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Intuitionistic fuzzy matrix equations

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ABSTRACT: The aim of this paper is to investigate an intuitionistic fuzzy matrix equations. The existence conditions of an intuitionistic fuzzy solution are given and also a method for computing the solution is derived. Finally some examples are presented to illustrate the effectiveness of the presented method.

Key Words: Intuitionistic fuzzy number, intuitionistic fuzzy matrix equations, inverse matrix.

Contents

1. Introduction

In 1965 Zadeh [23] proposed the concept of fuzzy sets with the purpose to model ambiguity, uncertainty, and vagueness in complicated systems. It can be considered as an extension of the usual (crisp) set theory. It has greater pliability to capture different aspects of incompleteness, imperfection and uncertainty in data about many situations. The membership μ of an element of a fuzzy set is a single value between 0 and 1. Therefore in reality, it may not always be true that the degree of non-membership ν of an element in a fuzzy set is equal to 1 minus the membership degree (1-μ) because there may be some uncertainty degree. Thus, since the fuzzy set has no means to characterize the neutral state, neither support nor opposition, then Atanassov [6] included the non-membership function and defined the degree of uncertainty as 1-μ-ν. He introduced the topic of intuitionistic fuzzy sets (IFSs) as an extension of the standard fuzzy sets [7]. Several applications of IFS theory in diverse fields have been carried out, and some recent such works can be found in [3,11,13,14,15,20,21,22].

Fuzzy linear systems (FLS) arise in many branches of science and technology such as economics, social sciences, telecommunications, image processing etc. [1,5,17]. In actual case the parameters may be uncertain or a vague estimation about the variables are known as those are found in general by some observation, experiment or experience. So, to overcome the uncertainty and vagueness, one may use the fuzzy numbers in place of the crisp numbers. The solution of FLS was first introduced in Friedman et al. [12] by proposing a general method for solving an n system which includes a crisp coefficients matrix with an arbitrary fuzzy number vector as right-hand side. The fuzzy system of equations were investigated by various authors using different approaches. Since then, many methods were proposed to solve FLS computationally [2,4,18,19]. Usually, in many applications some of the parameters in the problems are represented by intuitionistic fuzzy number rather than crisp, and hence it is important to develop mathematical models and numerical procedures that would appropriately treat intuitionistic fuzzy linear systems and solve them. An intuitionistic fuzzy system of linear equations may be written as $AX = b$, where, A crisp real matrix, b is intuitionistic fuzzy vector and X is unknown intuitionistic fuzzy vector. The main objective of intuitionistic fuzzy linear system to widen the scope of this system

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in scientific applications by removing the crispness assumption on the entries of coefficient matrix. A few of the literatures are reviewed for the sake of completeness of the problem. Solution of system of linear equations and dual linear systems involving intuitionistic fuzzy input parameters has been proposed by [8,10]. Computational methods for solving intuitionistic fuzzy linear systems have been discussed as well [9]. In this paper we present a general method for solving an intuitionistic fuzzy matrix equation witch we inspired by previous works.

The paper is organized in this way: In Sect.2 the main properties and definitions are introduced. Sect.3 is devoted to presenting a key result for this work, which gives a method for solving an intuitionistic fuzzy matrix equation and suitable assumptions to ensure the existence of an intuitionistic fuzzy solution and a method for computing this solution. In Sect.4, illustrative computational examples are showed. Finally Sect.5 offers the conclusion and suggest brief future research.

2. Preliminaries

Definition 2.1 *An intuitionistic fuzzy set (IFS) A in X is defined as an object of the following form*

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$$

where

- $\mu_A : X \to [0,1]$ degree of membership.
- $\nu_A : X \to [0,1]$ degree of non-membership.

and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for every $x \in X$.
$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the degree of non-determinacy (or uncertainty) of the element $x \in X$ to the intuitionistic fuzzy set A and $\pi_A(x) \in [0,1]$.
We denote by:

$$\mathrm{IF}_1 = \mathrm{IF}(\mathbb{R}) = \left\{ \langle u, v \rangle \ : \ \mathbb{R} \to [0,1]^2 , \mid \forall\, x \in \mathbb{R} \ \ 0 \leq u(x) + v(x) \leq 1 \right\}$$

An element $\langle u, v \rangle$ of IF_1 is said an intuitionistic fuzzy number if it satisfies the following conditions:

(i) $\langle u, v \rangle$ is normal i.e there exists x_0, $x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) u is fuzzy convex and v is fuzzy concave.

(iii) u is upper semi-continuous and v is lower semi-continuous.

(iv) $supp\, \langle u, v \rangle = \mathrm{cl}\{x \in \mathbb{R} \ : \mid v(x) < 1\}$ is bounded.

So we denote the collection of all intuitionistic fuzzy number by IF_1.
A Triangular Intuitionistic Fuzzy Number (TIFN) $\langle u, v \rangle$ is an intuitionistic fuzzy set in $\mathbb{R}$ with the following membership function u and non-membership function v :

$$u(y) = \begin{cases} \dfrac{y - a_1}{a_2 - a_1} & \text{if } a_1 \leq y \leq a_2 \\ \dfrac{a_3 - y}{a_3 - a_2} & \text{if } a_2 \leq y \leq a_3, \\ 0 & \text{otherwise} \end{cases}$$

$$v(y) = \begin{cases} \dfrac{a_2 - y}{a_2 - a_1'} & \text{if } a_1' \leq y \leq a_2 \\ \dfrac{y - a_2}{a_3' - a_2} & \text{if } a_2 \leq y \leq a_3', \\ 1 & \text{otherwise.} \end{cases}$$

where $a_1^{'} \leq a_1 \leq a_2 \leq a_3 \leq a_3^{'}$.
We define $0_{(1,0)} \in \mathrm{IF}_1$ as

$$0_{(1,0)}(t) = \begin{cases} (1,0) & t = 0 \\ (0,1) & t \neq 0 \end{cases}$$

Definition 2.2 *[8] An intuitionistic fuzzy number x in parametric form is a pair $x = \Big((\underline{x}^+, \overline{x^+}), (\underline{x}^-, \overline{x}^-)\Big)$ of functions $\underline{x}^-(r)$, $\overline{x}^-(r)$, $\underline{x}^+(r)$ and $\overline{x}^+(r)$ which satisfies the following requirements:*

1. *$\underline{x}^+(r)$ is a bounded monotonic increasing left continuous function,*
2. *$\overline{x}^+(r)$ is a bounded monotonic decreasing left continuous function,*
3. *$\underline{x}^-(r)$ is a bounded monotonic increasing left continuous function,*
4. *$\overline{x}^-(r)$ is a bounded monotonic decreasing left continuous function,*
5. *$\underline{x}^-(r) \leq \overline{x}^-(r)$ and $\underline{x}^+(r) \leq \overline{x}^+(r)$, for all $0 \leq r \leq 1$.*

This TIFN is denoted by $x = \langle a_1, a_2, a_3; a_1^{'}, a_2, a_3^{'} \rangle$.
Its parametric form is:

$$\begin{aligned} \underline{x}^+(r) &= a_1 + r(a_2 - a_1),\ \overline{x}^+(r) = a_3 - r(a_3 - a_2) \\ \underline{x}^-(r) &= a_1' + r(a_2 - a_1'),\ \overline{x}^-(r) = a_3' - r(a_3' - a_2) \end{aligned}$$

For two intuitionistic fuzzy numbers $x = \big(\underline{x}^+(r), \overline{x}^+(r), \underline{x}^-(r), \overline{x}^-(r)\big)$ and $y = \big(\underline{y}^+(r), \overline{y}^+(r), \underline{y}^-(r), \overline{y}^-(r)\big)$ and $k \in \mathbb{R}$, the equality, the addition and scaler-multiplication are defined as follows:

- $x = y$ if and only if $\underline{x}^+(r) = \underline{y}^+(r)$, $\overline{x}^+(r) = \overline{y}^+(r)$, $\underline{x}^-(r) = \underline{y}^-(r)$ and $\overline{x}^-(r) = \overline{y}^-(r)$
- $x + y = \big(\underline{x}^+(r) + \underline{y}^+(r), \overline{x}^+(r) + \overline{y}^+(r), \underline{x}^-(r) + \underline{y}^-(r), \overline{x}^-(r) + \overline{y}^-(r)\big)$
- $$kx = \begin{cases} (k\underline{x}^+(r), k\overline{x}^+(r), k\underline{x}^-(r), k\overline{x}^-(r)) & \text{si } 0 \leq k \\ (k\overline{x}^+(r), k\underline{x}^+(r), k\overline{x}^-(r), k\underline{x}^-(r)) & \text{si } k < 0 \end{cases}$$

3. Intuitionistic fuzzy matrix equation

Definition 3.1 *The matrix system*

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{p1} & \dots & x_{nn} \end{pmatrix} = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix} \tag{3.1}$$

where a_{ij}, $1 \leq i, j \leq n$ are real numbers, $b_{ik} \in IF_1$, $1 \leq i, k \leq n$ and the unknown $x_{jk} \in IF_1$, $1 \leq j, k \leq n$, is called an intuitionistic fuzzy linear matrix equations(IFLMEs).
Using matrix notation, we have

$$AX = B.$$

Definition 3.2 *An intuitionistic fuzzy matrix $X = \big(X_1, X_2, ..., X_n\big)$ given by $X_k = \big(X_{1k}, X_{2k}, ..., X_{nk}\big)^T$ and $X_{ik} = \big(\underline{x_{ik}^+}(r), \overline{x_{ik}^+}(r), \underline{x_{ik}^-}(r), \overline{x_{ik}^-}(r)\big)$, $1 \leq i, k \leq n$, $0 \leq r \leq 1$, is called solution of 3.1 if:*

$$AX_k = B_k, \quad 1 \leq k \leq n,$$

where $B_k = \big(B_{1k}, B_{2k}, ..., B_{nk}\big)^T$ and $B_{ik} = \big(\underline{b_{ik}^+}(r), \overline{b_{ik}^+}(r), \underline{b_{ik}^-}(r), \overline{b_{ik}^-}(r)\big)$, $1 \leq i, k \leq n$, $0 \leq r \leq 1$.

Theorem 3.1 *The intuitionistic fuzzy matrix equation can be extanded into two crisp matrix equations:*

$$\begin{pmatrix} s_{11} & \cdots & s_{1,2n} \\ \vdots & & \vdots \\ s_{2n,1} & \cdots & s_{2n,2n} \end{pmatrix} \begin{pmatrix} \underline{x}^+_{11} & \cdots & \underline{x}^+_{1n} \\ \vdots & & \vdots \\ \underline{x}^+_{n1} & \cdots & \underline{x}^+_{nn} \\ -\overline{x}^+_{11} & \cdots & -\overline{x}^+_{1n} \\ \vdots & & \vdots \\ -\overline{x}^+_{n1} & \cdots & -\overline{x}^+_{nn} \end{pmatrix} = \begin{pmatrix} \underline{b}^+_{11} & \cdots & \underline{b}^+_{1n} \\ \vdots & & \vdots \\ \underline{b}^+_{n1} & \cdots & \underline{b}^+_{nn} \\ -\overline{b}^+_{11} & \cdots & -\overline{b}^+_{1n} \\ \vdots & & \vdots \\ -\overline{b}^+_{n1} & \cdots & -\overline{b}^+_{nn} \end{pmatrix}$$

and

$$\begin{pmatrix} s_{11} & \cdots & s_{1,2n} \\ \vdots & & \vdots \\ s_{2n,1} & \cdots & s_{2n,2n} \end{pmatrix} \begin{pmatrix} \underline{x}^-_{11} & \cdots & \underline{x}^-_{1n} \\ \vdots & & \vdots \\ \underline{x}^-_{n1} & \cdots & \underline{x}^-_{nn} \\ -\overline{x}^-_{11} & \cdots & -\overline{x}^-_{1n} \\ \vdots & & \vdots \\ -\overline{x}^-_{n1} & \cdots & -\overline{x}^-_{nn} \end{pmatrix} = \begin{pmatrix} \underline{b}^-_{11} & \cdots & \underline{b}^-_{1n} \\ \vdots & & \vdots \\ \underline{b}^-_{n1} & \cdots & \underline{b}^-_{nn} \\ -\overline{b}^-_{11} & \cdots & -\overline{b}^-_{1n} \\ \vdots & & \vdots \\ -\overline{b}^-_{n1} & \cdots & -\overline{b}^-_{nn} \end{pmatrix},$$

where s_{ij}, $1 \le i, j \le 2n$, *are determined by:*
if $a_{ij} \ge 0$, *then* $s_{ij} = a_{ij}$ *and* $s_{n+i,n+j} = a_{ij}$,
if $a_{ij} < 0$, *then* $s_{i,j+n} = -a_{ij}$ *and* $s_{n+i,j} = -a_{ij}$,
otherwise $s_{ij} = 0$.

Proof: The original system can be rewritten in the block forms of matrix as follows:

$$A(X_1, ..., X_n) = (B_1, ..., B_n),$$

where B_j and X_j are respectively the jth column of matrix B and the jth column of the unknown matrix X.

Let $X_{ik} = \big(\underline{x}^+_{ik}(r), \overline{x}^+_{ik}(r), \underline{x}^-_{ik}(r), \overline{x}^-_{ik}(r)\big)$, $1 \le i, k \le n$, $0 \le r \le 1$, and $B_k = \big(B_{1k}, B_{2k}, ..., B_{nk}\big)^T$ and $B_{ik} = \big(\underline{b}^+_{ik}(r), \overline{b}^+_{ik}(r), \underline{b}^-_{ik}(r), \overline{b}^-_{ik}(r)\big)$, $1 \le i, k \le n$, $0 \le r \le 1$.
Then we get, for $1 \le k \le n$:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \big(\underline{x}^+_{1k}(r), \overline{x}^+_{1k}(r), \underline{x}^-_{1k}(r), \overline{x}^-_{1k}(r)\big) \\ \vdots \\ \big(\underline{x}^+_{nk}(r), \overline{x}^+_{nk}(r), \underline{x}^-_{nk}(r), \overline{x}^-_{nk}(r)\big) \end{pmatrix} = \begin{pmatrix} \big(\underline{b}^+_{1k}(r), \overline{b}^+_{1k}(r), \underline{b}^-_{1k}(r), \overline{b}^-_{1k}(r)\big) \\ \vdots \\ \big(\underline{b}^+_{nk}(r), \overline{b}^+_{nk}(r), \underline{b}^-_{nk}(r), \overline{b}^-_{nk}(r)\big) \end{pmatrix}.$$

Let $A = E + F$ where the elements e_{ij} of matrix E and the elements f_{ij} of matrix F are determined as follows:
if $a_{ij} \ge 0$, $e_{ij} = a_{ij}$ and $f_{ij} = 0$, $1 \le i, j \le n$,
if $a_{ij} < 0$, $f_{ij} = a_{ij}$ and $e_{ij} = 0$, $1 \le i, j \le n$.

$$\begin{pmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & & \vdots \\ e_{n1} & \cdots & e_{nn} \end{pmatrix} \begin{pmatrix} \big(\underline{x}^+_{1k}(r), \overline{x}^+_{1k}(r), \underline{x}^-_{1k}(r), \overline{x}^-_{1k}(r)\big) \\ \vdots \\ \big(\underline{x}^+_{nk}(r), \overline{x}^+_{nk}(r), \underline{x}^-_{nk}(r), \overline{x}^-_{nk}(r)\big) \end{pmatrix} + \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix} \begin{pmatrix} \big(\underline{x}^+_{1k}(r), \overline{x}^+_{1k}(r), \underline{x}^-_{1k}(r), \overline{x}^-_{1k}(r)\big) \\ \vdots \\ \big(\underline{x}^+_{nk}(r), \overline{x}^+_{nk}(r), \underline{x}^-_{nk}(r), \overline{x}^-_{nk}(r)\big) \end{pmatrix}$$

$$= \begin{pmatrix} \big(\underline{b}^+_{1k}(r), \overline{b}^+_{1k}(r), \underline{b}^-_{1k}(r), \overline{b}^-_{1k}(r)\big) \\ \vdots \\ \big(\underline{b}^+_{nk}(r), \overline{b}^+_{nk}(r), \underline{b}^-_{nk}(r), \overline{b}^-_{nk}(r)\big) \end{pmatrix}.$$

We extend into two linear systems

$$\begin{pmatrix} s_{11} & \dots & s_{1,2n} \\ \vdots & & \vdots \\ s_{2n,1} & \dots & s_{2n,2n} \end{pmatrix} \begin{pmatrix} \underline{x}^+_{1k} \\ \vdots \\ \underline{x}^+_{nk} \\ -\overline{x}^+_{1k} \\ \vdots \\ -\overline{x}^+_{nk} \end{pmatrix} = \begin{pmatrix} \underline{b}^+_{1k} \\ \vdots \\ \underline{b}^+_{nk} \\ -\overline{b}^+_{1k} \\ \vdots \\ -\overline{b}^+_{nk} \end{pmatrix}$$

and

$$\begin{pmatrix} s_{11} & \dots & s_{1,2n} \\ \vdots & & \vdots \\ s_{2n,1} & \dots & s_{2n,2n} \end{pmatrix} \begin{pmatrix} \underline{x}^-_{1k} \\ \vdots \\ \underline{x}^-_{nk} \\ -\overline{x}^-_{1k} \\ \vdots \\ -\overline{x}^-_{nk} \end{pmatrix} = \begin{pmatrix} \underline{b}^-_{1k} \\ \vdots \\ \underline{b}^-_{nk} \\ -\overline{b}^-_{1k} \\ \vdots \\ -\overline{b}^-_{nk} \end{pmatrix},$$

where $S = \begin{pmatrix} E & -F \\ -F & E \end{pmatrix}$, and $S = (s_{ij})$ where s_{ij}, $1 \leq i, j \leq 2n$, are determined by:
if $a_{ij} \geq 0$, then $s_{ij} = a_{ij}$ and $s_{n+i,n+j} = a_{ij}$,
if $a_{ij} < 0$, then $s_{i,j+n} = -a_{ij}$ and $s_{n+i,j} = -a_{ij}$,
otherwise $s_{ij} = 0$.

Finally, we get the extended matrix equations

$$\begin{pmatrix} s_{11} & \dots & s_{1,2n} \\ \vdots & & \vdots \\ s_{2n,1} & \dots & s_{2n,2n} \end{pmatrix} \begin{pmatrix} \underline{x}^+_{11} & \dots & \underline{x}^+_{1n} \\ \vdots & & \vdots \\ \underline{x}^+_{n1} & \dots & \underline{x}^+_{nn} \\ -\overline{x}^+_{11} & \dots & -\overline{x}^+_{1n} \\ \vdots & & \vdots \\ -\overline{x}^+_{n1} & \dots & -\overline{x}^+_{nn} \end{pmatrix} = \begin{pmatrix} \underline{b}^+_{11} & \dots & \underline{b}^+_{1n} \\ \vdots & & \vdots \\ \underline{b}^+_{n1} & \dots & \underline{b}^+_{nn} \\ -\overline{b}^+_{11} & \dots & -\overline{b}^+_{1n} \\ \vdots & & \vdots \\ -\overline{b}^+_{n1} & \dots & -\overline{b}^+_{nn} \end{pmatrix}$$

and

$$\begin{pmatrix} s_{11} & \dots & s_{1,2n} \\ \vdots & & \vdots \\ s_{2n,1} & \dots & s_{2n,2n} \end{pmatrix} \begin{pmatrix} \underline{x}^-_{11} & \dots & \underline{x}^-_{1n} \\ \vdots & & \vdots \\ \underline{x}^-_{n1} & \dots & \underline{x}^-_{nn} \\ -\overline{x}^-_{11} & \dots & -\overline{x}^-_{1n} \\ \vdots & & \vdots \\ -\overline{x}^-_{n1} & \dots & -\overline{x}^-_{nn} \end{pmatrix} = \begin{pmatrix} \underline{b}^-_{11} & \dots & \underline{b}^-_{1n} \\ \vdots & & \vdots \\ \underline{b}^-_{n1} & \dots & \underline{b}^-_{nn} \\ -\overline{b}^-_{11} & \dots & -\overline{b}^-_{1n} \\ \vdots & & \vdots \\ -\overline{b}^-_{n1} & \dots & -\overline{b}^-_{nn} \end{pmatrix}.$$

□

Remark 3.1 *The following example shows that the non-singularity of the matrix A is not sufficient to have the non-singularity of the matrix S:*

Example 3.1 $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ *is non-singular, while* $S = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ *is singular.*

Theorem 3.2 *[12] The matrix S is non-singular if and only if the matrix $E+F$ and $E-F$ are both non-singular.*

Theorem 3.3 *[12] If S is non-singular, then*

$$S^{-1} = \frac{1}{2}\begin{pmatrix} (E-F)^{-1} + (E+F)^{-1} & (E-F)^{-1} - (E+F)^{-1} \\ (E-F)^{-1} - (E+F)^{-1} & (E-F)^{-1} + (E+F)^{-1} \end{pmatrix}.$$

Theorem 3.4 *if $(E+F)$ and $(E-F)$ are both non-singular and B is an arbitrary intuitionistic fuzzy matrix. Then the matrix equation (3.1) has an intuitionistic fuzzy solution if S^{-1} is a nonnegative matrix .*

Proof: Let $1 \leq i, j \leq 2n$ and $S^{-1} = (t_{ij})$. Then for $1 \leq i, j \leq n$, we have:

$$\begin{aligned} \underline{x_{ij}^{+}} &= \sum_{k=1}^{n} t_{ik} \underline{b_{kj}^{+}} - \sum_{k=1}^{n} t_{i,n+k} \overline{b_{kj}^{+}}. \\ -\overline{x_{ij}^{+}} &= \sum_{k=1}^{n} t_{n+i,k} \underline{b_{kj}^{+}} - \sum_{k=1}^{n} t_{n+i,n+k} \overline{b_{kj}^{+}}. \\ \underline{x_{ij}^{-}} &= \sum_{k=1}^{n} t_{ik} \underline{b_{kj}^{-}} - \sum_{k=1}^{n} t_{i,n+k} \overline{b_{kj}^{-}}. \\ -\overline{x_{ij}^{-}} &= \sum_{k=1}^{n} t_{n+i,k} \underline{b_{kj}^{-}} - \sum_{k=1}^{n} t_{n+i,n+k} \overline{b_{kj}^{-}}. \end{aligned}$$

Since S^{-1} have the same structure as S i.e $S^{-1} = \begin{pmatrix} G & H \\ H & G \end{pmatrix}$ (see [12] for more detail), we obtain:

$$\begin{aligned} \overline{x_{ij}^{+}} &= -\sum_{k=1}^{n} t_{i,n+k} \underline{b_{kj}^{+}} + \sum_{k=1}^{n} t_{i,k} \overline{b_{kj}^{+}}. \\ \overline{x_{ij}^{-}} &= -\sum_{k=1}^{n} t_{i,n+k} \underline{b_{kj}^{-}} + \sum_{k=1}^{n} t_{i,k} \overline{b_{kj}^{-}}. \end{aligned}$$

Then, we get:

$$\overline{x_{ij}^{+}} - \underline{x_{ij}^{+}} = \sum_{k=1}^{n} t_{i,n+k} (\overline{b_{kj}^{+}} - \underline{b_{kj}^{+}}),$$

and

$$\overline{x_{ij}^{-}} - \underline{x_{ij}^{-}} = \sum_{k=1}^{n} t_{i,n+k} (\overline{b_{kj}^{-}} - \underline{b_{kj}^{-}}).$$

y_{kj}, $1 \leq k, j \leq n$ are fuzzy numbers.
Therfore:
$\overline{b_{kj}^{+}} - \underline{b_{kj}^{+}} \geq 0$ and $\overline{b_{kj}^{-}} - \underline{b_{kj}^{-}} \geq 0$, and since $t_{ij} \geq 0$, Then $\overline{x_{ij}^{+}} - \underline{x_{ij}^{+}} \geq 0$ and $\overline{x_{ij}^{-}} - \underline{x_{ij}^{-}} \geq 0$.
$\overline{b_{kj}^{+}}$ is monotonic decreasing and $\underline{b_{kj}^{+}}$ is monotonic increasing, then $\overline{x_{ij}^{+}}$ is monotonic decreasing and $\underline{x_{ij}^{+}}$ is monotonic increasing.
$\overline{b_{kj}^{-}}$ is monotonic decreasing and $\underline{b_{kj}^{-}}$ is monotonic increasing, then $\overline{x_{ij}^{-}}$ is monotonic decreasing and $\underline{x_{ij}^{-}}$ is monotonic increasing.
$\overline{b_{kj}^{+}}$ and $\underline{b_{kj}^{+}}$ are bounded left continuous , then $\overline{x_{ij}^{+}}$ and $\underline{x_{ij}^{+}}$ are bounded left continuous.
$\overline{b_{kj}^{-}}$ and $\underline{b_{kj}^{-}}$ are bounded left continuous , then $\overline{x_{ij}^{-}}$ and $\underline{x_{ij}^{-}}$ are bounded left continuous.

□

Theorem 3.5 *[16] The inverse of a nonnegative matrix A is nonnegative if and only if A is a generalized permutation matrix.*

4. Applications

To validate the proposed results, we present two examples computing the solutions for intuitionistic fuzzy matrix equations and we show some graphical representations of this solutions.

Example 4.1 *Consider the following intuitionistic fuzzy matrix equations:*

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} (-3+2r, 2-3r, -1-\frac{5}{2}r, \frac{7}{2}r-1) & (-1+2r, 4-3r, 1-\frac{5}{2}r, 1+\frac{7}{2}r) \\ (7r, 11-4r, 7-8r, 7+5r) & (1+4r, 10-5r, 5-6r, 5+6r) \end{pmatrix}.$$

The associated matrix equations are:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \underline{x}^+_{11} & \underline{x}^+_{12} \\ \underline{\underline{x}}^+_{21} & \underline{\underline{x}}^+_{22} \\ -\overline{\overline{x}}^+_{11} & -\overline{\overline{x}}^+_{12} \\ -\overline{x}^+_{21} & -\overline{x}^+_{22} \end{pmatrix} = \begin{pmatrix} -3+2r & -1+2r \\ 7r & 1+4r \\ 3r-2 & 3r-4 \\ 4r-11 & 5r-10 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \underline{x}^-_{11} & \underline{x}^-_{12} \\ \underline{\underline{x}}^-_{21} & \underline{\underline{x}}^-_{22} \\ -\overline{\overline{x}}^-_{11} & -\overline{\overline{x}}^-_{12} \\ -\overline{x}^-_{21} & -\overline{x}^-_{22} \end{pmatrix} = \begin{pmatrix} -1-\frac{5}{2}r & 1-\frac{5}{2}r \\ 7-8r & 5-6r \\ -\frac{7}{2}r+1 & -1-\frac{7}{2}r \\ -7-5r & -5-6r \end{pmatrix}.$$

Then we get

$$\begin{pmatrix} \underline{x}^+_{11} & \underline{x}^+_{12} \\ \underline{\underline{x}}^+_{21} & \underline{\underline{x}}^+_{22} \\ -\overline{\overline{x}}^+_{11} & -\overline{\overline{x}}^+_{12} \\ -\overline{x}^+_{21} & -\overline{x}^+_{22} \end{pmatrix} = \begin{pmatrix} r & r+1 \\ 2r & r \\ r-2 & 2r-4 \\ r-3 & r-2 \end{pmatrix} \text{ and } \begin{pmatrix} \underline{x}^-_{11} & \underline{x}^-_{12} \\ \underline{\underline{x}}^-_{21} & \underline{\underline{x}}^-_{22} \\ -\overline{\overline{x}}^-_{11} & -\overline{\overline{x}}^-_{12} \\ -\overline{x}^-_{21} & -\overline{x}^-_{22} \end{pmatrix} = \begin{pmatrix} 1-\frac{5}{4}r & 2-\frac{9}{8}r \\ 2-\frac{9}{4}r & 1-\frac{13}{8}r \\ -\frac{5}{4}r-1 & -\frac{15}{8}r-2 \\ -\frac{5}{4}r-2 & \frac{11}{8}r-1 \end{pmatrix}.$$

Finally we have the matrix solution $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ *given by:*

$x_{11} = (r, 2-r, 1-\frac{5}{4}r, \frac{5}{4}r+1)$,
$x_{12} = (2r, 3-r, 2-\frac{9}{4}r, 2+\frac{5}{4}r)$,
$x_{21} = (r+1, 4-2r, 2-\frac{9}{8}r, 2+\frac{15}{8}r)$,
$x_{22} = (r, 2-r, 1-\frac{13}{8}r, 1+\frac{11}{8}r)$.

Example 4.2 *Consider the following intuitionistic fuzzy matrix equations:*

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} (3r-3, 3-3r, -4r, 4r) & (4r-4, 6-6r, -7r, 7r) \\ (2r+1, 5-2r, 3-3r, 3+\frac{5}{2}r) & (3r, 7-4r, 3-5r, 3+5r) \end{pmatrix}.$$

The associated matrix equations are:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} \underline{x}^+_{11} & \underline{x}^+_{12} \\ \underline{\underline{x}}^+_{21} & \underline{\underline{x}}^+_{22} \\ -\overline{\overline{x}}^+_{11} & -\overline{\overline{x}}^+_{12} \\ -\overline{x}^+_{21} & -\overline{x}^+_{22} \end{pmatrix} = \begin{pmatrix} 3r-3 & 4r-4 \\ 2r+1 & 3r \\ 3r-3 & 6r-6 \\ 2r-5 & 4r-7 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} \underline{x}^-_{11} & \underline{x}^-_{12} \\ \underline{\underline{x}}^-_{21} & \underline{\underline{x}}^-_{22} \\ -\overline{\overline{x}}^-_{11} & -\overline{\overline{x}}^-_{12} \\ -\overline{x}^-_{21} & -\overline{x}^-_{22} \end{pmatrix} = \begin{pmatrix} -4r & -7r \\ 3-3r & 3-5r \\ -4r & -7r \\ -3-\frac{5}{2} & -3-5r \end{pmatrix}.$$

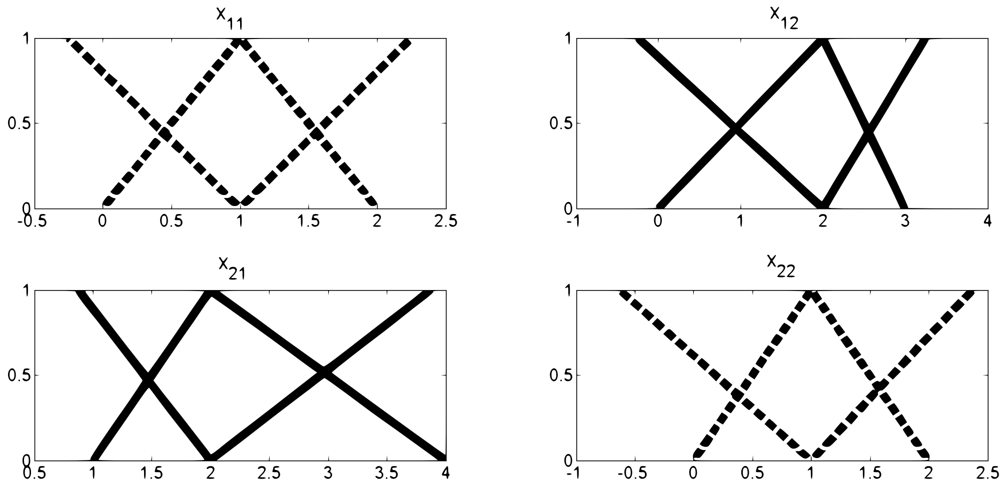

Figure 1: Intuitionistic fuzzy solution

Then we get

$$\begin{pmatrix} \underline{x}^+_{11} & \underline{x}^+_{12} \\ \underline{x}^+_{21} & \underline{x}^+_{22} \\ -\overline{x}^+_{11} & -\overline{x}^+_{12} \\ -\overline{x}^+_{21} & -\overline{x}^+_{22} \end{pmatrix} = \begin{pmatrix} r & r \\ r+1 & 2r \\ r-2 & 2r-3 \\ r-3 & 2r-4 \end{pmatrix} \text{ and } \begin{pmatrix} \underline{x}^-_{11} & \underline{x}^-_{12} \\ \underline{x}^-_{21} & \underline{x}^-_{22} \\ -\overline{x}^-_{11} & -\overline{x}^-_{12} \\ -\overline{x}^-_{21} & -\overline{x}^-_{22} \end{pmatrix} = \begin{pmatrix} 1-\frac{4}{3}r & 1-2r \\ 2-\frac{5}{3}r & 2-3r \\ -1-\frac{7}{6}r & -1-2r \\ -2-\frac{4}{3}r & -2-3r \end{pmatrix}.$$

Finally we have the intuitionistic fuzzy matrix solution $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ *given by:*

$x_{11} = (r, 2-r, 1-\frac{4}{3}r, 1+\frac{7}{6}r)$,
$x_{12} = (r, 3-2r, 1-2r, 1+2r)$,
$x_{21} = (r+1, 3-r, 2-\frac{5}{3}r, 2+\frac{4}{3}r)$,
$x_{22} = (2r, 4-2r, 2-3r, 2+3r)$.

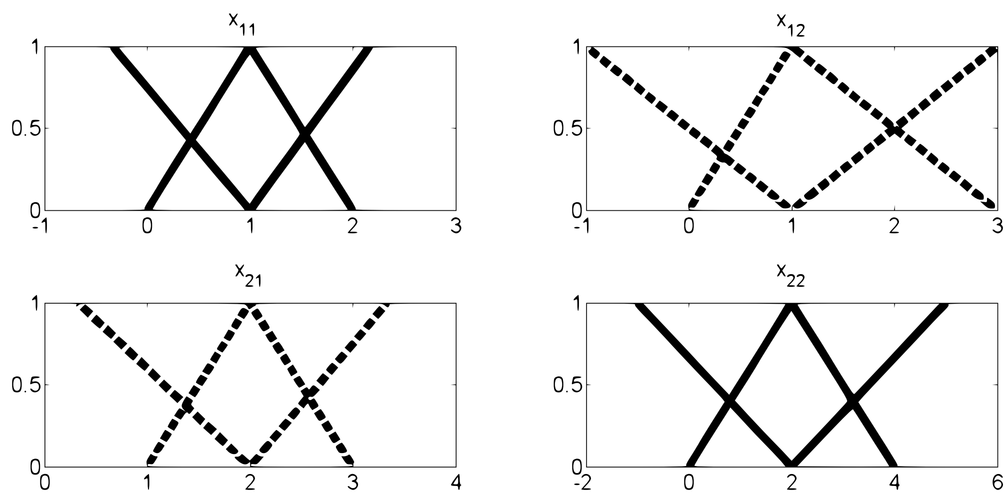

Figure 2: Intuitionistic fuzzy solution

5. Conclusion

In this work we proposed a general method for solving an intuitionistic fuzzy matrix equation $AX = B$ where the coefficients matrix A are real numbers, the unknown coefficients matrix X are intuitionistic fuzzy numbers and the coefficients matrix B are intuitionistic fuzzy numbers. We converted the original intuitionistic fuzzy matrix equation to two crisp matrix equations with the same matrix S. For our future research, we will develop the proposed approach for solving the inconsistent intuitionistic fuzzy matrix equations.

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