Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.67722

Multiplicity of weak solutions for a class of quasilinear elliptic Neumann problems using Variational methods

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ABSTRACT: The existence of infinitely many weak solutions for the strongly nonlinear elliptic equation of the form

$$\begin{cases} -\mathrm{div}\Big(w_1(x)|\nabla u|^{p(x)-2}\nabla u\Big) + w_0(x)|\ u\ |^{p(x)-2}u = f(x,u) + g(x,u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \gamma} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

is proved by applying a critical point variational principle obtained by B. Ricceri in weighted variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega, w_0, w_1)$.

Key Words: Neumann problem; Variational principle; Elliptic boundary value problem; Weighted variable exponent Lebesgue-Sobolev space $W^{1,p(\cdot)}(\Omega, w_0, w_1)$.

Contents

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ with boundary of class C^1, and let γ be the outward unit normal vector on the boundary $\partial\Omega$.

In recent years there has been an increasing interest in the study of various mathematical problems with variable exponent. These problems are interesting in applications and raise many difficult mathematical problems. The impulse for this, mainly come from their important applications in modeling real-world problems in electrorheological fluids and image processing, (see for example [6, 7, 10, 25, 27, 28]).

Our aim is to prove the existence of infinitely many weak solutions for the following degenerate $p(x)$-Laplacian equation with Neumann boundary value condition, this is a now topic.

$$(\mathcal{P})\begin{cases} -\mathrm{div}\Big(w_1(x)|\nabla u|^{p(x)-2}\nabla u\Big) + w_0(x)|\ u\ |^{p(x)-2}u = f(x,u) + g(x,u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \gamma} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Where $p \in L^\infty(\Omega)$ and satisfies the conditon

$$1 < p^- := \operatorname*{ess\,inf}_{\Omega} p(x) \le p^+ := \operatorname*{ess\,sup}_{\Omega} p(x) < \infty, \tag{1.1}$$

and $w_0(x), w_1(x)$ be a weight functions on Ω, i.e. each $w_0(x)$ and $w_1(x)$ is measurable a.e. strictly positive on Ω, satisfying some integrability conditions (see section 2). We refer the reader to [18, 19, 26] where the authors were concerned with Dirichlet problems.

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Submitted March 30, 2023. Published September 11, 2023
2010 *Mathematics Subject Classification:* 35J57, 35J60, 46E35.

In the classical Sobolev spaces, Ricceri [24], Anello and Cordaro [5] have proved the existence of solutions $(\mathcal{P})$ in the case $\Delta_p u$ and $w_0(x) = \lambda(x)$ is a positive function such that $\lambda(\cdot) \in L^\infty(\Omega)$ with $\lambda^- = \operatorname{ess}\inf_{x\in\Omega} \lambda(x) > 0$, and $p > N$. The existence of solutions of problem $(\mathcal{P})$ is proved by applying the following critical point theorem recently obtained by B. Ricceri as a consequence of a more general variational principle (see [23]).

In the Sobolev variable exponent setting, X. Fan and C. Ji in [14] have proved the existence of infinitely many solutions in the space with variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ in the particular case, $\Delta_{p(\cdot)} u$ and $w_0(x) = \lambda(x)$.

Even though the problem $(\mathcal{P})$ has been studied by some other authors in anisotropic variable exponent Sobolev spaces and weighted Sobolev space (see [1, 2, 3, 8, 9]), the hypotheses we use in this paper are totally different from those ones and so are our results.

The following theorem plays an important role in this paper.

Theorem 1.1 *(See [14], Theorem 2.2.) Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \longmapsto \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals. Assume also that Ψ is (strongly) continuous and satisfies $\lim_{\|u\|\to+\infty} \Psi(u) = +\infty$. For each $\rho > \inf_X \Psi$, put*

$$\varphi(\rho) = \inf_{u\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf_{v\in\overline{(\Psi^{-1}(]-\infty,\rho[))}} \Phi(v)}{\rho - \Psi(u)}, \tag{1.2}$$

where $\overline{(\Psi^{-1}(]-\infty,\rho[))}$ is the closure of $\Psi^{-1}(]-\infty,\rho[)$ in the weak topology. Then, the following conclusions hold:

***(a)** If there exist $\rho_0 > \inf_X \Psi$ and $u_0 \in X$ such that*

$$\Psi(u_0) < \rho_0 \tag{1.3}$$

and

$$\Phi(u_0) - \inf_{v\in\overline{(\Psi^{-1}(]-\infty,\rho_0[))}} \Phi(v) < \rho_0 - \Psi(u_0), \tag{1.4}$$

then the restriction of $\Psi + \Phi$ to $\Psi^{-1}(]-\infty,\rho_0[)$ has a global minimum.

***(b)** If there exists a sequence $\{r_n\} \subset (\inf_X \Psi, +\infty)$ with $r_n \to +\infty$ and a sequence $\{u_n\} \subset X$ such that for each n we have*

$$\Psi(u_n) < r_n \tag{1.5}$$

and

$$\Phi(u_n) - \inf_{v\in\overline{(\Psi^{-1}(]-\infty,r_n[))}} \Phi(v) < r_n - \Psi(u_n), \tag{1.6}$$

and in addition,

$$\liminf_{\|u\|\to+\infty} (\Psi(u) + \Phi(u)) = -\infty, \tag{1.7}$$

then there exists a sequence $\{v_n\}$ of local minima of $\Psi + \Phi$ such that $\Psi(v_n) \to +\infty$ as $n \to \infty$.

***(c)** If there exists a sequence $\{r_n\} \subset (\inf_X \Psi, +\infty)$ with $r_n \to \inf_X \Psi$ and a sequence $\{u_n\} \subset X$ such that for each n the condition (1.5) and (1.6) are satisfied, and in addition,*

$$\text{every global minimizer of } \Psi \text{ is not a local minimizer of } \Phi + \Psi, \tag{1.8}$$

then there exists a sequence $\{v_n\}$ of pairwise distinct local minimizers of $\Phi + \Psi$ such that $\lim_{n\to\infty} \Psi(v_n) = \inf_X \Psi$, and $\{v_n\}$ weakly converges to a global minimizer of Ψ.

This paper is organized as follows: In section 2, we present some preliminary knowledge on the weighted variable exponent Sobolev spaces $W^{1,p(\cdot)}(\Omega, w_0, w_1)$. We introduce in section 3 some assumptions for which our problem has solutions. In section 4, we prove the existence of infinitely many weak solutions for our Neumann elliptic problem. Finally, we conclude and provide some perspectives in section 5.

2. Preliminary

In this section we summarize notation, definitions and properties of our framework. The basic properties of the variable exponent Lebesgue-Sobolev spaces, that is, when $w(x) \equiv 1$ can be found from ([11, 13, 15, 16, 17, 20]).

Let Ω be a bounded domain in $\mathbb{R}^N$, we define:

$$\mathcal{C}_+(\overline{\Omega}) = \left\{\text{measurable function} \quad p(\cdot) : \overline{\Omega} \longrightarrow \mathbb{R} \quad \text{such that} \quad 1 < p^- \leq p^+ < \infty\right\},$$

where

$$p^- = \operatorname{ess\,inf}\left\{p(x) \,/\, x \in \overline{\Omega}\right\} \qquad \text{and} \qquad p^+ = \operatorname{ess\,sup}\left\{p(x) \,/\, x \in \overline{\Omega}\right\}.$$

Let w, w_0, w_1 are positive continuous functions defined in $\mathbb{R}^N$. For $p \in \mathcal{C}_+(\overline{\Omega})$, define

$$L^{p(\cdot)}(\Omega, w) = \left\{u(x) : uw^{\frac{1}{p(x)}} \in L^{p(\cdot)}(\Omega)\right\},$$

with the norm

$$\|u\|_{L^{p(\cdot)}(\Omega,w)} = \|u\|_{p(\cdot),\Omega,w} = \inf\left\{\sigma > 0 : \quad \int_\Omega w(x)\left|\frac{u(x)}{\sigma}\right|^{p(x)} dx \leq 1\right\}.$$

When $w(x) \equiv 1$, we use $L^{p(\cdot)}(\Omega)$ instead $L^{p(\cdot)}(\Omega, w)$ and use $\|u\|_{p(\cdot),\Omega}$ instead of $\|u\|_{p(\cdot),\Omega,w}$.

The weighted variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega, w_0, w_1)$ is defined by

$$W^{1,p(\cdot)}(\Omega, w_0, w_1) = \left\{u \in L^{p(\cdot)}(\Omega, w_0) : |\nabla u(x)| \in L^{p(\cdot)}(\Omega, w_1)\right\},$$

where the norm is

$$\|u\|_{W^{1,p(\cdot)}(\Omega,w_0,w_1)} = \|u\|_{p(\cdot),\Omega,w_0} + \|\nabla u(x)\|_{p(\cdot),\Omega,w_1}.$$

It is easy to see that the norm

$$|||u|||_{1,p(\cdot),\Omega,w_0,w_1} = \inf\left\{\mu > 0 : \quad \int_\Omega \left(w_0(x)\left|\frac{u(x)}{\mu}\right|^{p(x)} + w_1(x)\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)}\right) dx \leq 1\right\}, \tag{2.1}$$

is the equivalent norm. The theory of such spaces was developed in [18, 19, 21, 22, 26]. When $p(x)$ is a constant, some results have been proved in [4, 12]. If $w_0(x) \equiv w_1(x) \equiv 1$, we use $W^{1,p(\cdot)}(\Omega)$ instead of $W^{1,p(\cdot)}(\Omega, w_0, w_1)$ and use $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ instead of $\|u\|_{W^{1,p(\cdot)}(\Omega,w_0,w_1)}$.

Throughout the paper, we assume that w is a measurable positive and finite almost everywhere function in Ω satisfying that

(V1) $w, w_0, w_1 \in L^1_{loc}(\Omega)$ and $w^{\frac{-1}{p(x)-1}}, w_0^{\frac{-1}{p(x)-1}}, w_1^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega)$.

(V2) $w^{-\nu(x)}, w_0^{-\nu(x)}, w_1^{-\nu(x)} \in L^1(\Omega)$ with $\nu(x) \in \left]\frac{N}{p(x)-N}, \infty\right[\cap \left[\frac{1}{p(x)-1}, \infty\right[$ and $\nu^- > \frac{N}{p^- - N}$.

Proposition 2.1 ([19]) *The spaces $L^{p(\cdot)}(\Omega, w)$ and $W^{1,p(\cdot)}(\Omega, w)$ are separable and reflexive Banach spaces.*

Lemma 2.1 ([26]) *If we denote*

$$\rho(u) = \int_\Omega w(x) \mid u \mid^{p(x)} dx, \quad \forall u \in L^{p(\cdot)}(\Omega, w),$$

we have

(i) $\|u\|_{L^{p(\cdot)}(\Omega,w)} < 1(=1, > 1) \iff \rho(u) < 1(=1, > 1)$,

(ii) $\|u\|_{L^{p(\cdot)}(\Omega,w)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega,w)} \leq \rho(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega,w)}$,

(iii) $\|u\|_{L^{p(\cdot)}(\Omega,w)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega,w)} \leq \rho(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega,w)}$.

Lemma 2.2 ([21, 22]) *Set*

$$\zeta(u)=\int_\Omega\left(w_0(x)\Big|\frac{u(x)}{\mu}\Big|^{p(x)}+w_1(x)\Big|\frac{\nabla u(x)}{\mu}\Big|^{p(x)}\right)dx,\quad \forall u\in W^{1,p(\cdot)}(\Omega,w_0,w_1),$$

we obtain

(a) $\|\|u\|\|_{1,p(\cdot),\Omega,w_0,w_1}<1\Longrightarrow \|\|u\|\|^{p^+}_{1,p(\cdot),\Omega,w_0,w_1}\le\zeta(u)\le\|\|u\|\|^{p^-}_{1,p(\cdot),\Omega,w_0,w_1}$,

(b) $\|\|u\|\|_{1,p(\cdot),\Omega,w_0,w_1}>1\Longrightarrow \|\|u\|\|^{p^-}_{1,p(\cdot),\Omega,w_0,w_1}\le\zeta(u)\le\|\|u\|\|^{p^+}_{1,p(\cdot),\Omega,w_0,w_1}$.

The following compact embedding result is crucial.

Theorem 2.1 *Let $(V1)$ and $(V2)$ satisfied. Then we obtain the following result*

$$W^{1,p(\cdot)}(\Omega,w_0,w_1)\hookrightarrow\hookrightarrow C^0(\overline{\Omega}).$$

Proof Let $u\in W^{1,p(\cdot)}(\Omega,w_0,w_1)$, we denote by $p_1(x)=\frac{\nu(x)p(x)}{\nu(x)+1}<p(x)$. By the Hölder inequality in([18] Proposition 2.1) with parameters $q(x)=\frac{p(x)}{p_1(x)}=\frac{\nu(x)+1}{\nu(x)}$ and its conjugate $q'(x)=\nu(x)+1$ we have

$$\begin{aligned}\int_\Omega\Big|\nabla u\Big|^{p_1(x)}dx &=\int_\Omega\Big|\nabla u\Big|^{\frac{\nu(x)p(x)}{\nu(x)+1}}dx\\ &=\int_\Omega\Big|\nabla u\Big|^{\frac{p(x)\nu(x)}{\nu(x)+1}}w_1^{\frac{\nu(x)}{\nu(x)+1}}w_1^{\frac{-\nu(x)}{\nu(x)+1}}dx\\ &\le 2\Big\|w_1^{\frac{\nu(x)}{\nu(x)+1}}\Big|\nabla u\Big|^{\frac{p(x)\nu(x)}{\nu(x)+1}}\Big\|_{L^{\frac{\nu(x)+1}{\nu(x)}}(\Omega)}\Big\|w_1^{-\frac{\nu(x)}{\nu(x)+1}}\Big\|_{L^{\nu(x)+1}(\Omega)}.\end{aligned}$$

Assumption $(V2)$ and Lemma 2.1 imply that

$$\Big\|w_1^{-\frac{\nu(x)}{\nu(x)+1}}\Big\|_{L^{\nu(x)+1}(\Omega)}\le\left(\int_\Omega w_1^{-\nu(x)}(x)dx+1\right)^{\frac{1}{\nu^-+1}}\le C.$$

Thus we get

$$\int_\Omega\Big|\nabla u\Big|^{p_1(x)}dx\le C\Big\|w_1^{\frac{\nu(x)}{\nu(x)+1}}\Big|\nabla u\Big|^{\frac{p(x)\nu(x)}{\nu(x)+1}}\Big\|_{L^{\frac{\nu(x)+1}{\nu(x)}}(\Omega)}. \tag{2.2}$$

Without loss of generality, we can assume that $\int_\Omega\Big|\nabla u\Big|^{p_1(x)}dx>1$. (If not, it is easy to see from Lemma 2.1 that $u\in W^{1,p_1(\cdot)}(\Omega)$.) If $\int_\Omega w_i(x)\Big|\nabla u\Big|^{p(x)}dx<1$, then from (2.2) and Lemma 2.1 we have

$$\begin{aligned}\Big\|\nabla u\Big\|^{\frac{p^-\nu^-}{\nu^-+1}}_{L^{p_1(\cdot)}(\Omega)} &\le\int_\Omega\Big|\nabla u\Big|^{p_1(x)}dx\\ &\le C\left(\int_\Omega w_1(x)\Big|\nabla u\Big|^{p(x)}dx\right)^{\frac{\nu^-}{\nu^-+1}}\\ &\le C\Big\|\nabla u\Big\|^{\frac{p^-\nu^-}{\nu^-+1}}_{L^{p(\cdot)}(\Omega,w_1)},\end{aligned}$$

i.e.,

$$\Big\|\nabla u\Big\|_{L^{p_1(\cdot)}(\Omega)}\le C\Big\|\nabla u\Big\|_{L^{p(\cdot)}(\Omega,w_1)}. \tag{2.3}$$

On the other hand, if $\int_\Omega w_1(x)\Big|\nabla u\Big|^{p(x)}dx>1$, then from (2.2) and Lemma 2.1 we obtain

$$\begin{aligned}\Big\|\nabla u\Big\|^{\frac{p^-\nu^-}{\nu^-+1}}_{L^{p_1(\cdot)}(\Omega)} &\le\int_\Omega\Big|\nabla u\Big|^{p_1(x)}dx\\ &\le C\left(\int_\Omega w_1(x)\Big|\nabla u\Big|^{p(x)}dx\right)^{\frac{\nu^+}{\nu^++1}}\\ &\le C\Big\|\nabla u\Big\|^{\frac{p^+\nu^+}{\nu^++1}}_{L^{p(\cdot)}(\Omega,w_1)},\end{aligned}$$

i.e.,

$$\Big\|\nabla u\Big\|_{L^{p_1(\cdot)}(\Omega)} \leq C\Big\|\nabla u\Big\|^{\beta}_{L^{p(\cdot)}(\Omega,w_1)}, \tag{2.4}$$

where $\beta = \frac{p^+\nu^+}{\nu^++1}\cdot\frac{1+\nu^-}{p^-\nu^-}$. From the inequalities (2.3) and (2.4), we obtain $\nabla u \in L^{p(\cdot)}(\Omega, w_1)$. Therefore, we conclude that $W^{1,p(\cdot)}(\Omega, w_0, w_1) \hookrightarrow W^{1,p_1(\cdot)}(\Omega)$, by $(V2)$ we have $\nu^- > \frac{N}{p^- - N}$ then $p_1^- > N$. Since $W^{1,p_1(\cdot)}(\Omega)$ is continuously embedded in $W^{1,p_1^-}(\Omega)$, and $W^{1,p_1^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. We then deduce the result using the classic injections $W^{1,p(\cdot)}(\Omega, w_0, w_1) \hookrightarrow\hookrightarrow C^0(\overline{\Omega})$. This finishes the proof. □

Set

$$C_0 = \sup_{u\in W^{1,p(\cdot)}(\Omega,w_0,w_1)\setminus\{0\}} \frac{\|u\|_{L^\infty(\Omega)}}{\|\!|u\|\!|_{1,p(\cdot),\Omega,w_0,w_1}}, \tag{2.5}$$

then C_0 is a positive constant.

3. Essential assumptions

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with the boundary $\partial\Omega$ of class C^1, and let γ be is the outward unit normal to $\partial\Omega$.

Assume that $f, g : \Omega\times\mathbb{R} \longmapsto \mathbb{R}$ are Carathéodory functions satisfying,

$$\sup_{|t|\leq r} \mid f(x,t) \mid \in L^1(\Omega), \qquad \text{and} \qquad \sup_{|t|\leq r} \mid g(x,t) \mid \in L^1(\Omega) \qquad \text{for each} \quad r > 0. \tag{3.1}$$

We set

$$F(x,t) = \int_0^t f(x,s)ds \qquad \text{and} \qquad G(x,t) = \int_0^t g(x,s)ds. \tag{3.2}$$

We define, for any $u \in W^{1,p(\cdot)}(\Omega, w_0, w_1)$, the functionals

$$J(u) = \int_\Omega \frac{1}{p(x)}\left(w_1(x)\Big|\nabla u\Big|^{p(x)} + w_0(x) \mid u \mid^{p(x)}\right)dx, \tag{3.3}$$

$$\Psi(u) = J(u) - \int_\Omega G(x,u)dx \qquad \textit{and} \qquad \Phi(u) = -\int_\Omega F(x,u)dx. \tag{3.4}$$

We assume that G satisfies one of the following two conditions:

(G1) There exist $M > 0$, $\epsilon \in (0,1)$ and $\beta, \theta \in L^1(\Omega)$ with $\|\beta\|_{L^1(\Omega)} \neq 0$ such that

$$\text{for any} \quad |t| \geq M \qquad G(x,t) \leq \frac{(1-\epsilon)\beta(x)}{p^+ C_0^{p^-}\|\beta\|_{L^1(\Omega)}} \mid t \mid^{p^-} + \theta(x) \quad \text{a.e. in } \Omega,$$

(G2) There exist $M > 0$, $\epsilon \in (0,1)$ and $\theta' \in L^1(\Omega)$ such that

$$\text{for any} \quad |t| \geq M \qquad G(x,t) \leq \frac{(1-\epsilon)w_0(x)}{p(x)} \mid t \mid^{p(x)} + \theta'(x) \quad \text{a.e. in } \Omega.$$

From now on, we always assume that

(V3) $w_0 \in L^1(\Omega)$.

Definition 3.1 *A measurable function $u \in W^{1,p(\cdot)}(\Omega, w_0, w_1)$ is called a weak solution of the Neumann elliptic problem $(\mathcal{P})$ if*

$$\int_\Omega w_1(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u\nabla v dx + \int_\Omega w_0(x) \mid u \mid^{p(x)-2} uv dx = \int_\Omega f(x,u)v dx + \int_\Omega g(x,u)v dx,$$

for all $v \in W^{1,p(\cdot)}(\Omega, w_0, w_1)$.

Definition 3.2 *A function $F(x,t)$ satisfies the condition (S) if for each compact subset E of $\mathbb{R}$, there exists $\xi \in E$ such that*

$$F(x,\xi) = \sup_{t\in E} F(x,t) \quad \textit{for} \quad \textit{a.e.} \quad x \in \Omega. \tag{3.5}$$

4. Main results

This section contains the statement of the main results.

Taking u_0 and u_n in Theorem 1.1 as the constant value functions ξ_0 and ξ_n, and we assume that

$$\liminf_{|\xi|\to+\infty} \int_\Omega \left(\frac{w_0(x)}{p(x)} \mid \xi \mid^{p(x)} - G(x,\xi) - F(x,\xi) \right) dx = -\infty, \tag{4.1}$$

also, we will need the following condition

$$\int_\Omega \frac{w_0(x)}{p(x)} \mid \xi \mid^{p(x)} dx - \int_\Omega G(x,\xi)dx \leq d_1 \mid \xi \mid^{p^+} + d_2, \quad \forall \xi \in \mathbb{R}, \tag{4.2}$$

where d_1 and d_2 are two positive constants. It is easy to see that (4.2) holds true under the condition $(G1)$ or $(G2)$.

The following theorem is our first main result.

Theorem 4.1 *Let the conditions $(V1) - V(3)$, (4.1) − (4.2) be satisfied, and let $(G1)$ or $(G2)$ holds, and let F satisfy the condition (S). Suppose that $\{a_n\}$ and $\{b_n\}$ are two positive sequences such that*

$$\lim_{n\to\infty} b_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n^{p^+}}{b_n^{p^-}} = 0. \tag{4.3}$$

If there exists a positive function $h \in L^1(\Omega)$ and $\|h\|_{L^1(\Omega)} \neq 0$, such that for each n we have

$$F(x,a_n) + \frac{h(x)}{\|h\|_{L^1(\Omega)}} \left(d_0 \left(\frac{b_n}{C_0} \right)^{p^-} - d_1 a_n^{p^+} - d_2 \right) \geq \sup_{t\in[a_n,b_n]} F(x,t) \qquad \textit{a.e. in} \quad \Omega, \tag{4.4}$$

$$F(x,-a_n) + \frac{h(x)}{\|h\|_{L^1(\Omega)}} \left(d_0 \left(\frac{b_n}{C_0} \right)^{p^-} - d_1 a_n^{p^+} - d_2 \right) \geq \sup_{t\in[-b_n,-a_n]} F(x,t) \qquad \textit{a.e. in} \quad \Omega, \tag{4.5}$$

and the inequalities (4.4) and (4.5) are strict on a subset of Ω with positive measure, then there exists a sequence $\{v_n\}$ of local minima of $\Psi + \Phi$ such that $\lim_{n\to\infty} \Psi(v_n) = +\infty$. Consequently, the problem $(\mathcal{P})$ admits an unbounded sequence of weak solutions.

Proof of Theorem 4.1

Step 1 : Some technical lemmas

Lemma 4.1 ([18, 26]) *Assume that $(V1), (V2)$ and (3.1) are satisfied. Then,*

$$\Psi, \Phi \in C^1(W^{1,p(\cdot)}(\Omega, w_0, w_1), \mathbb{R})$$

and its Gâteaux derivatives are given by

$$\langle \Psi'(u), v\rangle = \int_\Omega w_1(x) \Big|\nabla u\Big|^{p(x)-2} \nabla u \nabla v dx + \int_\Omega w_0(x) \mid u \mid^{p(x)-2} uv dx - \int_\Omega g(x,u)v dx,$$

and

$$\langle \Phi'(u), v\rangle = -\int_\Omega f(x,u)v dx,$$

for any $u, v \in W^{1,p(\cdot)}(\Omega, w_0, w_1)$.

Lemma 4.2 ([18, 26]) *Assume that* $(V1),(V2)$ *and* (3.1) *hold. Then* Ψ,Φ *are sequentially weakly lower semicontinuous.*

Step 2 : Coercivity of Ψ

Proposition 4.1 *Assuming that* $G(x,t)$ *satisfies* (G_1) *or* (G_2), *then the functional* Ψ *is coercive, i.e.*

$$\Psi(u) \longrightarrow +\infty \qquad as \quad \|\|u\|\|_{1,p(\cdot),\Omega,w_0,w_1} \longrightarrow \infty \qquad for \quad u \in W^{1,p(\cdot)}(\Omega,w_0,w_1).$$

Proof
(G1) Assuming that the condition $(G1)$ is satisfied, then

$$G(x,t) \leq \frac{(1-\epsilon)\beta(x)}{p^+C_0^{p^-}\|\beta\|_{L^1(\Omega)}} \mid t\mid^{p^-} + \theta_1(x), \qquad \text{a.e. in } \Omega \qquad \text{for any} \quad |t|\geq M.$$

When $\|\|u\|\|_{1,p(\cdot),\Omega,w_0,w_1} > 1$, we obtain

$$\begin{aligned}
\Psi(u) &= J(u) - \int_\Omega G(x,u)dx \\
&= \int_\Omega \frac{1}{p(x)}\left(w_1(x)\Big|\nabla u\Big|^{p(x)} + w_0(x)\mid u\mid^{p(x)}\right)dx - \int_\Omega G(x,u)dx \\
&\geq \frac{1}{p^+}\|\|u\|\|^{p^-}_{1,p(\cdot),\Omega,w_0,w_1} - \frac{(1-\epsilon)}{p^+C_0^{p^-}\|\beta\|_{L^1(\Omega)}}\int_\Omega \beta\mid u\mid^{p^-}dx - \int_\Omega \theta_1(x)dx \quad (\text{by } \textit{Lemma } 2.2) \\
&\geq \frac{1}{p^+}\|\|u\|\|^{p^-}_{1,p(\cdot),\Omega,w_0,w_1} - \frac{(1-\epsilon)}{p^+C_0^{p^-}}\|u\|^{p^-}_{L^\infty(\Omega)} - c_1 \\
&\geq \frac{1}{p^+}\|\|u\|\|^{p^-}_{1,p(\cdot),\Omega,w_0,w_1} - \frac{(1-\epsilon)}{p^+}\|\|u\|\|^{p^-}_{1,p(\cdot),\Omega,w_0,w_1} - c_1 \quad (\text{using } 2.5) \\
&\geq \frac{\epsilon}{p^+}\|\|u\|\|^{p^-}_{1,p(\cdot),\Omega,w_0,w_1} - c_1.
\end{aligned}$$

(G2) Under the condition $(G2)$ we have

$$G(x,t) \leq \frac{(1-\epsilon)w_0(x)}{p(x)} \mid t\mid^{p(x)} + \theta_1^{'}(x), \qquad \text{a.e. in } \Omega \qquad \text{for any} \quad |t|\geq M,$$

when $|\|u\||_{1,p(\cdot),\Omega,w_0,w_1} > 1$, we obtain

$$\begin{aligned}
\Psi(u) &= J(u) - \int_\Omega G(x,u)dx \\
&= \int_\Omega \frac{1}{p(x)}\left(w_0(x)\mid u\mid^{p(x)} + w_1(x)\Big|\nabla u\Big|^{p(x)}\right)dx - \int_\Omega G(x,u)dx \\
&= \int_\Omega \frac{1}{p(x)}w_1(x)\Big|\nabla u\Big|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}w_0(x)|\,u\mid^{p(x)}dx - \int_\Omega\left(\frac{(1-\epsilon)w_0(x)}{p(x)}\mid u\mid^{p(x)} + \theta_1^{'}(x)\right)dx \\
&\geq \int_\Omega \frac{1}{p(x)}w_1(x)\Big|\nabla u\Big|^{p(x)}dx + \int_\Omega \frac{\epsilon w_0(x)}{p(x)}|\,u\mid^{p(x)}dx - c_2 \\
&\geq \frac{\epsilon}{p^+}\left(\int_\Omega w_1(x)\Big|\nabla u\Big|^{p(x)}dx + \int_\Omega w_0(x)\mid u\mid^{p(x)}dx\right) - c_2 \\
&\geq \frac{\epsilon}{p^+}\|\|u\|\|^{p^-}_{1,p(\cdot),\Omega,w_0,w_1} - c_2,
\end{aligned} \tag{4.6}$$

Thanks to (4.6) − (4.6), we conclude that Ψ is coercive. Moreover, there exist two positive constants d_0 and σ_0 such that

$$\Psi(u) \geq d_0\|\|u\|\|^{p^-}_{1,p(\cdot),\Omega,w_0,w_1} \quad \text{for} \quad \|\|u\|\|_{1,p(\cdot),\Omega,w_0,w_1} \geq \sigma_0. \tag{4.7}$$

□

Step 3 : A priori estimates

For $r > \inf_{W^{1,p(\cdot)}(\Omega,w_0,w_1)} \Psi$, we define

$$K(r) = \inf\Big\{\sigma > 0 : \Psi^{-1}(]-\infty, r[) \subset \overline{B_{W^{1,p(\cdot)}(\Omega,w_0,w_1)}(0,\sigma)}\Big\}, \tag{4.8}$$

where

$$B_{W^{1,p(\cdot)}(\Omega,w_0,w_1)}(0,\sigma) = \Big\{u \in W^{1,p(\cdot)}(\Omega,w_0,w_1) : |||u|||_{1,p(\cdot),\Omega,w_0,w_1} < \sigma\Big\},$$

and $\overline{B_{W^{1,p(\cdot)}(\Omega,w_0,w_1)}(0,\sigma)}$ denotes the closure of $B_{W^{1,p(\cdot)}(\Omega,w_0,w_1)}(0,\sigma)$ in $W^{1,p(\cdot)}(\Omega,w_0,w_1)$ with respect to the norm topology.
We have Ψ is coercive, then $0 < K(r) < +\infty$ for each $r > \inf_{W^{1,p(\cdot)}(\Omega,w_0,w_1)} \Psi$. In view of (4.7), we obtain

$$\Psi(u) < d_0 |||u|||^{p^-}_{1,p(\cdot),\Omega,w_0,w_1} \quad \Longrightarrow \quad |||u|||_{1,p(\cdot),\Omega,w_0,w_1} < \sigma_0.$$

Thanks to (4.8), we have
$\Psi^{-1}(]-\infty, r[) \subset \overline{B_{W^{1,p(\cdot)}(\Omega,w_0,w_1)}(0,K(r))}$ then $\overline{(\Psi^{-1}(]-\infty, r[))} \subset \overline{B_{W^{1,p(\cdot)}(\Omega,w_0,w_1)}(0,K(r))}$, and using (2.5), we get $\|u\|_{L^\infty(\Omega)} \leq C_0 |||u|||_{1,p(\cdot),\Omega,w_0,w_1}$ then

$$\overline{B_{W^{1,p(\cdot)}(\Omega,w_0,w_1)}(0,K(r))} \subset \big\{u \in C(\overline{\Omega}) : \|u\|_{L^\infty(\Omega)} \leq C_0 K(r)\big\}.$$

It follows that

$$\inf_{\overline{v \in (\Psi^{-1}(]-\infty,r[))}} \Phi(v) \geq \inf_{|||u|||_{1,p(\cdot),\Omega,w_0,w_1} \leq K(r)} \Phi(v) \geq \inf_{\|v\|_{L^\infty(\Omega)} \leq C_0 K(r)} \Phi(v). \tag{4.9}$$

By taking u_0 and u_n as constant value functions ξ_0 and ξ_n in Theorem 1.1 and using (4.9), we conclude the following Theorem 4.2, that relies on Theorem 1.1.

Theorem 4.2 *Let the conditions $(V1)$ and $(V2)$ be satisfied. Suppose that Ψ and Φ are as in (3.4), Φ is coercive, and $K(r)$ is as in (4.8).*
(a) *If there exist $\rho_0 > \inf_{W^{1,p(\cdot)}(\Omega,w_0,w_1)} \Psi$ and $\xi_0 \in \mathbb{R}$ such that*

$$\int_\Omega \frac{w_0(x)}{p(x)} \mid \xi_0 \mid^{p(x)} dx - \int_\Omega G(x,\xi_0)dx := e_0 < \rho_0, \tag{4.10}$$

and

$$\int_\Omega F(x,\xi_0)dx + (\xi_0 - e_0) > \sup_{v \in C(\overline{\Omega}), \|v\|_{L^\infty(\Omega)} \leq C_0 K(\rho_0)} \int_\Omega F(x, v(x))dx, \tag{4.11}$$

then the restriction of $\Psi + \Phi$ to $\Psi^{-1}(]-\infty, \rho_0[)$ has a global minimum.
(b) *If there exist a sequence $\{r_n\} \subset \Big(\inf_{W^{1,p(\cdot)}(\Omega,w_0,w_1)} \Psi, +\infty\Big)$ with $\lim_{n\to\infty} r_n \to +\infty$ and a sequence $\{\xi_n\} \subset \mathbb{R}$ such that for each n we have*

$$\int_\Omega \frac{w_0(x)}{p(x)} \mid \xi_n \mid^{p(x)} dx - \int_\Omega G(x,\xi_n)dx := e_n < r_n \tag{4.12}$$

and

$$\int_\Omega F(x,\xi_n)dx + (r_n - e_n) > \sup_{v \in C(\overline{\Omega}), \|v\|_{L^\infty(\Omega)} \leq C_0 K(r_n)} \int_\Omega F(x, v(x))dx, \tag{4.13}$$

and in addition (4.1) holds, then there exists a sequence $\{v_n\}$ of local minima of $\Psi + \Phi$ such that $\lim_{n\to\infty} \Psi(v_n) \to +\infty$.

(c) *If there exist a sequence* $\{r_n\} \subset \left(\inf_{W^{1,p(\cdot)}(\Omega,w_0,w_1)}, +\infty \right)$ *with* $\lim_{n\to\infty} r_n = \inf_{u\in W^{1,p(\cdot)}(\Omega,w_0,w_1)} \Psi(u)$ *and a sequence* $\{\xi_n\} \subset \mathbb{R}$ *such that for each* n*, the conditions (4.12) and (4.13) are satisfied, and in addition, the condition (1.8) is satisfied, then there exists a sequence* $\{v_n\}$ *of pairwise distinct local minima of* $\Psi + \Phi$ *such that* $\lim_{n\to\infty} \Psi(v_n) = \inf_{u\in W^{1,p(\cdot)}(\Omega,w_0,w_1)} \Psi(u)$*, [i.e, the sequence* $\{v_n\}$ *converges weakly to the global minimizer of* Ψ*].*

Proof Using (4.10), if there exist $\rho_0 > \inf_{u\in W^{1,p(\cdot)}(\Omega,w_0,w_1)} \Psi(u)$ and $\xi_0 \in \mathbb{R}$ such that

$$\int_\Omega \frac{w_0(x)}{p(x)} |\xi_0|^{p(x)}\, dx - \int_\Omega G(x,\xi_0)\, dx := e_0 < \rho_0 \quad \Longrightarrow \quad \Psi(\xi_0) < \rho_0,$$

therefore (1.3) holds.
Thanks to (4.11), we have

$$\int_\Omega F(x,\xi_0)\, dx + (\rho_0 - e_0) > \sup_{v\in C(\overline{\Omega}), \|v\|_{L^\infty(\Omega)} \le C_0 K(\rho_0)} \int_\Omega F(x,v)\, dx,$$

then

$$\rho_0 - \Psi(\xi_0) > -\int_\Omega F(x,\xi_0)\, dx + \sup_{v\in C(\overline{\Omega}), \|v\|_{L^\infty(\Omega)} \le C_0 K(\rho_0)} -\Phi(v).$$

Thanks to (4.9), we get

$$\rho_0 - \Psi(\xi_0) > \Phi(\xi_0) - \inf_{v\in \overline{\Psi^{-1}(]-\infty,\rho_0[)}} \Phi(v).$$

Therefore, the hypotheses (1.3) and (1.4) of Theorem 1.1(a) are satisfied. Then the restriction of $\Psi + \Phi$ to $\Psi^{-1}(]-\infty,\rho_0[)$ has a global minimum. Assuming that the hypotheses of Theorem 1.1(b) and Theorem 1.1(c) are satisfied, using the same approach we can conclude the proof Theorem 4.2. □

For the condition (4.11) in Theorem 4.2(a), we give the following proposition.

Proposition 4.2 *Assume that* $\rho_0 > \inf_{W^{1,p(\cdot)}(\Omega,w_0,w_1)} \Psi$*,* $\xi_0 \in \mathbb{R}$ *and (4.10) holds. If there exists a positive function* $\alpha \in L^1(\Omega)$ *with* $\|\alpha\|_{L^1(\Omega)} \ne 0$ *such that*

$$F(x,\xi_0) + \frac{\alpha(x)}{\|\alpha\|_{L^1(\Omega)}}(\rho_0 - e_0) > \sup_{|t|\le C_0 K(\rho_0)} F(x,t) \quad \textit{for a.e. } x\in\Omega, \tag{4.14}$$

and the inequality (4.14) is strict on a subset of Ω *with positive measure, then (4.11) holds.*

Proof Integrating (4.14) over Ω and noting that

$$\int_\Omega \sup_{|t|\le C_0 K(\rho_0)} F(x,t) dx \ge \sup_{v\in C(\overline{\Omega}), \|v\|_{L^\infty(\Omega)} \le C_0 K(\rho_0)} \int_\Omega F(x,v(x)) dx,$$

we obtain (4.11). □

The following proposition plays a crucial role in obtaining the main result in this section.

Proposition 4.3 *Assume that* Ψ *is coercive and (4.7) holds, for* $r \ge d_0 \sigma_0^{p^-}$ *we have*

$$K(r) \le \left(\frac{r}{d_0}\right)^{\frac{1}{p^-}}. \tag{4.15}$$

Proof Let $r \geq d_0 \sigma_0^{p^-}$ and $u \in W^{1,p(\cdot)}(\Omega, w_0, w_1)$ be such that $\Psi(u) < r$. When $|||u|||_{1,p(\cdot),\Omega,w_0,w_1} \geq \sigma_0$, by (4.7), one has

$$r > \Psi(u) \geq d_0 |||u|||_{1,p(\cdot),\Omega,w_0,w_1}^{p^-},$$

which implies that $|||u|||_{1,p(\cdot),\Omega,w_0,w_1} \leq \left(\frac{r}{d_0}\right)^{\frac{1}{p^-}}$.

When $|||u|||_{1,p(\cdot),\Omega,w_0,w_1} < \sigma_0$, it is clear that $|||u|||_{1,p(\cdot),\Omega,w_0,w_1} \leq \left(\frac{r}{d_0}\right)^{\frac{1}{p^-}}$. Using the definition of $K(r)$, we conclude (4.15). □

Step 4 : Proof of the statements (4.12) and (4.13)

We set $r_n = d_0 \left(\frac{b_n}{C_0}\right)^{p^-}$, then $\lim_{n\to\infty} r_n \to +\infty$, and thanks to (4.15) we obtain

$$C_0 K(r_n) \leq b_n. \tag{4.16}$$

Since F satisfies the condition (S), for each n, there exists $\xi_n \in [-a_n, a_n]$ such that

$$F(x, \xi_n) = \sup_{t\in[-a_n,a_n]} F(x,t), \text{for a.e.} x \in \Omega. \tag{4.17}$$

By (4.2), one has

$$\begin{aligned} e_n &= \int_\Omega \frac{w_0(x)}{p(x)} \mid \xi_n \mid^{p(x)} dx - \int_\Omega G(x,\xi_n) dx \\ &\leq d_1 \mid \xi_n \mid^{p^+} + d_2 \leq d_1 \mid a_n \mid^{p^+} + d_2. \end{aligned}$$

It follows from (4.3) that for n sufficiently large,

$$d_1 \mid a_n \mid^{p^+} + d_2 < d_0 \left(\frac{b_n}{C_0}\right)^{p^-} = r_n,$$

and consequently $e_n < r_n$, that is (4.12) holds. Without loss of generality, we may assume that for all n, (4.12) holds. By combining (4.4)-(4.5) and (4.17), we obtain

$$F(x,\xi_n) + \frac{h(x)}{\|h\|_{L^1(\Omega)}}(r_n - e_n) \geq \sup_{|t|\leq b_n} F(x,t) \quad \text{for a.e. } x \in \Omega, \tag{4.18}$$

and the inequality (4.18) is strict on a subset of Ω with positive measure. Using (4.16) and the Proposition 4.2, we obtain (4.13).

Therefore, all hypotheses of Theorem 4.2 (b) are satisfied, then the proof of the Theorem 4.1 is concluded. □

Our second main result is the following theorem

Theorem 4.3 *Assume that $(V1) - V(3)$ hold. Suppose that*

$$G(x,t) \leq 0 \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad \text{a.e.} \quad x \in \Omega, \tag{4.19}$$

there exist two positive constants M and ϵ such that

$$-G(x,t) \leq M \mid t \mid^{p^-} \quad \text{for} \quad t \leq \epsilon \quad \text{and} \quad \text{a.e.} \quad x \in \Omega, \tag{4.20}$$

The functional F satisfies the condition (S) and

$$\limsup_{|\xi|\to 0} \frac{\displaystyle\int_\Omega F(x,\xi)dx + \int_\Omega G(x,\xi)dx}{\mid \xi \mid^{p^-}} > \int_\Omega \frac{w_0(x)}{p(x)} dx. \tag{4.21}$$

Suppose that $\{a_n\}$ *and* $\{b_n\}$ *be two positive sequences such that*

$$\lim_{n\to\infty} b_n = 0 \quad and \quad \lim_{n\to\infty} \frac{a_n^{p^-}}{b_n^{p^+}} = 0, \tag{4.22}$$

and there exists a positive function $h \in L^1(\Omega)$ *with* $\|h\|_{L^1(\Omega)} \neq 0$, *such that for each* n *we have*

$$F(x, a_n) + \frac{h(x)}{\|h\|_{L^1(\Omega)}} \left(\frac{1}{p^+} \left(\frac{b_n}{C_0} \right)^{p^+} - d_3 a_n^{p^-} \right) \geq \sup_{t\in[a_n,b_n]} F(x,t) \qquad a.e. \ in \quad \Omega, \tag{4.23}$$

$$F(x, -a_n) + \frac{h(x)}{\|h\|_{L^1(\Omega)}} \left(\frac{1}{p^+} \left(\frac{b_n}{C_0} \right)^{p^+} - d_3 a_n^{p^-} \right) \geq \sup_{t\in[-b_n,-a_n]} F(x,t) \qquad a.e. \ in \quad \Omega, \tag{4.24}$$

and the inequalities (4.23) *and* (4.24) *are strict on a subset of* Ω *with positive measure, where* $d_3 = \int_\Omega \frac{w_0(x)}{p(x)} dx + M \mid \Omega \mid$. *Then there exists a sequence* $\{v_n\}$ *of pairwise distinct local minima of* $\Psi + \Phi$ *such that* $v_n \to 0$ *in* $W^{1,p(\cdot)}(\Omega, w_0, w_1)$ *and consequently, the problem* $(\mathcal{P})$ *admits a sequence of nonzero weak solutions which strongly converges to* 0 *in* $W^{1,p(\cdot)}(\Omega, w_0, w_1)$.

Proof of Theorem 4.3

Let us verify all the hypotheses of Theorem 4.2 (c). Using (4.19), for $|||u|||_{1,p(\cdot),\Omega,w_0,w_1} < 1$, we have

$$\begin{aligned} \Psi(u) &= J(u) - \int_\Omega G(x,u)dx \\ &= \int_\Omega \frac{1}{p(x)} \left(w_1(x) \Big| \nabla u \Big|^{p(x)} + w_0(x) \mid u \mid^{p(x)} \right) dx \\ &\geq \frac{1}{p^+} |||u|||^{p^+}_{1,p(\cdot),\Omega,w_0,w_1}. \end{aligned}$$

Then Ψ is coercive, $\inf_{W^{1,p(\cdot)}(\Omega,w_0,w_1)} \Psi = \Psi(0) = 0$ and 0 is the unique global minimizer of Ψ. Thanks to (4.21), we have

$$\begin{aligned} \limsup_{|\xi|\to 0} \left\{ \psi(\xi) + \Phi(\xi) \right\} &= \limsup_{|\xi|\to 0} \left\{ \int_\Omega \frac{w_0(x)}{p(x)} |\xi|^{p(x)}\, dx - \int_\Omega G(x,\xi)\, dx - \int_\Omega F(x,\xi)\, dx \right\} \\ &\leq \limsup_{|\xi|\to 0} \left\{ \int_\Omega \frac{w_0(x)}{p(x)} |\xi|^{p^-}\, dx - \int_\Omega G(x,\xi)\, dx - \int_\Omega F(x,\xi)\, dx \right\} < 0, \end{aligned}$$

then 0 is not a local minimizer of $\Psi + \Phi$, so (1.8) is satisfied.
For $r > 0$ sufficiently small, the condition $\Psi(u) < r$ implies that $|||u|||_{1,p(\cdot),\Omega,w_0,w_1} < (p^+ r)^{\frac{1}{p^+}}$, this shows that $K(r) \leq (p^+ r)^{\frac{1}{p^+}}$. Now put $r_n = \frac{1}{p^+} \left(\frac{b_n}{C_0} \right)^{p_+}$. Then $C_0 K(r_n) \leq b_n$. by (4.20), there exists a sequence $\{\xi_n\} \subset \mathbb{R}$ with $\xi_n \in [-a_n, a_n]$ such that for $\mid \xi_n \mid$ sufficiently small,

$$\begin{aligned} e_n &= \int_\Omega \frac{w_0(x)}{p(x)} |\xi_n|^{p(x)}\, dx - \int_\Omega G(x,\xi_n)\, dx \\ &\leq \left(\int_\Omega \frac{w_0(x)}{p(x)}\, dx + M|\Omega| \right) |\xi_n|^{p^-} \\ &= d_3 |\xi_n|^{p^-} \\ &= d_3 |a_n|^{p^-}. \end{aligned} \tag{4.25}$$

It follows from (4.22) that for n large enough,

$$d_3 |a_n|^{p^-} < \frac{1}{p^+} \left(\frac{b_n}{C_0} \right)^{p^+} = r_n,$$

and consequently $e_n < r_n$, that is (4.12) holds. Noting that F satisfies the condition (S), then thanks to (4.23) - (4.24) and (4.17) we can obtain that

$$F(x,\xi_n) + \frac{h(x)}{\|h\|_{L^1(\Omega)}}(r_n - e_n) \geq \sup_{|t|\leq b_n} F(x,t) \ \text{ a.e. in } \ \Omega, \tag{4.26}$$

and the inequality (4.26) is strict on a subset of Ω with positive measure. By Proposition 4.2 and (4.26) implies (4.13). Therefore, all hypotheses of Theorem 4.2 (c) are satisfied.
Consequently, there exists a sequence $\{v_n\}$ of pairwise distinct local minima of $\Psi+\Phi$ such that $\Psi(v_n) \to 0$, thus $|||u|||_{1,p(\cdot),\Omega,w_0,w_1} \longrightarrow 0$, which complete our proof. □

5. Conclusion and perspective

Through this paper, we have studied the existence of infinitely many weak solutions of a nonlinear elliptic partial differential equation of Neumann type in the weighted variable exponent Sobolev space, and we have shown the embedding $W^{1,p(\cdot)}(\Omega, w_0, w_1) \hookrightarrow\hookrightarrow C^0(\overline{\Omega})$, without assuming any condition on N and without using the log-Hölder continuity and by using the theory of critical points obtained by B. Ricceri, as a consequence of a more general variational principle.

So we are aware of a lot of open questions about this works for example the question of uniqueness, with totally different conditions, is very important and remains as an open question, therefore our future works will be devoted to this question. On the other hand, we will try to show the existence of infinitely many weak solutions for the problem $(\mathcal{P})$ in weighted Orlicz-Sobolev space and Musielak-Orlicz-Sobolev space.

Conflict of Interest statement

The authors certify that they have No Conflict of Interest in the subject matter or materials discussed in this manuscript.

References

1. A. Ahmed, T. Ahmedatt, H. Hjiaj and A. Touzani, *Existence of Infinitely Many Solutions for some Quasilinear $\vec{p}(x)$-Elliptic Neumann Problems*, Math. Bohem. 142 (2017), no. 3, 243-262.
2. A. Ahmed, Y. Akdim and A. Touzani, *Existence of solutions of the Neumann problem for a class of equations involving the p-Laplacian in Weighted Sobolev Spaces*, Applied Mathematical Sciences, Vol. 10, 2016, no. 40, 1953-1970.
3. A. Ahmed, H. Hjiaj and A. Touzani, *Existence of infinitely many weak solutions for a Neumann elliptic equations involving the $\vec{p}(x)$-laplacian operator*, Rend. Circ. Mat. Palermo (2). 64 (2015), no. 3, 459-473.
4. Y. Akdim, E. Azroul, and A. Benkirane, *Existence of solutions for quasilinear degenerated elliptic equations*, Electronic J. Diff. Eqns., Vol. 2001 No. 71, (2001) pp 1-19.
5. G. Anello, G. Cordaro, *An existence theorem for the Neumann problem involving the p-Laplacian*, J. Convex. Anal. 10 (2003) 185-198.
6. S.N. Antontsev, S.I. Shmarev, *A model porous medium equation with variable exponent of nonlinearity: Existence, uniqueness and localization properties of solutions*, Nonlinear Anal. 60 (2005) 515-545.
7. S.N. Antontsev, J.F. Rodrigues, *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara Sez. VII Sci. Mat. 52 (2006) 19-36.
8. M.-M. Boureanu, V.D. Radulescu, *Anisotropic Neumann problems in Sobolev spaces with variable exponent*, Nonlinear Analysis (2011), doi:10.1016/j.na.2011.09.033.
9. M. -M. Boureanu, C. Udrea and D. N. Udrea, *Anisotropic problems with variable exponents and constant Dirichlet condition*, Electron. J. Diff. Equ., 2013 (2013), 1-13.
10. Y. Chen, S. Levine, M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. 66 (4) (2006) 1383-1406.
11. L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Heidelberg, 2011.
12. P. Drabek, A.kufner, and F, Nicolosi, *Nonlinear elliptic equations, singular and degenerate cases*, preprint, 1996, University of West Bohemia.
13. X. L. Fan, *Solutions for $p(x)$-Laplacian Dirichlet problems with Singular coefficients*, J. Math. Anal. Appl. 312 (2005) 464-477.

14. X.L. Fan, C.Ji, *Existence of infinitely many solutions for a Neumann problem involving the $p(x)$-Laplacian*, C. J. Math. Anal. Appl. 334 (2007) 248-260.

15. X.L. Fan and D. Zhao, *On the generalised Orlicz-Sobolev Space $W^{k,p(x)}(\Omega)$*, J. Gansu Educ. College 12 (1) (1998) 1-6.

16. X. L. Fan, D. Zhao, *On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. 263 (2001) 424-446.

17. P. Harjulehto and P. Hästö, *Sobolev Inequalities for Variable Exponents Attaining the Values* 1 *and* n, Publ. Mat., 52 (2008), no. 2, 347-363.

18. K. Ho, I. Sim, *Existence and some properties of solutions for degenerate elliptic equations with exponent variable*, Department of Mathematics, University of Ulsan, Nonlinear Analysis 98 (2014) 146-164.

19. Y.H. Kim, L. Wang, C. Zhang, *Global bifurcation of a class of degenerate elliptic equations with variable exponents*, J. Math. Anal. Appl. 371 (2010) 624-637.

20. O. Kováčik, J. Rákosník, *On spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, Czechoslovak Math. J. 41 (1991) 592-618.

21. Q. Liu and D. Liu, *Existence and multiplicity of solution to a $p(x)$-laplacian equation with nonlinear boundaey condition on unbounded domain*, Differ. Equ. Appl. 5 (2013), 595-611.

22. Q. Liu, *Compact Trace in Weighted Variable Exponent Sobolev Spaces $W^{1,p(x)}(\Omega; v_0, v_1)$*, J. Math. Anal. Appl., 348 (2008), 760-774.

23. B. Ricceri, *A general variational principle and some of its applications*, J. Comput. Appl.Math. 113 (2000) 401-410.

24. B. Ricceri, *Infinitely many solutions of the Neumann problem for elliptic equations involving the p-Laplacian*, Bull. London Math. Soc. 33 (2001) 331-340.

25. M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory, Springer-Verlag*, Berlin, 2000.

26. I. Sim, Y.H. Kim, *Existence of solutions and positivity of the infimum eigenvalue for degenerate elliptic equations with variable exponents*, DCDS (2013) in press.

27. V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR Izv. 9 (1987) 33-66.

28. V.V. Zhikov, *On the density of smooth functions in Sobolev-Orlicz spaces*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 226 (2004), 67-81.

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