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Super-Recurrence for Backward Shifts

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ABSTRACT: In this paper, we characterize the super-recurrence of backward shifts acting on the weighted sequence spaces $\ell^p(\mathbb{Z},\nu)$ for $1 \leq p < \infty$ and $c_0(\mathbb{Z},\nu)$, where $v := (v_n)_n$ is a strictly positive sequence of weights. As a result, we show that supercyclic and super-recurrent backward shifts are equivalent. We also prove that there are no super-recurrent backward shifts neither on $\ell^\infty(\mathbb{N},\nu)$ nor on $\ell^\infty(\mathbb{Z},\nu)$.

Key Words: Weighted sequence spaces, backward shifts, super-recurrence, supercyclicity.

Contents

1. Introduction and preliminaries

Throughout this paper, and unless otherwise stated, X will denote an infinite-dimensional separable complex Fréchet space which is over the field of complex numbers $\mathbb{C}$. Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators acting on X. We denote $I = \mathbb{N}$ or $\mathbb{Z}$, where $\mathbb{Z}$ and $\mathbb{N}$ are the sets of all integers and positive integers, respectively. Recall that the space of all (real or complex) sequences is defined as follows

$$\mathbb{K}^I := \left\{(x_k)_k \ ; x_k \in \mathbb{K},\ k \in I\right\},$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The natural concept of convergence is that of coordinatewise convergence.

Let B be the unilateral (respectively, bilateral) backward shift on a Fréchet sequence space indexed over $\mathbb{N}$ (respectively, indexed over $\mathbb{Z}$) which is defined by

$$B(x_k)_{k\in I} := (x_{k+1})_{k\in I}.$$

Similarly, the unilateral (respectively, bilateral) weighted backward shift on a Fréchet sequence space indexed over $\mathbb{N}$ (respectively, indexed over $\mathbb{Z}$) is defined by

$$B_{\mathbf{w}}(x_k)_{k\in I} := (w_{k+1}x_{k+1})_{k\in I},$$

where $\mathbf{w} = (w_k)_{k\in I}$ is called a weight sequence of $B_{\mathbf{w}}$. The sequence $(w_k)_{k\in I}$ will be assumed to be a bounded sequence of nonnegative real numbers.

For a strictly nonnegative sequence of weights $\nu := (v_k)_k$, the weighted space $\ell^p(I,\nu)$ is defined by

$$\ell^p(I,\nu) := \left\{(x_k)_{k\in I} \ ; \sum_{k\in I} |x_k|^p v_k < \infty\right\}; \quad 1 \leq p < \infty.$$

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$\ell^p(I,\nu)$ is a Banach space, endowed with the norm $\|x\|_{p,v} := \big(\sum_{k\in I}|x_k|^p v_k\big)^{\frac{1}{p}}$.

When $p=\infty$, we define the weighted space of bounded sequences $\ell^\infty(I,\nu)$ and its closed subspace of null sequences $c_0(I,\nu)$ by

$$\ell^\infty(I,\nu) := \left\{(x_k)_{k\in I}\ ;\ \sup_{k\in I}|x_k|v_k < \infty\right\}$$

and

$$c_0(I,\nu) := \left\{(x_k)_{k\in I}\ ;\ \lim_{k\to\infty}|x_k|v_k = 0\right\}.$$

$\ell^\infty(I,\nu)$ and $c_0(I,\nu)$ are Banach spaces under the sup-norm $\|x\|_\infty := \sup_{k\in I}|x_k|v_k$ and the induced norm respectively.

To have the boundedness of backward shifts on weighted spaces, it is required that the weights satisfy

$$\sup_{k\in I}\frac{v_k}{v_{k+1}} < +\infty.$$

Condition that will always be assumed to hold.

The concept of hypercyclicity, supercyclicity and recurrence play an important role in the linear dynamic behavior of an operator $T\in\mathcal{B}(X)$. We refer the readers to these survey papers [2,7,8,11,12,14,15] and to the interesting books [3,10].

Recall that an operator $T\in\mathcal{B}(X)$ is hypercyclic if there is some $x\in X$ such that the orbit of x under T, that is, the set

$$\operatorname{Orb}(T,x) := \{T^n x; n\geq 0\},$$

is dense in X. $T\in\mathcal{B}(X)$ is supercyclic if there is some $x\in X$ such that the projective orbit of x under T, that is, the set

$$\mathbb{C}.\operatorname{Orb}(T,x) := \{\lambda T^n x; \lambda\in\mathbb{C}, n\geq 0\},$$

is dense in X. In each case, such a vector x is called a hypercyclic vector and a supercyclic vector for T, respectively. The set of all hypercyclic (respectively, supercyclic) vectors for T is denoted by $\operatorname{HC}(T)$ (respectively, $\operatorname{SC}(T)$). The hypercyclicity and the supercyclicity are often verified by equivalent formulations which consist in showing that the orbit (respectively, projective orbit) of any non-empty open subset of X under T is dense in X.

The first example of a hypercyclic operator in infinite-dimensional Banach space was constructed by Rolewicz [15] in 1969 where the example is: the multiples λB $(\lambda\in\mathbb{C})$ of the backward shift operator B on $\ell^p(\mathbb{N})$; $1\leq p<\infty$ are hypercyclic if and only if $|\lambda|>1$.

Later on, the concept of supercyclicity was defined by Hilden and Wallen [12] in 1974 where they proved that every unilateral weighted backward shift on a Banach space is supercyclic. This concept was extensively studied by many authors, for example Salas [16] gave a characterization of supercyclic bilateral weighted backward shifts via the well known Supercyclicity Criterion that is, a sufficient condition for supercyclicity. For more details about hypercyclicity, supercyclicity and Supercyclicity Criterion we refer the reader to these works [3,6,10,14,16].

An oldest important notion in the theory of dynamical systems, is that recurrence. The study of this notion started with Poincaré in the measure setting in 1890 with the Poincaré Recurrence Theorem [13]. Latterly, Birkhoff continued Poincaré's work in the topological setting in 1920 with Birkhoff recurrence theorem. Recently, this notion was studied systematically in the linear dynamics in two fundamental papers by G. Costakis, A. Manoussos, I. Parissis [7] and G. Costakis, I. Parissis [8]. Following these works, an operator $T\in\mathcal{B}(X)$ is said to be recurrent if for each non-empty open subset U of X, there exists a nonnegative integer n such that

$$T^n(U)\cap U\neq\emptyset.$$

A vector $x\in X$ is called a recurrent vector for T if there exists a strictly increasing sequence of nonnegative integers $(n_k)_k$ such that

$$T^{n_k}x\longrightarrow x \quad \text{as} \quad k\longrightarrow\infty.$$

Rec(T) denote the set of all recurrent vectors for T and we have that T is recurrent if and only if Rec(T) is dense in X. This notion has received the attention of several authors, in different types of generality for example, see [5]. Recently, M. Amouch et O. Benchiheb [1] generalized the class of recurrent operators to a large class of operators called super-recurrent operators. We say that $T \in \mathcal{B}(X)$ is super-recurrent if for each non-empty open subset U of X, there exist some $n \geq 1$ and some $\lambda \in \mathbb{C}$ such that

$$\lambda T^n(U) \cap U \neq \emptyset.$$

A vector $x \in X$ is called a super-recurrent vector for T if there exist a strictly increasing sequence of nonnegative integers $(n_k)_k$ and a sequence of nonzero complex numbers $(\lambda_k)_k$ such that

$$\lambda_k T^{n_k} x \longrightarrow x \text{ as } k \longrightarrow \infty.$$

SRec(T) denotes the set of all super-recurrent vectors for T. Recurrence implies super-recurrence. However, the converse does not hold in general.
We summarize the relations linking all the notions mentioned above in the following diagram.

$$\begin{tikzcd}
\text{Recurrence} \arrow[r, Rightarrow] \arrow[d, Rightarrow, bend left, "\times\ {[3,\ \text{Proposition 1.1}]}"] & \text{Super-recurrence} \arrow[l, Rightarrow, bend right, "{[1,\ \text{Remark 2.2}]}"', "\times"] \arrow[d, Rightarrow, bend left, "\times\ {[1,\ \text{Remark 2.2}]}"] \\
\text{Hypercyclicity} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \text{Supercyclicity} \arrow[u, Rightarrow] \arrow[l, Rightarrow, bend left, "\times", "{[3,\ \text{Example 1.15}]}"']
\end{tikzcd}$$

The paper is organized as follows, in Section 2 we characterize the super-recurrent bilateral backward shifts by means of the weight sequence. Consequently, we show that super-recurrent and supercyclic bilateral backward shifts acting on $\ell^p(\mathbb{Z}, \nu)$ and $c_0(\mathbb{Z}, \nu)$ are equivalent. Finally in Section 4, we extend the weighted shift's result established by George Costakis, Antonios Manoussos and Ioannis Parissis in [7, Theorem 5.1] by proving that there are no super-recurrent shifts neither on $\ell^\infty(\mathbb{N}, \nu)$ nor on $\ell^\infty(\mathbb{Z}, \nu)$.

2. Weighted ℓ^p-spaces and super-recurrent backward shifts

In order to provide a characterization of the super-recurrent bilateral backward shift on $\ell^p(\mathbb{Z}, \nu)$ for $1 \leq p < \infty$ and $c_0(\mathbb{Z}, \nu)$ in terms of the weights, we first need the following lemma.

Lemma 2.1 *[1] Let $T \in \mathcal{B}(X)$. The following assertions are equivalent*

1. *T is super-recurrent;*
2. *For all $x \in X$, there exist a sequence $(n_k)_k$ of nonnegative integers, a sequence $(x_k)_k$ of elements of X and a sequence $(\lambda_k)_k$ of nonzero complex numbers such that*

$$x_k \underset{k \to +\infty}{\longrightarrow} x \quad \textit{and} \quad \lambda_k T^{n_k}(x_k) \underset{k \to +\infty}{\longrightarrow} x.$$

Theorem 2.2 *Let B be a bilateral backward shift acting on $\ell^p(\mathbb{Z}, \nu)$; $1 \leq p < \infty$ or $c_0(\mathbb{Z}, \nu)$. Then, B is super-recurrent, if and only if, for all $i \in \mathbb{Z}$, there exist a sequence of nonnegative integers $(n_k)_k$ and a sequence of nonzero complex numbers $(\lambda_k)_k$ such that*

$$\lim_{k \to +\infty} \frac{1}{|\lambda_k|} v_{i+n_k} = 0 \quad \textit{and} \quad \lim_{k \to +\infty} |\lambda_k| v_{i-n_k} = 0.$$

Proof: Suppose that B is super-recurrent. Fix $i \in \mathbb{Z}$ and $0 < \varepsilon < \min\{1, v_i\}$. By virtue of Lemma 2.1, there exist $x \in \ell^p(\mathbb{Z}, \nu)$, an integer $n > 2i$ and $\lambda \in \mathbb{C}\backslash\{0\}$ such that

$$\|x - e_i\|_{p,v} < \varepsilon \quad \text{and} \quad \|\lambda B^n x - e_i\|_{p,v} < \varepsilon;$$

where $(e_i)_{i\in\mathbb{Z}}$ denote the canonical basis of $\ell^p(\mathbb{Z}, \nu)$. Then, we obtain from the first inequality

$$|x_i - 1|v_i < \varepsilon \quad \text{and} \quad |x_{i+n}|v_{i+n} < \varepsilon.$$

We have from the second inequality

$$|\lambda x_{n+i} - 1|v_i < \varepsilon \quad \text{and} \quad |\lambda x_i|v_{i-n} < \varepsilon.$$

Using these inequalities we get

$$\frac{v_{i+n}}{|\lambda|} < \frac{\varepsilon v_i}{v_i - \varepsilon} \quad \text{and} \quad |\lambda|v_{i-n} < \frac{\varepsilon v_i}{v_i - \varepsilon}$$

Since ε is arbitrary, the result follows.

Conversely, let $S : \ell^p(\mathbb{Z}, \nu) \longrightarrow \ell^p(\mathbb{Z}, \nu)$ be the forward shift defined by $Se_i = e_{i+1}$. Fix $i \in \mathbb{Z}$, according to assumptions, there exist a sequence of nonnegative integers $(n_k)_k$ and a sequence of nonzero complex numbers $(\lambda_k)_k$ such that

$$|\lambda_k| \, \|B^{n_k} e_i\|_{p,v} = |\lambda_k| \, \|e_{i-n_k}\|_{p,v} = |\lambda_k| v_{i-n_k} \underset{k\to+\infty}{\longrightarrow} 0;$$

and

$$\|y_k\|_{p,v} := \frac{1}{|\lambda_k|} \|S^{n_k} e_i\|_{p,v} = \frac{1}{|\lambda_k|} \|e_{i+n_k}\|_{p,v} = \frac{1}{|\lambda_k|} v_{i+n_k} \underset{k\to+\infty}{\longrightarrow} 0.$$

Set $x_k := y_k + e_i$. Then,

$$\|x_k - e_i\|_{p,v} = \|y_k\|_{p,v} \underset{k\to+\infty}{\longrightarrow} 0 \text{ and } \lambda_k B^{n_k} y_k = \lambda_k B^{n_k}(\frac{1}{\lambda_k} S^{n_k} e_i) = e_i.$$

Thus,

$$\|\lambda_k B^{n_k} x_k - e_i\|_{p,v} = \|(\lambda_k B^{n_k} e_i) + (\lambda_k B^{n_k} y_k - e_i)\|_{p,v} \underset{k\to+\infty}{\longrightarrow} 0.$$

By Lemma 2.1, the result follows. □

Corollary 2.3 *Let B be a bilateral backward shift, acting on $\ell^p(\mathbb{Z}, \nu)$; $1 \leq p < \infty$ or $c_0(\mathbb{Z}, \nu)$. The operator B is super-recurrent, if and only if, it is supercyclic.*

Proof: Assume that T is super-recurrent. Then, for all $i \in \mathbb{Z}$, there exist a sequence of nonnegative integers $(n_k)_k$ such that $\liminf\limits_{k\to\infty} v_{i+n_k} v_{i-n_k} = 0$. Hence, T is supercyclic. □

Remark 2.4 *Using conjugacy, these results can be generalized to weighted shifts. Indeed, let $B_{\mathbf{w}}$ be a weighted backward shift acting on $\ell^p(\mathbb{Z})$. The operator $\phi_v : \ell^p(\mathbb{Z}, \nu) \to \ell^p(\mathbb{Z})$ is defined by $\phi_v(x_k)_k = (x_k v_k)_k$ with*

$$v_k = (w_1 \cdots w_k)^{-1} \text{ for } k \geq 1, \; v_k = w_{k+1} \cdots w_0 \text{ for } k \leq -1 \text{ and } v_0 = 1.$$

It is clear that $B_{\mathbf{w}} \circ \phi_v = \phi_v \circ B$, which means that the following diagram commutes

$$\begin{tikzcd} \ell^p(\mathbb{Z}, \nu) \arrow[r, "B"] \arrow[d, "\phi_v"'] & \ell^p(\mathbb{Z}, \nu) \arrow[d, "\phi_v"] \\ \ell^p(\mathbb{Z}) \arrow[r, "B_{\mathbf{w}}"'] & \ell^p(\mathbb{Z}) \end{tikzcd}$$

Since the conjugacy preserves the super-recurrence [1], the previous result yields the following corollary.

Corollary 2.5 *Let $\mathbf{w} = (w_k)_{k\in\mathbb{Z}}$ be the weight sequence and $B_{\mathbf{w}}$ be a bilateral weighted backward shift on $\ell^p(\mathbb{Z})$ for $1 \le p < \infty$. Then $B_{\mathbf{w}}$ is super-recurrent if and only if, $B_{\mathbf{w}}$ is supercyclic.*

Proof: Notice that, according to [16, Theorem 3.1], $B_{\mathbf{w}}$ is supercyclic if and only if for any $i \in \mathbb{Z}$,

$$\liminf_{k\to+\infty} (w_1 \cdots w_{i+k})^{-1} \times (w_0 \cdots w_{i-k+1}) = 0.$$

The result follows at once from the previous corollary since the sequence $(v_k)_k$ satisfies $v_{i+k} = (w_1 \cdots w_{i+k})^{-1}$ and $v_{i-k} = w_0 \cdots w_{i-k+1}$. □

Remarks 2.6

1. *The proofs for the case $c_0(\mathbb{N}, \nu)$ are similar using the supremum norm.*
2. *Every unilateral backward shift acting on $\ell^p(\mathbb{N}, \nu)$ for $1 \le p < \infty$ or $c_0(\mathbb{N}, \nu)$ is super-recurrent.*
3. *Since the sequence spaces $\ell^p(\mathbb{Z}, \nu)$ for $0 < p < 1$ are not local convex, we can not include them in our study.*

3. Weighted ℓ^∞-space and super-recurrent backward shifts

As we mentioned in the previous section, there exist super-recurrent backward shifts on $\ell^p(I, \nu)$; for $1 \le p < \infty$. However, $\ell^\infty(I, \nu)$ do not support any super-recurrent backward shift.

Theorem 3.1 *There are no super-recurrent unilateral or bilateral backward shifts neither on $\ell^\infty(\mathbb{N}, \nu)$ nor on $\ell^\infty(\mathbb{Z}, \nu)$, respectively.*

Proof: For the sake of contradiction, assume that $B : \ell^\infty(\mathbb{N}, \nu) \to \ell^\infty(\mathbb{N}, \nu)$ is a super-recurrent unilateral backward shift. Let $M \ge 1$ and define the vector $x := (x_1, \frac{M+1}{v_2}, x_3, \cdots)$ such that for all $j > 1$, $|x_j| > \frac{1}{2v_j}$. By virtue of [1, Theorem 3.8] we have, $\mathrm{SRec}(B)$ is dense in $\ell^\infty(\mathbb{N}, \nu)$. Which means that there exist a super-recurrent vector y, a nonnegative integer n and a nonzero complex number λ such that

$$\|y - x\|_\infty = \sup_{k\ge1} |y_k - x_k| v_k < \frac{1}{2} \quad \text{and} \quad \|\lambda B^n y - y\|_\infty = \sup_{k\ge1} |\lambda y_{n+k} - y_k| v_k < \frac{1}{2}.$$

From the first inequality, it follows that

$$v_1|y_1| < \frac{1}{2} + v_1|x_1| \quad \text{and} \quad \frac{1}{2} + M < v_2|y_2|.$$

And from the second inequality we obtain

$$v_1|\lambda y_{n+1}| < \frac{1}{2} + v_1|y_1| \quad \text{and} \quad \frac{-1}{2} + v_2|y_2| < v_2|\lambda y_{n+2}|.$$

Using these inequalities we get

$$v_1|\lambda y_{n+1}| < 1 + v_1|x_1| \quad \text{and} \quad M < v_2|\lambda y_{n+2}|.$$

Hence

$$\frac{v_1 M}{v_2(1 + v_1|x_1|)} < \frac{|y_{n+2}|}{|y_{n+1}|}.$$

Looking at the $(n+1)$-th and $(n+2)$-th coordinates in the first inequality we get

$$\frac{|y_{n+2}|}{|y_{n+1}|} < \frac{v_{n+1}(\frac{1}{2} + v_{n+2}|x_{n+2}|)}{v_{n+2}(\frac{-1}{2} + v_{n+1}|x_{n+1}|)}.$$

Therefore

$$\frac{v_{n+1}}{v_{n+2}} > \left(\frac{\frac{-1}{2} + v_{n+1}|x_{n+1}|}{\frac{1}{2} + v_{n+2}|x_{n+2}|}\right)\left(\frac{v_1 M}{v_2(1+v_1|x_1|)}\right).$$

Since M can be chosen to be arbitrarily large, $\sup\limits_{k\geq 1}\frac{v_k}{v_{k+1}}$ goes to infinity, which is a contradiction. □

By conjugacy, we can reformulate the result of Theorem 3.1 in terms of weighted backward shifts acting on classical unweighted sequence spaces $\ell^\infty(\mathbb{N})$ and $\ell^\infty(\mathbb{Z})$, as the following shows.

Corollary 3.2 *There are no super-recurrent unilateral or bilateral weighted backward shifts neither on* $\ell^\infty(\mathbb{N})$ *nor on* $\ell^\infty(\mathbb{Z})$, *respectively.*

Proof: Applying Theorem 3.1 and following the same argument as Remark 2.4, we get immediately the proof. □

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