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On joint spectra of families of operator pencils

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ABSTRACT: In this paper, we introduce and study basic properties of several types of joint spectra of n-tuples of operator pencils in both Banach algebra and Banach space. We give spectral properties (i.e. compactness, the spectral mapping theorem), as well as an example of the upper and lower triangular matrices. In addition, we describe the joint spectrum of the tensor product of several operators.

Key Words: Joint spectrum, operator pencils, bounded operator, spectral mapping theorem, tensor product.

Contents

1. Introduction

Considering the complex Banach algebra space $\mathcal{A}$ with identity e and let $a = (a_1, a_2, \dots, a_n)$ be an n-tuple of elements of $\mathcal{A}$. In the literature various authors studied various concepts of a joint spectrum of a, see [5,7,8,9,15]. The joint Hart spectrum of a denoted by $\sigma(a)$ is the union of the left joint spectrum $\sigma^{\mathrm{l}}(a)$ and the right joint spectrum $\sigma^{\mathrm{r}}(a)$ and we have $\sigma(a) = \sigma^{\mathrm{l}}(a) \cup \sigma^{\mathrm{r}}(a)$. Let $\mathbb{C}$ be the complex space, the left joint spectrum is defined as the set of those n-tuple of $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n$ for which the left ideal generated by the system $a - \lambda e = (a_1 - \lambda_1 e, a_2 - \lambda_2 e, \dots, a_n - \lambda_n e)$, is proper and the right joint spectrum can be defined analogously. In [8,9], R. Harte proved many spectral properties for this joint spectra i.e. compactness, non emptiness, the projection property and the spectral mapping theorem.
Nevertheless, many authors studied the joint spectrum of n-tuple of operator $T = (T_1, \dots, T_n)$ in $\mathcal{B}(X)$, where $\mathcal{B}(X)$ is the set of all bounded linear operator in Banach space X with identity I see, for instance [10,11,14,15,17]. The joint spectrum $\sigma(T)$ is the set of all $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n$ for which at least one of the equation $\sum_{i=1}^{n} (T_i - \lambda_i I) C_i = I$ and $\sum_{i=1}^{n} C_i (T_i - \lambda_i I) = I$ has no solution $C = (C_i)_{1 \le i \le n}$ in $\mathcal{B}(X)^n$. The joint spectrum of n-tuple $T = (T_1, \dots, T_n)$ is the union of the left joint spectrum and the right joint spectrum which we denote by $\sigma_l(T)$ (resp. $\sigma_r(T)$). For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, is not in $\sigma_l(T)$, if and only if, there exists $(B_1, \dots, B_n) \in \mathcal{B}(X)^n$ such that $\sum_{i=1}^{n} (T_i - \lambda_i I) B_i = I$, and λ is not in $\sigma_r(T)$, if and only if, there exists $(B_1, \dots, B_n) \in \mathcal{B}(X)^n$ such that $\sum_{i=1}^{n} B_i (T_i - \lambda_i I) = I$.

In [17], W. Zelazko studied the joint approximative spectrum of an n-tuple $T = (T_1, \dots, T_n)$ denoted by $\sigma_{\mathrm{a}}(T)$, it is defined as the set of all $\lambda \in \mathbb{C}^n$, such that there is a sequence of vectors $x_n \in X$ with $\|x_n\| = 1$ and $\sum_{j=1}^{n} \|(T_j - \lambda_j)x_n\| \to 0, n \to \infty$. The commutant $\mathcal{B}'$ of $\mathcal{B}(X)$ consists of all operators $B \in \mathcal{B}(X)$ such that $AB = BA$ for all $A \in \mathcal{B}(X)$, and the bicommutant $\mathcal{B}''$ of $\mathcal{B}(X)$ is defined as $(\mathcal{B}')'$. A vector $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$, belong to the commutant spectrum of n-tuple $T = (T_1, \dots, T_n)$ (we write $\lambda \in \sigma'(T)$) if and only if there are no operators $(C_i)_{1 \le i \le n} \in \mathcal{B}'$ such that $\sum_{i=1}^{n} C_i (T_i - \lambda_i I) = I$, and we say that a point λ belong to the bicommutant spectrum of T (we write $\lambda \in \sigma''(T)$), if and only if,

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the last equality can not hold for all operators $(C_i)_{1\leq i\leq n} \in \mathcal{B}''$. All these spectra coincide in the case of a single operator with its usual spectrum.

In this paper, we consider several types of joint spectra of families of element pencils in $\mathcal{A}$. Let $a = (a_1, a_2, \ldots, a_n)$ be an n-tuple of elements of $\mathcal{A}$ and s be a single element in $\mathcal{A}$. For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$, let $a - \lambda s = (a_1 - \lambda_1 s, a_2 - \lambda_2 s, \ldots, a_n - \lambda_n s)$ be an n-tuple of element pencils in the Banach algebra $\mathcal{A}$. We start by giving the definition of joint Harte spectrum of n-tuple of element pencils $a - \lambda s$ and we give the basic properties (i.e. compactness, and the spectral mapping theorem). Also, we give a relation ship between the joint spectrum of n-tuple of element pencils in commutative Banach algebra space and the set of linear multiplicative functional. Therefore, we provide generalizations of the corresponding concepts in [8] and [9] respectively, where s is equal to the identity e.

In addition, by similar arguments, we describe in Section 3 several joint spectra for n-tuple of operator pencils in Banach space X. Let $T - \lambda S = (T_1 - \lambda_1 S, \ldots, T_n - \lambda_n S)$. $T - \lambda S$ is an n-tuple of operator pencils where S is a bounded operator and $T = (T_1, \cdots, T_n)$ in $\mathcal{B}(X)^n$. We define the joint left, right, approximative, commutant and bicommutant spectra of the system $T - \lambda S$ in Banach space X and we give the basic properties. Thus, the novelty of our paper lies within generalizing the definitions of some previously introduced joint spectra, in the case where S is equal to the identity I(see [14] and [17]). This work offers a generalization of several properties and theorems presented in [1,4,6,18,19], focusing on the case where T represents a single operator within the bounded linear operators space $\mathcal{L}(X)$, in the same of essential spectrum (see [1,2,3]), the paper gives the results of joint spectrum.

The rest of this paper is organized as follows: In section 2, we introduce the Harte joint spectra of n-tuple of element pencils in Banach algebra and prove their main spectral properties. We give as well a relationship between the joint spectrum of n-tuple of element pencils in commutative Banach algebra and the set of linear multiplicative functionals. Section 3 deals with the joint left, right, approximative, commutant and bicommutant spectra of an n-tuple of operator pencils in Banach space and provide its properties. In the same section, we give an example of upper and lower triangular complex matrix. Finally, in section 4, we describe the bicommuttant joint spectrum of the tensor product of several operators.

2. Joint spectrum of element pencils in Banach algebra

Let us begin this part by recalling some definitions. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ be elements of $\mathcal{A}^n$. We have

$$a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i.$$

If $a \cdot b = 1$ we say that a is a left inverse for b in $\mathcal{A}^n$, and b is a right inverse for a in $\mathcal{A}^n$.

Let $a = (a_1, a_2, \cdots, a_n)$ be an n-tuple of elements of $\mathcal{A}^n$, and $0 \neq b \in \mathcal{A}^n$. If

$$ba_1 = ba_2 = \cdots = ba_n = 0,$$

then b will be called a left annihilator for a in $\mathcal{A}^n$. If

$$a_1 b = a_2 b = \cdots = a_n b = 0,$$

then b is the right annihilator for a in $\mathcal{A}^n$.

For a sequence $(v) = (v_k)_{k=1}^{\infty}$ and $a = (a_1, a_2, \ldots, a_n)$ be an n-tuple of elements of $\mathcal{A}^n$, such that

$$\inf_k \|v_k\| > 0 \quad \text{and} \quad \lim_k \|v_k a_j\| = 0 \quad \text{for all } j = 1, 2, \ldots, n, \tag{2.1}$$

we say that v is an approximate left annihilator for $a \in \mathcal{A}^n$. On the other hand, if

$$\inf_k \|v_k\| > 0 \quad \text{and} \quad \lim_k \|a_j v_k\| = 0 \quad \text{for all } j = 1, 2, \ldots, n, \tag{2.2}$$

we say that v is an approximate right annihilator for $a \in \mathcal{A}^n$.

Definition 2.1 Let $a = (a_1, a_2, \ldots, a_n)$ be in $\mathcal{A}^n$ and s be a single element in $\mathcal{A}$. The joint Harte spectrum of the system $a - \lambda s = (a_1 - \lambda_1 s, \cdots, a_n - \lambda_n s)$, and we write the s-joint spectrum for a denoted by $\sigma_s(a)$ is defined by:

$$\sigma_s(a) = \sigma_s^{\mathrm{l}}(a) \cup \sigma_s^{\mathrm{r}}(a),$$

where $\sigma_s^{\mathrm{l}}(a)$ is the set of the joint left spectrum of an n-tuple element pencils $a - \lambda s$ and we write the s-joint left spectrum as follows

$$\sigma_s^{\mathrm{l}}(a) = \{\lambda \in \mathbb{C}^n : \quad a - \lambda s \text{ is not left invertible in } \mathcal{A}^n\},$$

and $\sigma_s^{\mathrm{r}}(a)$ is the set of the joint right spectrum of an n-tuple element pencils $a - \lambda s$ and we write the s-joint right spectrum as follows

$$\sigma_s^{\mathrm{r}}(a) = \{\lambda \in \mathbb{C}^n : a - \lambda s \text{ is not right invertible in } \mathcal{A}^n\}.$$

We define also in this part the s-joint point spectrum as the joint point spectrum of an n-tuple of element pencils $a - \lambda s$ denoted by $\pi_{\mathcal{A},s}(a)$ for $a \in \mathcal{A}^n$ and we have

$$\pi_{\mathcal{A},s}(a) = \pi_{\mathcal{A},s}^{l}(a) \cup \pi_{\mathcal{A},s}^{r}(a),$$

where $\pi_{\mathcal{A},s}^{l}(a)$ (resp. $\pi_{\mathcal{A},s}^{r}(a)$) is the s-left (resp. s-right) point spectrum and it is given by the following sets:

$$\pi_{\mathcal{A},s}^{l}(a) = \{\lambda \in \mathbb{C}^n,\ a - \lambda s \text{ has a left annihilators in } \mathcal{A}^n\},$$

$$\big(\text{resp. } \pi_{\mathcal{A},s}^{r}(a) = \{\lambda \in \mathbb{C}^n,\ a - \lambda s \text{ has a right annihilators in } \mathcal{A}^n\}\big).$$

Moreover, we set the s-joint approximate point spectrum as the joint approximate point spectrum of an n-tuple of element pencils $a - \lambda s$ we denote by $\tau_{\mathcal{A},s}(a)$, and we have

$$\tau_s(a) = \tau_{\mathcal{A},s}(a) = \tau_{\mathcal{A},s}^{l}(a) \cup \tau_{\mathcal{A},s}^{r}(a),$$

where $\tau_{A,s}^{l}(a)$ (resp. $\tau_{A,s}^{r}(a)$) is the s-left (resp. s-right) approximate point spectrum given as follows:

$$\tau_{\mathcal{A},s}^{l}(a) = \{\lambda \in \mathbb{C}^n, \quad a - \lambda s \ \text{ has approximate left annihilators in } \mathcal{A}^n\},$$

$$\big(\text{resp. } \tau_{A,s}^{r}(a) = \{\lambda \in \mathbb{C}^n, \quad a - \lambda s \ \text{ has approximate right annihilators in } \mathcal{A}^n\}\big). \qquad \diamondsuit$$

In this paragraph, we show the results only for the s-joint left spectrum since we have:

$$\sigma_{\mathcal{A},s}^{r}(a) = \sigma_{A^{\wedge},s^*}^{l}(a),$$

where $A^{\wedge}$ denotes the algebra obtained by "reversing products" in $\mathcal{A}$, and s^* is the adjoint for s.

We start to prove the following inclusions.

Proposition 2.1 *Let $a = (a_1, a_2, \ldots, a_n) \in \mathcal{A}^n$ and $s \in \mathcal{A}$. Let $\mathcal{A}_1$ be a closed subalgebra of $\mathcal{A}$ containing the identity. If $a_j \in \mathcal{A}_1$ for all $1 \leq j \leq n$, then*

(1) $\pi_{\mathcal{A},s}(a) \subseteq \tau_{\mathcal{A},s}(a) \subseteq \sigma_{\mathcal{A},s}(a)$.

(2) $\sigma_{\mathcal{A},s}(a) \subseteq \sigma_{\mathcal{A}_1,s}(a), \pi_{\mathcal{A}_1,s}(a) \subseteq \pi_{\mathcal{A},s}(a)$ *and* $\tau_{\mathcal{A}_1,s}(a) \subseteq \tau_{\mathcal{A},s}(a)$. $\diamondsuit$

Proof: (1) Let $\lambda \in \pi_{\mathcal{A},s}^{l}(a)$ and let b be a right annihilator for $a - \lambda s$. Now, we choose the sequence $(v_k)_k$ such that $v_k = b$ for all $1 \leq k \leq n$, then we obtain that $(v_k)_k$ is an approximate right annihilator for $a - \lambda s$. Hence applying Equation (2.1), we have $\lambda \in \tau_{\mathcal{A},s}^{l}(a)$. Therefore

$$\pi_{\mathcal{A},s}^{l}(a) \subset \tau_{\mathcal{A},s}^{l}(a).$$

Now, to show that

$$\pi^r_{\mathcal{A},s}(a) \subset \tau^r_{\mathcal{A},s}(a),$$

it sufficed to apply the Equation (2.2).

Let $\lambda \notin \sigma^l_{\mathcal{A},s}(a)$, then there exists $b \in \mathcal{A}^n$ such that

$$b(a-\lambda s) = \sum_{j=1}^{n} b_j \left(a_j - \lambda_j s\right) = 1. \tag{2.3}$$

Note that, $\lambda \in \tau^l_{\mathcal{A},s}(a)$ if and only if

$$\inf_{\|b\|=1} \sum_{j=1}^{n} \|\left(a_j - \lambda_j s\right) b_j\| = 0. \tag{2.4}$$

For an arbitrary sequence $(v_k)_k \in \mathcal{A}$ and by Equation (2.3), we have

$$\|v_k\| \leqq \sum_{j=1}^{n} \|b_j\| \, \|\left(a_j - \lambda_j s\right) v_k\| \text{ for all } k = 1, 2, 3, \ldots,$$

which contradicts Equation (2.4). Hence $\lambda \notin \tau^l_{\mathcal{A},s}(a)$. Consequently, we prove that

$$\tau^l_{\mathcal{A},s}(a) \subseteq \sigma^l_{\mathcal{A},s}(a).$$

Using the same argument we obtain the result for the right spectrum, and we have

$$\tau^r_{\mathcal{A},s}(a) \subseteq \sigma^r_{\mathcal{A},s}(a).$$

(2) It is easy to see that any invertible, annihilator or approximate annihilator for $a - \lambda s$ in $\mathcal{A}_1$ is also an invertible or approximate annihilator in $\mathcal{A}$. □

In the following theorem, we show that $\sigma_{\mathcal{A},s}(a)$ and $\tau_{\mathcal{A},s}(a)$ are compact subsets of $\mathbb{C}^n$.

Theorem 2.1 *Using the notation introduced above. If $s \neq 0$ and $\|s\| < 1$, then $\sigma_{\mathcal{A},s}(a)$ and $\tau_{\mathcal{A},s}(a)$ are compact in $\mathbb{C}^n$.* ♢

Proof: Let $\lambda_k \notin \sigma_s(a_k)$, for any $k \in \{1, 2, \ldots, n\}$, then there exists $b_k \in A$ such that

$$b_k(a_k - \lambda_k s) = (a_k - \lambda_k s)b_k = 1.$$

We take $b_k = (a_k - \lambda_k s)^{-1}$, $b_j = 0$ $(j \neq k)$, then we get $b = (b_1, b_2, .., b_n) \in \mathcal{A}^n$. Hence, we infer that:

$$\sum_{k=1}^{n} b_k \left(a_k - \lambda_k s\right) = \sum_{k=1}^{n} \left(a_k - \lambda_k s\right) b_k = 1.$$

Therefore $\lambda \notin \sigma_s\left(a_1, a_2, ..., a_n\right)$ and we obtain

$$\sigma_s\left(a_1, \ldots ...., a_n\right) \subset \sigma_s\left(a_1\right) \times \sigma_s\left(a_2\right) \times \ldots \times \sigma_s\left(a_n\right). \tag{2.5}$$

Since each $\sigma_s\left(a_i\right)$ is a bounded subset of $\mathbb{C}^n$, it follows from Equation (2.5) that $\sigma_s\left(a_1, a_2, ..., a_n\right)$ is a bounded subset of $\mathbb{C}^n$. It remains now to show that $\sigma_s\left(a\right)$ is closed.

We start to prove the s-left spectrum is closed. Let $\lambda \notin \sigma^l_s\left(a\right)$, then there exists $b \in \mathcal{A}$ such that $b(a - \lambda s) = 1$, we choose ε as follows $\varepsilon = \frac{1}{\|b\|\|s\|}(s \neq 0, b \neq 0)$ and let $\lambda' \in \mathcal{B}(\lambda, \varepsilon)$, where $\mathcal{B}(\lambda, \varepsilon)$ is the ball from center λ and radius ε, then we infer that $\|\lambda - \lambda'\| < \varepsilon$, on the other hand, we have

$$\|1 - b \cdot (a - \lambda' s)\| = \|b(\lambda' - \lambda)s\| < \|\lambda' - \lambda\|\|b\|\|s\| < 1.$$

It follows that $b(a-\lambda' s)$ is invertible in $\mathcal{A}$, as a result $(b(a-\lambda' s))^{-1}(b_1, b_2, \ldots, b_n)$ is a left invertible for $a-\lambda' s$, and we conclude that $\lambda' \notin \sigma_s^l(a)$, and here we show that the complement of $\sigma_s^l(a)$ is open and with the same argument we show that $\sigma_s^r(a)$ is closed. Since $\sigma_s(a)$ is the union of the s-joint left and right spectrum, then we have $\sigma_s(a)$ is a subset that is closed and bounded in $\mathbb{C}^n$.

The same argument for $\tau_{A,s}(a)$, if $\lambda \notin \tau_{A,s}^l(a)$ then by Equation (2.4), there is $C_k > 0$ for which

$$\sum_{j=1}^{n} \|(a_j - \lambda_j s)\, b\| \geq k\|b\| \quad \text{for} \quad b = b_1 \quad \text{in } \mathcal{A}.$$

For $\varepsilon' = \dfrac{k}{2\|s\|}$ and let $\lambda'' \in \mathcal{B}(\lambda, \varepsilon')$ is the ball of center λ and radius ε'. Hence, we infer that

$$\begin{aligned}\sum_{j=1}^{n} \left\|\left(a_j - \lambda_j'' s\right) b\right\| &= \sum_{j=1}^{n} \left\|(a_j - \lambda_j s)\, b - (\lambda_j'' - \lambda_j) s b\right\| \\ &\geq \sum_{j=1}^{n} \|(a_j - \lambda_j s)\, b\| - \sum_{j=1}^{n} \left\|(\lambda_j'' - \lambda_j) s b\right\|.\end{aligned}$$

Consequently, we get that:

$$\sum_{j=1}^{n} \left|\left(a_j - \lambda_j'' s\right) b\right| \geq \frac{1}{2} k\|b\|. \tag{2.6}$$

Using the Equation (2.6), we obtain that $\lambda'' \notin \tau_{A,s}^l(a)$, then we have that $\tau_{A,s}^l(a)$ is closed and with the same principle we show that $\tau_{A,s}^r(a)$ is closed. Which give that

$$\tau_{A,s} = \tau_{A,s}^r(a) \cup \tau_{A,s}^l(a) \quad \text{is bounded closed in } \mathbb{C}^n.$$

□

Example 2.1 *The following example show that the s-joint Harte spectrum can be empty. Even in the simplest situations in the algebra A of complex* 2×2 *matrices. Take* $a = (a_1, a_2)$ *with*

$$a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and we consider

$$s = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is clair that $\sigma_s(a_1) = \{0\}$ *and* $\sigma_s(a_2) = \{0\}$ *and since* $\sigma_s(a_1, a_2) \subset \sigma_s(a_1).\sigma_s(a_2)$, *then*

$$\sigma_s(a) \subset \{0, 0\}.$$

In the other hand we have

$$a_2 a_1 + a_1 a_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we obtain $\{0, 0\} \notin \sigma_s(a)$.

2.1. The spectral mapping theorem

In this part, we treat the case of a commutative Banach algebra for that, we consider the non trivial commutative Banach algebra $\mathcal{A}'$ with identity e. For a, b in $\mathcal{A}$, if $ab = ba$, then $a, b \in \mathcal{A}'$. the aim is to find a relationship between the s-joint spectrum and the linear multiplicative functional on $\mathcal{A}'$. A linear functional φ on $\mathcal{A}'$ is called multiplicative if $\varphi \neq 0$ and

$$\varphi(ab) = \varphi(a)\varphi(b), \quad a, b \in \mathcal{A}'.$$

Let $\mathcal{M}$ be the set of the linear multiplicative functionals on $\mathcal{A}'$. In [[13], Theorem 9.10], Schechter proved that a complex number λ is in $\sigma(a)$ if and only if there is a φ on $\mathcal{M}$ such that $\varphi(a) = \lambda$.

Theorem 2.2 *Let a and s in $\mathcal{A}'$, a complex number λ is in $\sigma_s(a)$ if and only if there exists φ in $\mathcal{M}$ such that $\varphi(a) = \lambda\varphi(s)$.* ♢

Before starting to prove this theorem, we need the following result.

Proposition 2.2 *If $H \neq \mathcal{A}'$ is an ideal in $\mathcal{A}'$, then there exists φ in $\mathcal{M}$ such that φ vanishes on H.* ♢

Proof: of Theorem 2.2 Let $\lambda \notin \sigma_s(a)$, then there exists $b \in \mathcal{A}$ such that $b(a - \lambda s) = e$, for any $\varphi \in \mathcal{M}$, we have that

$$\varphi(b)(\varphi(a) - \lambda\varphi(s)) = \varphi(e).$$

Note that $\varphi(e) = 1$, thus $\varphi(b)(\varphi(a) - \lambda\varphi(s)) = 1$, this shows that we can not have $\varphi(a) = \lambda\varphi(s)$.

Now, suppose that $\lambda \in \sigma_s(a)$, therefore $a - \lambda s$ does not have an inverse. Thus, $b(a - \lambda s) \neq e$ for all $b \in \mathcal{A}'$. The set of all elements of the form $b(a - \lambda s)$ is an ideal $H \neq \mathcal{A}'$. Applying Proposition 2.2, there exists $\varphi \in \mathcal{M}$ which vanishes on H. Hence we get that $\varphi(a) = \lambda\varphi(s)$. □

We can generalize this result of an n-tuple $a = (a_1, \cdots, a_n)$ and we have the following theorem.

Corollary 2.1 *Let $a = (a_1, \cdots, a_n)$ be n-tuple of element in $\mathcal{A}'$ and s be in $\mathcal{A}'$. A vector $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$ is in $\sigma_s(a)$ if and only if there exists φ on $\mathcal{M}$ such that $\varphi(a_k) = \lambda_k\varphi(s)$, for all $1 \leq k \leq n$.*

Let $P(t_1, \cdots, t_n) = \sum \alpha_{k_1,\cdots,k_n} t_1^{k_1} \cdots t_n^{k_n}$ be a polynomial of n variables. A homogeneous polynomial is a polynomial whose non zero terms all have the same degree. A polynomial is homogenous if and only if it defines a homogenous function, this means that if a multivariate polynomial P is homogenous of degree d, then

$$P(\lambda x_1, \ldots, \lambda x_n) = \lambda^d P(x_1, \ldots, x_n), \tag{2.7}$$

for every λ in an field containing the coefficients of P.

Theorem 2.3 *Let P be a homogeneous polynomial of degree k and let $s \in \mathcal{A}$, then we have*

$$\sigma_s[P(a_1, \cdots, a_n)] = \varphi(s)^{k-1} P[\sigma_s(a_1, \cdots, a_n)].$$ ♢

Proof: Applying Theorem 2.2, one has,

$$\mu \in \sigma_s[P(a_1, \cdots, a_n)] \quad \text{iff, there exists a functional } \varphi \in \mathcal{M} \text{ such that } \varphi[P(a_1, \cdots, a_n)] = \mu\varphi(s).$$

$$\varphi[P(a_1, \cdots, a_n)] = P[\varphi(a_1), \cdots, \varphi(a_n)].$$

On the other hand, by Corollary 2.1, we obtain that

$(\lambda_1, \cdots, \lambda_n) \in \sigma_s(a_1, \cdots, a_n)$ if and only if there exists $\varphi \in \mathcal{M}$ such that $\varphi(a_k) = \lambda_k\varphi(s), \quad 1 \leq k \leq n.$

Consequently, we have

$$P[\varphi(s)\lambda_1, \cdots, \varphi(s)\lambda_n] = \mu\varphi(s).$$

Now, we apply Equation (2.7) to get,

$$P[\varphi(s)\lambda_1, \cdots, \varphi(s)\lambda_n] = \varphi(s)^k P[\lambda_1, \cdots, \lambda_n].$$

Therefore, we infer that:

$$\varphi(s)^k P[\lambda_1, \cdots, \lambda_n] = \mu\varphi(s).$$

Finally, we get that:

$$\mu = \varphi(s)^{k-1} P[\lambda_1, \cdots, \lambda_n] \quad \textit{with} \ (\lambda_1, \cdots, \lambda_n) \in \sigma_s(a_1, \cdots, a_n).$$ □

Remark 2.1 *If we consider as P a homogeneous polynomial of degree 1, then we have*

$$\sigma_s[P(a_1, \cdots, a_n)] = P[\sigma_s(a_1, \cdots, a_n)].$$ ♢

3. Joint spectra of an n-tuple of operator pencils in Banach space

In this section we define the joint left, right, commutant, bicommutant, and approximative spectrum of n-tuple of operator pencils in Banach space X.

Definition 3.1 Let $T = (T_1, \ldots, T_n)$ be n-tuple of commuting operators in X and S be a single bounded operator on X we denote by $\sigma''_S(T)$ the bicommutant joint spectrum of an n-tuple of operator pencils $T - \lambda S$ and we write similarly the S-joint bicommutant spectrum that is defined as the set of all points $\lambda \in \mathbb{C}^n$ such that the closed $\{T_i - \lambda_i S, \ \ \forall 1 \leq i \leq n\}$ is a proper ideal in the algebra $\mathcal{B}''$.

$$\lambda \notin \sigma''_S(T) \text{ if and only if there exists } (C_1, \ldots, C_n) \in \mathcal{B}'' \text{ such that}$$

$$\sum_{i=1}^{n} C_i \left(T_i - \lambda_i S\right) = I. \tag{3.1}$$

With the same notation, we denote by $\sigma'_S(T)$ the commutant joint spectrum of n-tuple of operator pencils $T - \lambda S$ and we write the S-joint commutant spectrum is the set of all point $\lambda \in \mathbb{C}^n$ such that the set $\{T_i - \lambda_i S, \ \ \forall 1 \leq i \leq n\}$ contained in a proper (two-sided).

$\lambda \notin \sigma'_S(T)$ if and only if there exists $C_1, \ldots, C_n \in \mathcal{B}'$ such that the equation (3.1) is satisfied. $\diamondsuit$

Proposition 3.1 *Let $T = (T_1, \ldots, T_n)$ be n-tuple for commuting operators in $\mathcal{B}(X)$ and let $S \in \mathcal{B}(X)$, then we have the following inclusion*

$$\sigma'_S(T) \subset \sigma''_S(T). \qquad \diamondsuit$$

Definition 3.2 Let $T = (T_1, \ldots, T_n)$ be n-tuple of commuting operators in $\mathcal{B}(X)$ and $S \in \mathcal{B}(X)$. We define the joint approximative spectrum of n-tuple of operator pencils $T - \lambda S$, and similarly we write the S-joint approximative spectrum $\sigma_{S,\mathrm{a}}(T)$ is the set of all $\lambda \in \mathbb{C}^n$ such that there exists a sequence of vectors $x_n \in X$ with $\|x_n\| = 1$ and $\sum_{j=1}^{n} \|(T_j - \lambda_j S)x_n\| \to 0, n \to \infty$. $\diamondsuit$

The following proposition gives a useful tool for the characterization of $\sigma_{S,\mathrm{a}}(T)$.

Proposition 3.2 *Let $T = (T_1, \ldots, T_n)$ be n-tuple for commuting operators in $\mathcal{B}(X)$ and let $S \in \mathcal{B}(X)$, then we have the following equivalence :*

$$\lambda \in \sigma_{S,\mathrm{a}}(T) \textit{ if and only if } \inf\left\{ \sum_{j=1}^{n} \|(T_j - \lambda_j S)y\|, \ y \in X, \ \|y\| = 1 \right\} = 0. \qquad \diamondsuit$$

In the following result we gives the relation between the S-joint approximative spectrum and the S-joint commutant spectrum.

Proposition 3.3 *Let $T = (T_1, \ldots, T_n)$ be n-tuple for commuting operators in $\mathcal{B}(X)$ and let $S \in \mathcal{B}(X)$, then we have, $\sigma_{S,\mathrm{a}}(T) \subset \sigma'_S(T)$.* $\diamondsuit$

Proof: We assume that $\lambda \in \sigma_{S,\mathrm{a}}(T)$ and $\lambda \notin \sigma'_S(T)$. We can find $(C_1, \ldots, C_n) \in \mathcal{B}^{'}$ such that Equality (3.1) holds true. Or $\lambda \in \sigma_{S,\mathrm{a}}(A)$, then there exists a sequence $(x_n) \subset X$ such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \sum_{j=1}^{n} \|(A_j - \lambda_j S)x_n\| = 0$. Applying Equality (3.1) for (x_n) we obtain, $\sum_{i=1}^{n} C_i \left(T_i - \lambda_i S\right) x_n = x_n$, hence

$$\left\| \sum_{i=1}^{n} C_i \left(T_i - \lambda_i S\right) x_n \right\| = \|x_n\|. \tag{3.2}$$

Which is not possible since the norm of the left-hand in (3.2) elements tend to 0 with x_n, while the norms of the right-hand side in(3.2) are always equal to 1. $\square$

Definition 3.3 Let $T = (T_1, \ldots, T_n)$ be mutually of commuting operators and let $S \in \mathcal{B}(X)$. We define the joint left spectrum and the joint right spectrum of an n-tuple of operator pencils $T - \lambda S$ and similarly we write the S-joint left spectrum and the S-joint right spectrum of an n-tuple T, we denote by $\sigma_{S,l}(T)$ (resp. $\sigma_{S,r}(T)$), the set for $\lambda \in \mathbb{C}^n$ such that the family $(T_i - \lambda_i S, \ \forall\, 1 \leq i \leq n)$ generates in the algebra $\mathcal{B}(X)$ a proper left (resp. right) ideal. Hence $\lambda \notin \sigma_{S,l}(T)$ if and only if there exists $(B_1, \ldots, B_n) \in \mathcal{B}(X)^n$ such that

$$\sum_{i=1}^{n} (T_i - \lambda_i S)\, B_i = I,$$

and, $\lambda \notin \sigma_{S,r}(T)$ if and only if there exists $(B_1, \ldots, B_n) \in \mathcal{B}(X)^n$ such that

$$\sum_{i=1}^{n} B_i\, (T_i - \lambda_i S) = I. \qquad \diamondsuit$$

Clearly for n-tuple $T = (T_1, \ldots, T_n) \in \mathcal{B}^n$ and $S \in \mathcal{B}(X)$, we have $\sigma_{S,l}(T) \subset \sigma'_S(T)$ and $\sigma_{S,r}(T) \subset \sigma'_S(T)$. We move now to defining the joint spectrum of an $n-$tuple of operator pencils $T - \lambda S$ and similarly we write the S-joint spectrum of T.

Definition 3.4 Let S and T be as above. The S-joint spectrum for T denoted by $\sigma_S(T)$ is the union of the S-joint left spectrum and the S-joint right spectrum and we have

$$\sigma_S(T) = \sigma_{S,l}(T) \cup \sigma_{S,r}(T),$$

then $\lambda \notin \sigma_S(T)$ if and only if there exist $(B_1, \ldots, B_n)$ such that:

$$\sum_{i=1}^{n} B_i\, (T_i - \lambda_i S) = \sum_{i=1}^{n} (T_i - \lambda_i S)\, B_i = I. \qquad \diamondsuit$$

Theorem 3.1 *Let $T = (T_1, \ldots, T_n)$ in $\mathcal{B}(X)^n$ and $S \neq 0$ in $\mathcal{B}(X)$. All the sets $\sigma_{S,l}(T)$, $\sigma_{S,r}(T)$, $\sigma_S(T)$, $\sigma'_S(T)$, and $\sigma_{S,a}(T)$ are compact subsets of $\mathbb{C}^n$.* $\diamondsuit$

Proof: We prove first that these sets are closed in $\mathbb{C}^n$. We start by $\sigma_{S,a}(T)$. Suppose that $\lambda \notin \sigma_{S,\mathrm{a}}(T)$, then there exists a $\delta > 0$ such that, for all $x \in X$ with $\|x\| = 1$ we have

$$\sum_{j=1}^{n} \|(T_j - \lambda_j S)x\| > \delta.$$

Now, consider the following neighborhood of λ :

$$\mathcal{V}_S = \left\{ \lambda' \in \mathbb{C}^n : \left|\lambda'_j - \lambda_j\right| \|S\| < \frac{\delta}{2n}, j = 1, \ldots, n \right\}.$$

For $\lambda' \in \mathcal{V}_S$ and $x \in X$ with $\|x\| = 1$ we have

$$\begin{aligned}
\sum_{j=1}^{n} \left\|T_j x - \lambda'_j S x\right\| &\geqslant \sum_{j=1}^{n} \left(\|T_j x - \lambda_j S x\| - \left\|\left(\lambda_j - \lambda'_j\right) S x\right\| \right) \\
&= \sum_{j=1}^{n} \|T_j x - \lambda_j S x\| - \sum_{j=1}^{n} \left|\lambda_j - \lambda'_j\right| \|S\| > \delta - \frac{\delta}{2},
\end{aligned}$$

then we obtain $\sum_{j=1}^{n} \left\|T_j x - \lambda'_j S x\right\| > \dfrac{\delta}{2}$. Hence $\lambda' \notin \sigma_{S,\mathrm{a}}(T)$ and $\sigma_{S,\mathrm{a}}(T)$ is closed in $\mathbb{C}^n$.

Now, let $\lambda \notin \sigma_{S,l}(T)$ then there exist $B_1, \ldots, B_n$ such that $\sum_{i=1}^{n} (T_i - \lambda_i S) B_i = I$. We can find $\delta > 0$ such that for any $\mu \in \mathbb{C}^n$ with $|\mu_i - \lambda_i| < \frac{\delta}{\|S\|}$ for $i = 1, \ldots, n$, we have

$$\left\| \sum_{i=1}^{n} B_i (T_i - \mu_i S) - I \right\| < 1.$$

This implies that the element $\sum_{i=1}^{n} B_i (T_i - \mu_i S)$ has an inverse Q, in $\mathcal{B}(X)$, and so

$$\sum_{i=1}^{n} Q B_i (T_i - \mu_i S) = I.$$

Which means $\mu \notin \sigma_{S,l}(T)$. Thus, we have shown that the complement of $\sigma_{S,l}(T)$ in $\mathbb{C}^n$ is open.

The same approach we prove that $\sigma_{S,r}(T)$ is closed in $\mathbb{C}^n$. Since $\sigma_S(T)$ is the union of two closed subsets in $\mathbb{C}^n$, then $\sigma_S(T)$ is also closed in $\mathbb{C}^n$.

We prove now that the S-joint commutant spectrum is closed. Indeed, let $\lambda \notin \sigma'_S(T)$ and if λ' is in the neighborhood of λ given by the inequality

$$\sum_{j=1}^{n} \left|\lambda_j - \lambda'_j\right| \|S\| \|B_j\| < 1,$$

then the operator

$$\mathcal{Q} = \sum_{j=1}^{n} \left(T_j - \lambda'_j S\right) B_j,$$

belongs to $\mathcal{B}'$ and has an inverse belonging to $\mathcal{B}'$. Since the operator

$$I - \mathcal{Q} = \sum_{j=1}^{n} \left(\lambda_j - \lambda'_j\right) S B_j,$$

belongs to $\mathcal{B}'$ and $\|I - \mathcal{Q}\| < 1$. Thus,

$$\sum_{j=1}^{n} \left(T_j - \lambda'_j S\right) B_j \mathcal{Q}^{-1} = I,$$

then, $\lambda' \notin \sigma'_S(T)$. One similarly proves that the S-joint bicommutant spectrum is closed.

For n-tuple $T = (T_1, \cdots, T_n)$, all this spectra are contained in $\sigma_S(T)$, and we have

$$\sigma_S(T_1, \cdots, T_n) \subset \prod_{i=1}^{n} \sigma_S(T_i),$$

since $\sigma_S(T_i)$ is compact in $\mathbb{C}$ for all $1 \leq i \leq n$, then we obtain $\sigma_S(T_1, \cdots, T_n)$ is a bounded subset in $\mathbb{C}^n$. Finally all this spectra are closed and bounded subset in $\mathbb{C}^n$, hence compact subsets in $\mathbb{C}^n$. □

Example 3.1 *Let $T = (T_1, \cdots, T_n)$ be n-tuple of $m \times m$ "upper triangular' complex matrices:*

$$T_j = \begin{pmatrix} T_j(11) & T_j(12) & \ldots & T_j(1m) \\ 0 & T_j(22) & \ldots & T_j(2m) \\ 0 & \ldots & \ldots & . \\ 0 & 0 & \ldots & T_j(mm) \end{pmatrix},$$

and S be an upper triangular complex matrix with the following form :

$$S=\begin{pmatrix} S_{11} & S_{12} & \dots & S_{1m} \\ 0 & S_{22} & \dots & S_{2m} \\ 0 & \dots & \dots & . \\ 0 & 0 & \dots & S_{mm} \end{pmatrix} \quad \text{with} \quad S_{ii}\neq 0 \ \forall\ 1\leq i\leq n.$$

The S-joint spectrum for T is given by the set:

$$\sigma_S(T)=\left\{\frac{T_1(kk)}{S_{11}},\frac{T_2(kk)}{S_{22}},\cdots,\frac{T_n(kk)}{S_{mm}},\quad \forall\ 1\leq k\leq n\right\}.$$

Let

$$\lambda_j=\frac{T_j(kk)}{S_{ii}},\ \forall \ \ 1\leq j\leq n \ \ \text{and} \ \ 1\leq i,k\leq m.$$

For arbitrary $U=(U_1,\cdots,U_n)$ of n-tuple of $m\times m$ upper triangular matrices.

We have $U(T-\lambda S)=U\left(S^{-1}T-\lambda I\right)$ and $(T-\lambda S)U=\left(S^{-1}T-\lambda I\right)U$, with S^{-1} the inverse matrix of S. Since S is an upper triangular matrix then it is the same for S^{-1} and $S^{-1}T$, with $\left(S^{-1}T\right)_{ii}=\dfrac{T_{jj}(kk)}{S_{ii}}$ for all $1\leq i\leq n$. Then the matrix $B=S^{-1}T$ be a n-tuple of $m\times m$ upper triangular matrices with

$$B_{ii}=\tfrac{T_j(kk)}{S_{ii}},\quad \forall\ 1\leq j\leq n \ \ \text{and} \ \ 1\leq i,k\leq m. \qquad \diamondsuit$$

The proof is sealed by applying the result of R. Harte in [8] example 2.3.

Example 3.2 *Let $T=(T_1,\cdots,T_n)$ be n-tuple of $m\times m$ lower triangular complex matrices and S be $m\times m$ lower triangular complex matrices. $\sigma_S(T)$, is given by the set of diagonal the entries of the matrix $S^{-1}T$.* $\diamondsuit$

4. Tensor product of the joint spectra of pencil operators

Let $X_1,\cdots,X_n$ be complex Banach spaces and let W be the completion of $X_1\otimes\cdots\otimes X_n$ with respect to some cross-norm (cf., e.g., [12,16]). Let I_k be the identity operator and A_k an arbitrary bounded operator on $X_k, 1\leq k\leq n$. Set:

$$\begin{array}{rcl} T_1 &=& A_1\otimes I_2\otimes\cdots\otimes I_n, \\ T_2 &=& I_1\otimes A_2\otimes I_3\cdots\otimes I_n, \\ T_k &=& I_1\otimes\cdots\otimes I_{k-1}\otimes A_k\otimes I_{k+1}\cdots\otimes I_n. \end{array}$$

The operators T_i obviously commute, and we have $\sigma\left(T_k\right)=\sigma\left(A_k\right), 1\leq k\leq n$.

A. T. Dash and M. Schechter proved in [7] that, for n-tuple $(T_1,\cdots,T_n)$ in $\mathcal{B}''$, a complex vector $(\lambda_1,\cdots,\lambda_n)$ is in $\sigma''\left(T_1,\cdots,T_n\right)$ if and only if $\lambda\in\sigma(A_k),\ 1\leq k\leq n$ and we have

$$\sigma''\left(T_1,\cdots,T_n\right)=\prod_{k=1}^{n}\sigma\left(T_k\right)=\prod_{k=1}^{n}\sigma\left(A_k\right).$$

The following theorem aims to give a relation between $\sigma''_S(T)$ and $\sigma_S(A_k)$ for all $1\leq k\leq n$, what is in the following theorem.

Theorem 4.1 *Let $(T_1,\cdots,T_n)\in\mathcal{B}''$ and $S\in\mathcal{B}(X)$ such that $0\notin\sigma(S)$, then we have:*

$$\sigma_S\left(T_1,\cdots,T_n\right)=\textstyle\prod_{k=1}^{n}\sigma_S\left(T_k\right)=\sigma_S(I)^{n-1}\prod_{k=1}^{n}\sigma_S(A_k)=\sigma(S)^{n-1}\prod_{k=1}^{n}\sigma_S(A_k). \qquad \diamondsuit$$

Proof: First of all, by giving the relation between $\sigma_S(T_k)$ and $\sigma_S(A_k)$ we have

$$\sigma_S(T_1)=\sigma_S(A_1\otimes I_2\otimes\cdots\otimes I_n)=\sigma_S(A_1).\sigma_S(I_2)\cdots\sigma_S(I_n).$$

Since $\sigma_S(I_2) = \cdots = \sigma_S(I_n)$ and $\sigma_S(I_k) = \sigma(S)$, for all $1 \leq k \leq n$, we obtain

$$\sigma_S(T_1) = \sigma_S(A_1).(\sigma(S))^{n-1}.$$

Finally, for all $1 \leq k \leq n$, on gets

$$\sigma_S(T_k) = \sigma_S(A_k).\sigma(S)^{n-1}.$$

A complex vector $(\lambda_1, \cdots, \lambda_n)$ is in the S-joint resolvent $\rho_S(T_1, \cdots, T_n)$ of the T_k if there exist $C_1, \cdots, C_n \in \mathcal{B}''$ such that

$$\sum_{k=1}^{n} C_k\left(T_k - \lambda_k S\right) = I = I_1 \otimes \cdots \otimes I_n.$$

Otherwise it is in $\sigma''_S(T_1, \cdots, T_n)$. We prove now the first inclusion, let $\lambda_k \in \rho_S(T_k)$ for some k, then there exist $C_k \in \mathcal{B}''$ such that

$$C_k\left(T_k - \lambda_k S\right) = I.$$

Setting

$$C_k = (T_k - \lambda_k S)^{-1} \quad \text{and} \quad C_j = 0 \quad \text{for} \quad j \neq k.$$

We obtain

$$\sum_{k=1}^{n} C_k\left(T_k - \lambda_k S\right) = I,$$

hence $(\lambda_1, \cdots, \lambda_n)$ is in $\rho_S(T_1, \cdots, T_n)$.

For the second inclusion. Assume that $\lambda_k \in \sigma_S(T_k)$ for each k, this implies that $(\lambda_1, \cdots, \lambda_n)$ is in $\sigma_S(T_1, \cdots, T_n)$. For $\lambda_k = 0$, then $0 \in \sigma_S(T_k) = \sigma_S(A_k) * \sigma(S)^{n-1}$, since $0 \notin \sigma(S)^{n-1}$ we get $0 \in \sigma_S(A_k)$, $\forall\, 1 \leq k \leq n$.
In this case we have either a sequence $\{U_{km}\}$ of elements in X_k such that:

$$\|U_{km}\| = 1, \|A_k U_{km}\| \to 0 \text{ as } m \to \infty, \tag{4.1}$$

or a sequence $\{U'_{km}\}$ of elements in X'_k where X'_k denotes the dual space of X_k and A'_k is the conjugate of A_k, such that

$$\|U'_{km}\| = 1, \|A'_k U'_{km}\| \to 0 \text{ as } m \to \infty. \tag{4.2}$$

For $1 \leq k \leq t$ and U'_{km} in X'_k we set

$$\|U'_{km}\| = 1 \;\; and \;\; U'_{km}\left(U_{km}\right) = 1, \tag{4.3}$$

and for $t < k \leq n$, let $U_{km} \in X_k$ be such that,

$$\|U_{km}\| = 1 \;\; and \;\; \left|U'_{km}\left(U_{kn}\right)\right| > 1 - \frac{1}{m}. \tag{4.4}$$

We choose now the sequence $(U_m)_m$ in X_k defined by:

$$U_m = U_{1m} \otimes \cdots \otimes U_{1m},$$

and U'_m in X'_k defined by:

$$U'_m = U'_{1m} \otimes \cdots \otimes U'_{nm}.$$

Applying (4.3) and (4.4) we get:

$$\left|U'_m\left(U_m\right)\right| \geq \left(1 - \frac{1}{m}\right)^n \to 1 \text{ as } m \to \infty.$$

Now suppose that $(0,\cdots,0)\in\rho_S(T_1,\cdots,T_n)$ then there exist $C_1,\cdots,C_n\in\mathcal{B}''$ such that

$$\sum_{1}^{n} C_k T_k = I. \tag{4.5}$$

On the other hand we have:

$$U'_m\left(\sum_{k=1}^{n} C_k T_k U_m\right)=\sum_{k=1}^{t} U'_m\left(C_k T_k U_m\right)+\sum_{k=t+1}^{n} T'_k U'_m\left(C_k U_m\right),$$

and by applying (4.5) we obtain:

$$\left|U'_m\left(U_m\right)\right|\leq\sum_{k=1}^{t}\|C_k\|\,\|A_k U_{km}\|+\sum_{k=t+1}^{n}\|C_k\|\,\|A'_k U'_{km}\|\ \to 0 \text{ as } m\to\infty.$$

This contradicts the fact that $\left|U'_m\left(U_m\right)\right|\geq\left(1-\frac{1}{m}\right)^n\to 1$ as $m\to\infty$. Which yields $(0,\cdots,0)\in\sigma''_S\left(T_1,\cdots,T_n\right)$. □

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