaternionic Polynomial

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ABSTRACT: This paper investigates the position of zeros of quaternionic polynomials. Recently, it was demonstrated that a quaternionic polynomial with real and positive coefficients obeying monotonicity contains all of its zeros in a four-dimensional closed unit ball. In this work, we identify new regions for the zeros of lacunary-type quaternionic polynomials and establish closed balls, centered at one, that encompass all the zeros of such polynomials.

Key Words: Quaternions, Zeros, Lacunary-type polynomial.

Contents

1. Introduction

Regarding the location of zeros of a polynomial Cauchy [3] proved the following famous result, known as Cauchy's classic result:

Theorem A. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then all the zeros of p lie in

$$|z| < 1 + \max_{1\leq v\leq n-1} \left|\frac{a_v}{a_n}\right|.$$

Cauchy's bound for a polynomial's zeros is beneficial, but it can offer a very broad region when the coefficients are huge in absolute value. As a result, better constraints for a polynomial's zeros are required. Because there is a continuous relationship between the zeros and coefficients of a polynomial, it is preferable to constrain the coefficients of a polynomial to obtain better bounds. In this regard, the Eneström-Kakeya Theorem (see [4], [12], [13]) is an elegant result on the position of zeros of a polynomial with restricted coefficients. G. Eneström appears to have been the first to obtain this phenomenon while exploring a problem in pension fund theory. In 1912, S. Kakeya [11] published a paper in the Tôhoku Mathematical Journal that included the following more detailed result:

Theorem B. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n such that $0 < a_0 \leq a_1 \leq ... \leq a_n$, then all the zeros of p lie in $|z| \leq 1$.

In the literature, for example see ([1], [9], [10], [12], [13]), there exist various extensions and generalizations of Eneström-Kakeya Theorem. In 1967, Joyal, Labelle, and Rahman [10] published a result which might be considered the foundation of the studies which we are currently studying. The Eneström-Kakeya Theorem, as stated in Theorem B, deals with polynomials with non-negative coefficients which form a monotone sequence. Joyal, Labelle, and Rahman generalized Theorem B by dropping the condition of non-negativity and maintaining the condition of monotonicity. Namely, they proved:

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Theorem C. If $p(z)=\sum_{v=0}^{n}a_vz^v$ is a polynomial of degree n such that $a_0\leq a_1\leq ...\leq a_n$, then all the zeros of p lie in $|z|\leq\frac{1}{|a_n|}(|a_0|+a_n-a_0)$.
Of course, when $a_0\geq 0$, then Theorem C reduces to Theorem B.

2. Preliminary

Preliminaries on a quaternionic variable's regular functions are provided in this part and will be helpful in the follow-up. A novel theory of regularity for functions, specifically for polynomials of a quaternionic variable, was created in a recent study (for example, see [2], [5], and [6]) and is very helpful in replicating many significant characteristics of holomorphic functions. The discreteness of the zero sets of holomorphic functions of a complex variable is one of their fundamental characteristics (except when the function vanishes identically). All limitations to complex lines of a regular function of a quaternionic variable have discrete zero sets or disappear indistinguishably since they are all holomorphic. The structure of a quaternionic regular function's zero sets and the factorization property of zeros were detailed in the preliminary steps. Gentili and Stoppato [5] gave a necessary and sufficient condition for a quaternionic regular function to have a zero at a point in terms of the coefficients of the function's power series expansion. Before we go to our results, we need to go over some basics about quaternions and quaternionic polynomials.
William Rowan Hamilton introduced quaternions in 1843 as an extension of complex numbers to four dimensions. The set of all quaternions is denoted by $\mathbb{H}$ in honour of Sir Hamilton and is generally represented in the form $q=\alpha+i\beta+j\gamma+k\delta\in\mathbb{H}$, where $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ and i,j,k are the fundamental quaternion units, such that $i^2=j^2=k^2=ijk=-1$. There is a conjugate for each quaternion q denoted by $q*$ and is defined as $q^*=\alpha-i\beta-j\gamma-k\delta$. Furthermore, the norm (or length) of a quaternion q can be calculated using

$$||q||=\sqrt{qq^*}=\sqrt{\alpha^2+\beta^2+\gamma^2+\delta^2}.$$

The closed ball of radius $r>0$ with center $q_0\in\mathbb{H}$ is defined as $B(q_0,r)=\{q\in\mathbb{H}\mid |q-q_0|\leq r\}$. The quaternions are the standard example of a non-commutative division ring and also forms a four dimensional vector space over $\mathbb{R}$ with $\{1,i,j,k\}$ as a basis.

Quaternions have found a permanent place in engineering and computer description of moving objects. In physics, their use has been controversial. Although they appear natural to the description of 4-dimensional space and entities therein, they have not been widely used in the 20th century. However, in quantum field theory, quaternions have always been present in the guise of "spinors". The application of quaternions is mentioned below.
Computer Graphics: Given two orientations in the 3-dimensions, programmers traditionally used linear interpolation between the corresponding Euler angles to model a rotation. Such an algorithm can cause singularities, and problems such as gimbal License: CC-BY-NC REFERENCES lock (rotation in one axis is momentarily forbidden) may be encountered, which would severely affect the smoothness of the animation. The quaternions on the other hand generate a more realistic animation. A technique which is currently gaining favour is called spherical linear interpolation (SLERP) and uses the fact that the set of all unit quaternions form a unit sphere. By representing the quaternions of key frames as points on the unit sphere, a SLERP defines the intermediate sequence of rotations as a path along the great circle between the two points on the sphere [16].
Physics: The quaternions have found use in to express the Lorentz Transform making them useful for work on Special and General Relativity [8]. Their properties as generators of rotation make them incredibly useful for Newtonian Mechanics, scattering experiments such as crystallography, and quantum mechanics (Particle spin is an emerge property of the mathematics) [8].

The indeterminate for a quaternionic polynomial is defined as q. Without commutativity, we are left with the polynomial aq^n and the polynomial $a_0qa_1q\cdots qa_n$, $a=a_0a_1\cdots a_n$. To address this issue, we use the standard that polynomials have indeterminate on the left and coefficients on the right, resulting in the quaternionic polynomial $p_1(q)=\sum\limits_{s=0}^{m}q^sa_s$. For such a p_1 and $p_2(q)=\sum\limits_{s=0}^{n}q^sb_s$, the regular product of

p_1 and p_2 is defined as $\big(p_1 * p_2\big)(q) = \sum\limits_{i,j=0}^{n,m} q^{i+j} a_i b_j$. This is consistent with the definition of the regular product for a quaternionic variable's power series (see definition 3.1 of [5]). If p_1 has real coefficients, then $*$ multiplication is equivalent to point-wise multiplication. In general, the product rule $*$ is associative rather than commutative. Polynomials behave differently in the lack of commutativity than they do in the real or complex situation. A real or complex polynomial of degree n, for example, can have no more than n zeros, according to the Factor theorem, which asserts that a being a zero of $p(z)$ is equal to $z - a$ being a divisor of $p(z)$. The Factor Theorem, on the other hand, only holds in a commutative ring (see Theorem III. 6.6 of [7]). The second degree polynomial $q^2 + 1$ in the Quaternion case has an unlimited number of zeros, notably $q_0 = i\ or j\ or k$ and all those given by $w_0 = h^{-1} q_0 h \quad \forall h \in \mathbb{H}$.
We define the set of quaternionic polynomials with quaternion coefficients by

$$\mathbb{P}_n := \left\{ p\ ;\ \ p(q) = \sum_{s=0}^{n} q^s a_s\ ,\ \ \ q \in \mathbb{H} \right\}$$

where $a_s \in \mathbb{H},\ \ 0 \le s \le n$. As we know an nth degree quaternion polynomial has infinite number of zeros and to locate all those zeros will be interesting. In this direction, Carney et al. [2] recently proved the following extension of Theorem B for the quaternionic polynomial $p \in \mathbb{P}_n$. More precisely, they proved the following result:
Theorem D. If $p \in \mathbb{P}_n$ is a quaternionic polynomial of degree n with real coefficients satisfying $0 < a_0 \le a_1 \le ... \le a_n$, then all the zeros of p lie in $|q| \le 1$.

In the same paper, they proved the following result which replaces the condition of monotonicity on the real coefficients by monotonicity in the real and imaginary parts of the quaternion coefficients:
Theorem E. If $p \in \mathbb{P}_n$ is a quaternionic polynomial of degree n where $a_s = \alpha_s + \beta_s i + \gamma_s j + \delta_s k \in \mathbb{H}\ ;\ 0 \le s \le n$ and

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \quad ; \quad \beta_0 \le \beta_1 \le \cdots \le \beta_n$$
$$\gamma_0 \le \gamma_1 \le \cdots \le \gamma_n \quad ; \quad \delta_0 \le \delta_1 \le \cdots \le \delta_n \quad ;$$

then all the zeros of p lie in

$$|q| \le \frac{\big(|\alpha_0| - \alpha_0 + a_n\big) + \big(|\beta_0| - \beta_0 + \beta_n\big) + \big(|\gamma_0| - \gamma_0 + \gamma_n\big) + \big(|\delta_0| - \delta_0 + \delta_n\big)}{|a_n|}.$$

For $0 \le \mu \le n-1$, we denote by $\mathbb{P}_{n,\mu}$ the class of quaternionic polynomials.
$p(q) = q^n a_n + q^\mu a_\mu + q^{\mu-1} a_{\mu-1} + \cdots + q a_1 + a_0$ where $a_s \in \mathbb{H},\ \ 0 \le s \le n$ are quaternion coefficients, having some missing terms, and we call these polynomials the lacunary type of quaternionic polynomials. For $\mu = n-1$, the class $\mathbb{P}_{n,\mu}$ reduces to the class $\mathbb{P}_n$.

Recently, various authors have obtained a number of results on the location of zeros of quaternionic polynomials (see [14], [15], [17]) and various results have been extended for the location of zeros of complex polynomials to quaternions. In this paper, we obtain new regions for the location of zeros of lacunary-type polynomials with quaternionic variable and quaternionic coefficients. The regions obtained are closed balls in $\mathbb{H}$ with nearly identical radii, as determined by various authors, but with different centers, and they encompass all the zeros of quaternionic polynomials. More precisely, we prove the following result:

3. Main Results

Theorem 1. All zeros of the polynomial $p \in \mathbb{P}_{n,\mu}$ lie in

$$|q - 1| \le \max\left(2, \frac{|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M_\mu}{|a_n|}\right),$$

where

$$M_\mu = \sum_{s=0}^{\mu} \big[|\alpha_{s+1} - \alpha_s| + |\beta_{s+1} - \beta_s| + |\gamma_{s+1} - \gamma_s| + |\delta_{s+1} - \delta_s|\big] \quad \text{and} \quad a_{\mu+1} = 0.$$

Taking $\mu = n-1$ in Theorem 1, we get the following result.

Corollary 1. All zeros of the polynomial $p \in \mathbb{P}_n$ lie in

$$|q-1| \leq \max\Big(2, \frac{|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M}{|a_n|}\Big),$$

where

$$M = \sum_{s=0}^{n-1} \big[|\alpha_{s+1} - \alpha_s| + |\beta_{s+1} - \beta_s| + |\gamma_{s+1} - \gamma_s| + |\delta_{s+1} - \delta_s|\big].$$

If the real and imaginary parts of the coefficients are non-negative and satisfy monotonicity, then we obtain from Corollary 1

Corollary 2. If $p \in \mathbb{P}_n$ is a quaternionic polynomial of degree n where $a_s = \alpha_s + \beta_s i + \gamma_s j + \delta_s k \in \mathbb{H}\ ;\ 0 \leq s \leq n$ satisfy

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \quad ; \quad \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n$$
$$\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_n \quad ; \quad \delta_0 \leq \delta_1 \leq \cdots \leq \delta_n \quad ;$$

then all the zeros of p lie in

$$|q-1| \leq \max\left(2,\ \frac{\big(|\alpha_0| - \alpha_0 + a_n\big) + \big(|\beta_0| - \beta_0 + \beta_n\big) + \big(|\gamma_0| - \gamma_0 + \gamma_n\big) + \big(|\delta_0| - \delta_0 + \delta_n\big)}{|a_n|}\right).$$

Taking $a_s = \alpha_s \in \mathbb{R}\ ,\ \forall\ 0 \leq s \leq n$ and $a_n \geq a_{n-1} \geq \cdots \geq a_0$ so that $\beta_s = \gamma_s = \delta_s = 0$ for $0 \leq s \leq n$ in Corollary 1, we get the following Enström type result for polynomials over quaternion settings.

Corollary 3. If $p \in \mathbb{P}_n$ is of degree n with real coefficients satisfying $a_n \geq a_{n-1} \geq \cdots \geq a_0$, then all zeros of p lie in

$$|q-1| \leq \max\Big(2, \frac{|a_0| + a_n - a_0}{|a_n|}\Big).$$

4. Lemmas

For the proof of our main result, we need the following lemmas:

Lemma 1. If $f(q) = \sum_{v=0}^{\infty} q^v a_v$ and $g(q) = \sum_{v=0}^{\infty} q^v b_v$ be two given quaternionic power series with radii of convergence greater than R. The regular product of $f(q)$ and $g(q)$ is defined as $(f * g)(q) = \sum_{v=0}^{\infty} q^v c_v$, where $c_v = \sum_{s=0}^{\infty} a_v b_{v-l}$. Let $|q_0| < R$, then $(f * g)(q_0) = 0$ if and only if either $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g(f(q_0)^{-1} q_0 f(q_0)) = 0$.

Above lemma for the zeros of regular product of power series is due to G. Gentili and C. Stoppato [6]. The following lemma is introduced by Gentili and Struppa for regular functions [5].

Lemma 2. Maximum Modulus Theorem: Let $B = B(0, r)$ be a ball in $\mathbb{H}$ with center 0 and radius r and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ has a relative maximum at a point $a \in B$, then f is constant on B.

5. Proof of Theorem

Proof of Theorem 1. Since $1-q$ is a quaternionic polynomial of degree one with real coefficients, we have by the definition of $*$ multiplication

$$\begin{aligned} p(q) * (1-q) &= \big(q^n a_n + q^\mu a_\mu + q^{\mu-1} a_{\mu-1} + \cdots + q a_1 + a_0\big) * (1-q) \\ &= -q^{n+1} a_n + q^n a_n - q^{\mu+1} a_\mu + q^\mu a_\mu - q^\mu a_{\mu-1} + \cdots + q a_1 - q_0 + a_0 \\ &= f(q) - q^{n+1} a_n + q^n a_n, \end{aligned} \tag{5.1}$$

where $f(q) = a_0 + \sum_{s=0}^{\mu} q^{s+1}(a_{s+1} - a_s)$ and $a_{\mu+1} = 0$. Note that

$$\begin{aligned}\left|a_{s+1} - a_s\right| &= \left|(\alpha_{s+1} - \alpha_s) + (\beta_{s+1} - \beta_s)i + (\gamma_{s+1} - \gamma_s)j + (\delta_{s+1} - \delta_s)k\right| \\ &\leq \left|\alpha_{s+1} - \alpha_s\right| + \left|\beta_{s+1} - \beta_s\right| + \left|\gamma_{s+1} - \gamma_s\right| + \left|\delta_{s+1} - \delta_s\right|,\end{aligned}$$

therefore for $|q| = 1$, we have

$$\begin{aligned}|f(q)| &\leq \left|a_0 + \sum_{s=0}^{\mu} q^{s+1}(a_{s+1} - a_s)\right| \\ &\leq |a_0| + \sum_{s=0}^{\mu} |q|^{s+1}\left|a_{s+1} - a_s\right| \\ &= |a_0| + \sum_{s=0}^{\mu} \left|a_{s+1} - a_s\right| \\ &\leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{s=0}^{\mu}\Big\{\left|\alpha_{s+1} - \alpha_s\right| + \left|\beta_{s+1} - \beta_s\right| + \left|\gamma_{s+1} - \gamma_s\right| + \left|\delta_{s+1} - \delta_s\right|\Big\} \\ &= |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M_\mu, \end{aligned} \tag{5.2}$$

where

$$M_\mu = \sum_{s=0}^{\mu}\Big\{\left|\alpha_{s+1} - \alpha_s\right| + \left|\beta_{s+1} - \beta_s\right| + \left|\gamma_{s+1} - \gamma_s\right| + \left|\delta_{s+1} - \delta_s\right|\Big\}.$$

Since

$$\max_{|q|=1}\left|q^n * f(\frac{1}{q})\right| = \max_{|q|=1}\left|q^n f(\frac{1}{q})\right| = \max_{|q|=1}\left|f(\frac{1}{q})\right| = \max_{|q|=1}\left|f(q)\right|$$

it follows that $|q^n * f(\frac{1}{q})|$ has same bound on $|q| = 1$ as of $|f(q)|$, hence we obtain from inequality (5.2) that

$$\left|q^n * f(\frac{1}{q})\right| = \left|q^n f(\frac{1}{q})\right| \leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M_\mu \quad \text{for} \quad |q| = 1.$$

This implies, by Maximum Modulus Theorem (Lemma 2),

$$\left|q^n f(\frac{1}{q})\right| \leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M_\mu \quad \text{for} \quad |q| \leq 1.$$

Replacing q by $\frac{1}{q}$, we obtain

$$\frac{1}{|q^n|}|f(q)| \leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M_\mu \quad \text{for} \quad |q| \geq 1.$$

That is, for $|q| \geq 1$

$$|f(q)| \leq \Big\{|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M_\mu\Big\}|q|^n. \tag{5.3}$$

With the help of inequality (5.3), we obtain from equation (5.1) that for $|q| \geq 1$

$$\begin{aligned}\left|p(q) * (1 - q)\right| &= \left|f(q) - q^{n+1}a_n + q^n a_n\right| = \left|f(q) - q^n(q - 1)a_n\right| \\ &\geq |q|^n|a_n||q - 1| - \left|f(q)\right| \\ &\geq |q|^n|a_n||q - 1| - \Big(|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M_\mu\Big)|q|^n \\ &= |q|^n\Big\{|a_n||q - 1| - \Big(|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M_\mu\Big)\Big\}.\end{aligned}$$

This implies that for $|q| \geq 1$, $|p(q) * (1-q)| > 0$, i.e, $p(q) * (1-q) \neq 0$ if

$$|q-1| > \frac{|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M_\mu}{|a_n|}.$$

Therefore all the zeros of $p(q) * (1-q)$ whose modulus is greater than or equal to 1 lie in

$$|q-1| \leq \frac{|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M_\mu}{|a_n|}.$$

Those zeros of $p(q) * (1-q)$ whose modulus is less than 1 already lie in $|q-1| \leq 2$, it follows that all the zeros of $p(q) * (1-q)$ lie in

$$|q-1| \leq \max\Big(2, \frac{|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M_\mu}{|a_n|}\Big).$$

By Lemma 1, $p(q) * (1-q) = 0$ if and only if either $p(q) = 0$ or $p(q) \neq 0$ implies $1 - p(q)^{-1}qp(q) = 0$. Notice that $1 - p(q)^{-1}qp(q) = 0$ is equivalent to $p(q)^{-1}qp(q) = 1$ and if $p(q) \neq 0$, this implies that $q = 1$. So the only zeros of $p(q) * (1-q) = 0$ are zeros of $p(q)$ and $q = 1$. Consequently, we conclude that all zeros of $p(q)$ lie in

$$|q-1| \leq \max\Big(2, \frac{|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + M_\mu}{|a_n|}\Big).$$

This completes the proof of Theorem 1.

6. Conclusions

A new bound for the zeros of lacunary-type quaternionic polynomials has been established without any restrictions on the coefficients of the polynomials. This is significant, as many prior results might have relied on certain conditions or constraints on the coefficients. Furthermore, an Eneström-Kakeya type result has been derived and a new region has been obtained for the location of zeros of quaternionic polynomials, which further enhances our understanding of the behavior of zeros of quaternionic polynomials.

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