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On Some Existence Results for Nonlinear Fractional Differential Equations with Hybrid Proportional Caputo Derivatives

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ABSTRACT: The aim of this manuscript is to prove the existence and uniqueness of certain solutions to a class of boundary value issues for a nonlocal boundary condition generalized nonlinear hybrid Caputo fractional differential equation. Some fundamental methods of φ-fractional calculus as well as mixed Lipshitz and Caratheodory conditions are used to prove our main results. Additionally, a few basic φ-fractional differential inequalities are proved. The final section of this study provides a nontrivial case as an application to show our theoretical results.

Key Words: $\varphi-$fractional integral, $\varphi-$Caputo fractional derivative, fixed point, Carathéodory function.

Contents

1. Introduction

The main crux of this work is to study the existence and uniqueness of solutions for the following nonlinear conformable fractional evolution equation:

$$\begin{cases} {}^{C}D^{\beta;\varphi}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)),\ t \in J = [0,T], \\ \\ x'(0) = 0, \quad a\dfrac{x(0)}{f(0,x(0))} + b\dfrac{x(T)}{f(T,x(T))} = c. \end{cases} \tag{1.1}$$

Where ${}^{C}D^{\beta;\varphi}$ is the φ-Caputo fractional derivative of order $2 < \beta \leq 3$,
$f \in \mathcal{C}^2(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$.

In various applied sciences, fractional differential equations have gained prominence as potent tools for modeling a range of phenomena. Recently, these equations have been employed extensively to enhance the accuracy of descriptions for phenomena that exhibit both discrete and continuous behaviors. Fractional-order models have found applications across diverse scientific and engineering fields, such as material theory, transport processes, earthquakes, electrochemical processes, wave propagation, signal theory, biology, electromagnetic theory, fluid flow phenomena, thermodynamics, mechanics, geology, astrophysics, economics, and control theory(see [3,10,12,14,16,22,24]). Recently, there has been a notable surge in interest surrounding fractional differential equations, particularly in the context of boundary

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value problems associated with nonlinear fractional differential equations. These equations prove valuable for modeling and articulating non-homogeneous physical phenomena in their distinct forms. Almeida et al. [5] investigated the existence and uniqueness results of nonlinear fractional differential equations involving a Caputo-type fractional derivative with respect to another function by using fixed point theorems and Picard iteration metho. Zhang in [27] proved the existence and uniqueness results for nonlinear fractional boundary value problem involving Caputo type fractional derivatives by using some fixed point theorems. Many researchers have obtained some interesting results on the existence and uniqueness of solutions of boundary value problems for fractional differential equations involving different fractional derivatives such as Riemann-Liouville [19], Caputo [2], Hilfer [18], Erdelyi-Kober [21] and Hadamard [1]. There is a certain type of kernel dependency included in all those definitions. Therefore, a fractional derivative with respect to another function known as the ψ-Caputo derivative was introduced in order to study fractional differential equations in a general manner. For specific selections of ψ, we can obtain some well-known fractional derivatives, such as the Caputo, Caputo-Hadamard, or Caputo-Erdélyi-Kober fractional derivatives, which are dependent on a kernal. From the viewpoint of applications, this approach also seems appropriate. With the help of a good selection of a "trial" function ψ, the $\psi-$Caputo fractional derivative allows some measure of control over the modeling of the phenomenon under consideration. For more details, the reader may also consult [6,7,11,12,23,26,28] and the references therein.
Our manuscript is structured as follows. In Section 2, we give some basic notations of $\varphi-$Caputo fractional derivatives and $\varphi-$fractional integrals, definitions of $\varphi-$fractional calculus and important results that will be used in subsequent parts of the paper. In Section 3, we establish an existence result for φ-fractional hybrid differential equations with nonlocal boundary conditions (1.1) by using the mixed Lipschitz and Caratheodory conditions and we proved some fundamental fractional differential inequalities in Section 4. As application, an illustrative example is presented in Section 5 followed by conclusion in Section 6.

2. Preliminaries

We start this section by introducing some necessary definition and basic results required for further developments.
Assume that $X = \mathcal{C}(J, \mathbb{R})$ is a Banach space of all continuous functions from $J = [0, T], T > 0$ into $\mathbb{R}$ with the norm

$$||y|| = \sup\{|y(t)|, t \in J\}.$$

Let $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ denote the class of function $g : J \times \mathbb{R} \to \mathbb{R}$ such that

1. The map $t \mapsto g(t, x)$ is measurable for $x \in \mathbb{R}$.
2. The map $x \mapsto g(t, x)$ is continuous for each $t \in J$.

The class $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ is called the Caratheodory class of functions on $J \times \mathbb{R}$ wich are Lebesgue integrable when bounded by a Lebesgue integrable function on J.
Let $L^1(J, \mathbb{R})$ denote the space of Lebesgue integrable real-valued functions on J equipped with norm $||.||_{L^1}$ defined by

$$||x||_{L^1} = \int_0^T |x(s)| ds.$$

We define the multiplication in X by $(xy)(t) = x(t)y(t), \forall x, y \in X$.
Clearly, $X = \mathcal{C}(J, \mathbb{R})$ is a Banach algebra with respect to the above norm and multiplication.

Theorem 1 *[9] Let S be a non-empty, closed, convex and bounded subset of the Banach algebra X, and let $A : X \to X, B : X \to X$ be two operators such that*

1. *A is Lipschitzian with a Lipschitz constant β.*
2. *B is completely continuous.*
3. *$x = AxBy \Rightarrow x \in S$, for all $y \in S$.*

4. $\beta M < 1$, where $M = ||B(S)||$.

Then, the operator equation $AxBx = x$ has a solution in S.

Now, we give some results and properties from the theory of fractional calculus.

Definition 1 *(φ-Riemann-Liouville fractional integral [4])*
Let $\beta > 0$, f be an integrable function defined on $[a,b]$ and $\varphi : [a,b] \to \mathbb{R}$ that is an increasing differentiable function such that $\varphi'(t) \neq 0$, for all $t \in [a,b]$.
The φ-Riemann-Liouville fractional integral operator of order β of a function f is defined by

$$I_a^{\beta;\varphi} f(t) = \frac{1}{\Gamma(\beta)} \int_a^t \Psi'(s)(\varphi(t) - \varphi(s))^{\beta-1} f(s) ds.$$

Definition 2 *(φ-Riemann-Liouville fractional derivative [4])*
Let $n \in \mathbb{N}$, $f, \varphi \in \mathcal{C}^n([a,b])$ be two functions such that φ is increasing with $\varphi'(t) \neq 0$, for all $t \in [a,b]$. φ-Riemann-Liouville fractional derivative of order β of a function f is defined by

$$\begin{aligned} D_a^{\beta;\varphi} f(t) &= \left(\frac{1}{\varphi'(t)} \frac{d}{dt}\right)^n (I_a^{n-\beta;\varphi} f(t)) \\ &= \frac{1}{\Gamma(n-\beta)} \left(\frac{1}{\varphi'(t)} \frac{d}{dt}\right)^n \int_a^t \Psi'(s)(\varphi(t) - \varphi(s))^{n-\beta-1} f(s) ds, \end{aligned}$$

where $n = [\beta] + 1$ and $[\beta]$ denotes the integer part of β.

Definition 3 *(φ-Caputo fractional derivative [4])*
Let $n \in \mathbb{N}$, $f, \varphi \in \mathcal{C}^n([a,b])$ be two functions such that φ is increasing with $\varphi'(t) \neq 0$, for all $t \in [a,b]$. φ-Caputo fractional derivative of order β of a function f is defined by

$$\begin{aligned} {}^C D_a^{\beta;\varphi} f(t) &= (I_a^{n-\beta;\varphi} f_\varphi^{[n]})(t) \\ &= \frac{1}{\Gamma(n-\beta)} \int_a^t \Psi'(s)(\varphi(t) - \varphi(s))^{n-\beta-1} f_\varphi^{[n]}(s) ds, \end{aligned}$$

where $n = [\beta] + 1$, for $\beta \notin \mathbb{N}$. And $f_\varphi^{[n]}(t) = \left(\frac{1}{\varphi'(t)} \frac{d}{dt}\right)^n f(t)$ on $[a,b]$.

From the definition, it is clear that when $\beta = n \in \mathbb{N}$, we have

$${}^C D_a^{\beta;\varphi} f(t) = f_\varphi^{[n]}(t).$$

We note that if $f \in \mathcal{C}^n([a,b])$. The φ-Caputo fractional derivative of order β of f is determined as

$${}^C D_a^{\beta;\varphi} f(t) = D_a^{\beta;\varphi} \left(f(t) - \sum_{k=0}^{n-1} \frac{f_\varphi^{[k]}(a^+)}{k!} (\varphi(t) - \varphi(a))^k \right).$$

Theorem 2 *[4] Let $f \in \mathcal{C}^n([a,b])$ and $\beta > 0$. Then we have*

$$I_a^{\beta;\varphi}\, {}^C D_a^{\beta;\varphi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f_\varphi^{[k]}(a^+)}{k!} (\varphi(t) - \varphi(a))^k.$$

In particular, given $\beta \in (0,1)$ we have:

$$I_a^{\beta;\varphi}\, {}^C D_a^{\beta;\varphi} f(t) = f(t) - f(a).$$

In order to prove the existence and uniqueness results of the problem (1.1), we assume the following assumptions throughout the rest of this paper.

(H_0) $x \mapsto \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$, for all $t \in J$.

(H_1) There exists a constant $L > 0$ such that:

$$|f(t,x) - f(t,y)| \leq L|x-y|, \text{ for all } t \in J \text{ and } x, y \in \mathbb{R}.$$

(H_2) The function g satisfies the following growth condition for constants $P, Q > 0$ and $p \in (0,1)$ such that:

$$|g(t,x)| \leq P|x|^p + Q, \text{ for each } t \in J \text{ and each } x \in \mathbb{R}.$$

3. Main Results

Existence and uniqueness results

In this section, we prove the existence result for the problem (1.1) on the closed and bounded interval $J = [0,T], T > 0$ under mixed Lipschitz and Carateodory conditions on the non linearities involved in it. First of all, we define what we mean by a solution for the boundary fractional hybrid differential equations (1.1).

Definition 4 *A function $x \in \mathcal{C}^2(J,\mathbb{R})$ is said to be a solution of (1.1) if*

i) The function $t \mapsto \frac{x}{f(t,x)} \in \mathcal{C}^2(J,\mathbb{R})$ for each $x \in X$,

ii) x satisfies (1.1).

For the existence of solutions for the problem (1.1), we need the following Lemma.

Lemma 1 *Assume that hypothesis (H_0)holds and a, b, c are real constants with $a + b \neq 0$. For a given $h \in L^1(J,\mathbb{R})$, the unique solution of the hybrid fractional differential equation*

$$\begin{cases} {}^C D^{\beta;\varphi}\left(\dfrac{x(t)}{f(t,x(t))}\right) = h(t), \; t \in J = [0,T], \\ a\dfrac{x(0)}{f(0,x(0))} + b\dfrac{x(T)}{f(T,x(T))} = c. \end{cases} \tag{3.1}$$

is given by

$$\begin{aligned} x(t) = f(t,x(t))\Big\{ & c_1[\varphi(t) - \frac{a}{a+b}\varphi(T) - \frac{a}{a+b}\varphi(0)] \\ & + c_2[(\varphi(t)-\varphi(0))^2 - \frac{b}{a+b}(\varphi(T)-\varphi(0))^2]. \\ & + \frac{1}{\Gamma(\beta)}\int_0^{\boldsymbol{t}} \varphi'(s)(\varphi(\boldsymbol{t}) - \varphi(s))^{\beta-1}h(s)ds \\ & - \frac{b}{a+b}\frac{1}{\Gamma(\beta)}\int_0^{\boldsymbol{T}} \varphi'(s)(\varphi(\boldsymbol{T}) - \varphi(s))^{\beta-1}h(s)ds \\ & + \frac{c}{a+b}\Big\}. \end{aligned}$$

Where $c_1 = \dfrac{1}{\varphi'(0)}\left(\dfrac{x(0)}{f(0,x(0))}\right)'$ and $c_2 = \dfrac{1}{2\Psi'(0)}\left(\dfrac{x(0)}{f(0,x(0))}\right)''$.

Proof 1 *Taking the φ-Riemann-Liouville fractional integral of order β to the first equation of (3.1), and using Theorem 2, we get*

$$\frac{x(t)}{f(t,x(t))} = \frac{x(0)}{f(0,x(0))} + \underbrace{\frac{1}{\varphi'(0)}\left(\frac{x(0)}{f(0,x(0))}\right)'}_{c_1}(\varphi(t)-\varphi(0)) + \underbrace{\frac{1}{2}\frac{1}{\varphi'(0)}\left(\frac{x(0)}{f(0,x(0))}\right)''}_{c_2}(\varphi(t)-\varphi(0))^2 + I^{\beta;\varphi}h(t).$$

Then,

$$b\frac{x(T)}{f(T,x(T))} = b\frac{x(0)}{f(0,x(0))} + bc_1(\varphi(T)-\varphi(0)) + bc_2(\varphi(T)-\varphi(0))^2 + bI^{\beta;\varphi}h(T).$$

implies that

$$\frac{x(0)}{f(0,x(0))} = \frac{1}{a+b}\left\{c - bc_1(\varphi(T)-\varphi(0)) - bc_2(\varphi(T)-\varphi(0))^2 - bI^{\beta;\varphi}h(T)\right\}.$$

Consequently,

$$\begin{aligned} x(t) = f(t,x(t))\Big\{ & c_1[\varphi(t) - \frac{a}{a+b}\varphi(T) - \frac{a}{a+b}\varphi(0)] \\ & + c_2[(\varphi(t)-\varphi(0))^2 - \frac{b}{a+b}(\varphi(T)-\varphi(0))^2] \\ & + \frac{1}{\Gamma(\beta)}\int_0^{\boldsymbol{t}} \varphi'(s)(\varphi(\boldsymbol{t})-\varphi(s))^{\beta-1}h(s)ds \\ & - \frac{b}{a+b}\frac{1}{\Gamma(\beta)}\int_0^{\boldsymbol{T}} \varphi'(s)(\varphi(\boldsymbol{T})-\varphi(s))^{\beta-1}h(s)ds \\ & + \frac{c}{a+b}\Big\}. \end{aligned}$$

Theorem 3 *Assume that hypothesis $(\boldsymbol{H_0})$-$(\boldsymbol{H_2})$ hold and $a+b \neq 0$, then the hybrid fractional differential equation (1.1) has a solution defined on J provided that $Lw < 1$. Where*

$$\begin{aligned} w = & \frac{|c_1|}{|a+b|}\Big\{|\varphi(T)| + |b||\varphi(T)| + |a||\varphi(0)|\} + \frac{|c_2|}{|a+b|}\{(|b|+1)(\varphi(T)-\varphi(0))^2\Big\} \\ & + \frac{(P+Q)}{\Gamma(\beta+1)}\Big(1+|b|)(\varphi(T)-\varphi(0)\Big)^{\beta} + \frac{|c|}{|a+b|}. \end{aligned}$$

Proof 2 *We define a subset S of X by*

$$S = \{x \in X, ||x|| \le N\},$$

where $N = \frac{F_0 w}{1-Lw}$, and $F_0 = \sup_{t\in J}|f(t,0)|$. By an application of Lemma 1, equation (1.1) is equivalent to the nonlinear hybrid integral equation

$$\begin{aligned} x(t) &= [f(t,x(t)]\Big\{c_1[\varphi(t) - \frac{a}{a+b}\varphi(T) - \frac{a}{a+b}\varphi(0)] \\ &+ c_2[(\varphi(t)-\varphi(0))^2 - \frac{b}{a+b}(\varphi(T)-\varphi(0))^2] \\ &+ \frac{1}{\Gamma(\beta)}\int_0^{\boldsymbol{t}} \varphi'(s)(\varphi(\boldsymbol{t})-\varphi(s))^{\beta-1}g(s,x(s))ds \\ &- \frac{b}{a+b}\frac{1}{\Gamma(\beta)}\int_0^{\boldsymbol{T}} \varphi'(s)(\varphi(\boldsymbol{T})-\varphi(s))^{\beta-1} \\ &\times g(s,x(s))ds + \frac{c}{a+b}\Big\},\ t \in J. \end{aligned} \tag{3.2}$$

Where $c_1 = \frac{1}{\varphi'(0)}\left(\frac{x(0)}{f(0,x(0))}\right)'$ *and* $c_2 = \frac{1}{2\Psi'(0)}\left(\frac{x(0)}{f(0,x(0))}\right)''$.
Define two operators $A : X \to X$ *and* $B : S \to X$ *by:*

$$Ax(t) = f(t, x(t)), \ t \in J, \tag{3.3}$$

and

$$\begin{aligned} Bx(t) &= c_1[\varphi(t) - \frac{a}{a+b}\varphi(T) - \frac{a}{a+b}\varphi(0)] + c_2[(\varphi(t)-\varphi(0))^2 \\ &- \frac{b}{a+b}(\varphi(T)-\varphi(0))^2] + \frac{1}{\Gamma(\beta)}\int_0^{\boldsymbol{t}} \varphi'(s)(\varphi(\boldsymbol{t})-\varphi(s))^{\beta-1} g(s,x(s))ds \\ &- \frac{b}{a+b}\frac{1}{\Gamma(\beta)}\int_0^{\boldsymbol{T}} \varphi'(s)(\varphi(\boldsymbol{T})-\varphi(s))^{\beta-1} \times g(s,x(s))ds + \frac{c}{a+b}. \end{aligned} \tag{3.4}$$

Then the hybrid integral equation (3.2) *is transformed into the operator equation as*

$$x(t) = Ax(t)Bx(t), \ t \in J.$$

We show that the operators A *and* B *satisfy all the conditions of Theorem 1 in sevral steps.*

Step 1: *Let* $x, y \in X$. *Then by hypothesis* $(\boldsymbol{H_1})$,

$$|Ax(t) - Ay(t)| = |f(t,x(t)) - f(t,y(t))| \le L|x(t)-y(t)|, \ t \in J.$$

Step 2: *Let* $x_n, x \in S$ *with* $\lim_{n\to\infty} ||x_n - x|| = 0$. *It is trivial to see that* x_n *is bounded subset of* S. *As a result, we see* $||x|| \le N$. *It easy to see that* $g(s,x_n(s)) \to g(s,x(s))$, *as* $n \to \infty$ *due to the continuity of* g.
On the other hand taking $(\boldsymbol{H_2})$ *into consideration we get the following inequality:*

$$\frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}||g(s,x_n(s)-g(s,x(s))|| \le 2\frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}(PN^p+Q).$$

We notice that since the function $s \mapsto 2\frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}(PN^p+Q)$ *is Lebesgue integrable over* $[0,t]$, *the same thing about* $s \mapsto 2\frac{\varphi'(s)(\varphi(T)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}(PN^p+Q)$ *is Lebesgue integrable over* $[0,T]$. *This fact together with is Lebesgue dominated convergence theorem implies that*

$$\int_0^t \frac{\varphi'(s)(\varphi(t)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}||g(s,x_n(s)-g(s,x(s))|| \underset{n\to\infty}{\longrightarrow} 0,$$

$$\int_0^{\boldsymbol{T}} \frac{\varphi'(s)(\varphi(\boldsymbol{T})-\varphi(s))^{\beta-1}}{\Gamma(\beta)}||g(s,x_n(s)-g(s,x(s))|| \underset{n\to\infty}{\longrightarrow} 0.$$

It follows that $||Bx_n - Bx|| \to 0$ *as* $n \to \infty$. *Which implies the continuity of the operator* B.
In order to show that B *is compact. For* $x \in S$, *we have*

$$\begin{aligned} |Bx(t)| \le |c_1|\left\{|\varphi(T)| + \frac{|b|}{|a+b|}|\varphi(T)| + \frac{|a|}{|a+b|}|\varphi(0)|\right\} + |c_2|\left\{(\varphi(T)-\varphi(0))^2\right. \\ \left. + \frac{|b|}{|a+b|}(\varphi(T)-\varphi(0))^2\right\} + \frac{|b|}{|a+b|}\frac{(P||x||^p+Q)}{\Gamma(\beta+1)}(\varphi(T)-\varphi(0))^\beta + \frac{|c|}{|a+b|}. \end{aligned}$$

Therefore,

$$\begin{aligned} ||Bx|| \le |c_1|\left\{|\varphi(T)| + \frac{|b|}{|a+b|}|\varphi(T)| + \frac{|a|}{|a+b|}|\varphi(0)|\right\} + |c_2|\left\{(\varphi(T)-\varphi(0))^2\right. \\ \left. + \frac{|b|}{|a+b|}(\varphi(T)-\varphi(0))^2\right\} + \frac{|b|}{|a+b|}\frac{(PN^p+Q)}{\Gamma(\beta+1)}(\varphi(T)-\varphi(0))^\beta + \frac{|c|}{|a+b|}, \end{aligned}$$

which shows that B is uniformly bounded.
Now, for equi-continuity of B take $t_1, t_2 \in J$ with $t_1 < t_2$, and let $x \in S$. Thus we get

$$|Bx(t_2) - Bx(t_1)| \leq |c_1|(\varphi(t_2) - \varphi(t_1)) + |c_2|(\varphi(t_1) - \varphi(0))^2 - (\varphi(t_2) - \varphi(0))^2 + \frac{(PN^p + Q)}{\Gamma(\beta+1)}(\varphi(t_2) - \varphi(0))^\beta - (\varphi(t_1) - \varphi(0))^\beta.$$

From the last estimate, we deduce that $||Bx(t_2) - Bx(t_1)|| \to 0$ when $t_2 \to t_1$. Therefore, B is equicontinuous. Thus by Ascoli-Arzela theorem [25], the operator B is compact.

Step 3: *Let $x \in X$ and $y \in S$ be arbitrary such that $x = AxBy$. Then*

$$|x(t)| \leq |Ax(t)By(t)| \leq (L|x(t)| + F_0)|By(t)|$$
$$|x(t)| \leq \frac{F_0|By(t)|}{1 - L|By(t)|}.$$

Without any loss of generality, let take $p = 0$. Then

$$|By(y)| \leq \frac{|c_1|}{|a+b|}\{|\varphi(T)| + |b||\varphi(T)| + |a||\varphi(0)|\} + \frac{|c_2|}{|a+b|}\{(|b|+1)(\varphi(T) - \varphi(0))^2\} + \frac{(P+Q)}{\Gamma(\beta+1)}(1+|b|)(\varphi(T) - \varphi(0))^\beta + \frac{|c|}{|a+b|}.$$

Therefore,

$$||x|| \leq \frac{F_0 w}{1 - Lw},$$

with

$$w = \frac{|c_1|}{|a+b|}\{|\varphi(T)| + |b||\varphi(T)| + |a||\varphi(0)|\} + \frac{|c_2|}{|a+b|}\{(|b|+1)(\varphi(T) - \varphi(0))^2\} + \frac{(P+Q)}{\Gamma(\beta+1)}(1+|b|)(\varphi(T) - \varphi(0))^\beta + \frac{|c|}{|a+b|}.$$

Step 4: *Finally, it remains to show that $LM < 1$ with $M = ||B(s)||$. Then,*

$$M = ||B(s)|| = \sup\{||Bx||, x \in S\} \leq w,$$
$$LM \leq Lw < 1, \ (L > 0).$$

Thus all the conditiond of Theorem 1 are satisfied and hence the operator equation $AxBx = x$ has a solution in S. As a result, (1.1) has a solution defined on J.

$\varphi-$fractional hybrid differential inequalities

In this section, we discuss a fundamental result relative to strict inequalities for the problem (1.1). We begin with the definition of the class $C_p([0,T], \mathbb{R})$.

Definition 5 *[20] $m \in C_p([0,T], \mathbb{R}), p \in (0,1)$ means that $m \in C([0,T], \mathbb{R})$ and $t^p m \in C([0,T], \mathbb{R})$ with $p + q = 1$.*

Definition 6 *[20] For all $m \in C_p([0,T], \mathbb{R})$. The φ-Riemann-Liouville derivative of $m(t)$ is defined as*

$$D^{q;\varphi} m(t) = \frac{1}{\Gamma(p)} \frac{1}{\varphi'(t)} \frac{d}{dt} \int_0^t \Psi'(s)(\varphi(t) - \varphi(s))^{p-1} m(s) ds,$$

where $p \in (0,1), q > 0$ and $\varphi \in C([0,T], \mathbb{R})$ and increasing with $\varphi'(t) \neq 0$.

Lemma 2 *Let $m \in C_p([0,T],\mathbb{R})$. Suppose that for all $t_1 \in [0,T]$ we have $m(t) < 0$ with $0 < t < t_1$, then,*

$$D^{q;\varphi} m(t_1) \geq 0,\ q > 0.$$

Proof 3 *Consider $m \in C_p([0,T],\mathbb{R})$, suppose that $t_1 \in [0,T]$ we have $m(t) < 0$ with $0 < t < t_1$. Then, $m(t)$ and $t^p m(t)$ are continuous on $[0,T]$.*
Since $m(t)$ is continuous on $[0,T]$, given any t_1 such that $0 < t_1 < T$, there exists a $k(t_1) > 0$ and $h > 0$ such that

$$-k(t_1) \leq m(s) \leq k(t_1), \text{where } 0 < t_1 - h \leq s \leq t_1 + h < T. \tag{3.5}$$

Because we have

$$D^{q;\varphi} m(t) = \frac{1}{\Gamma(p)} \frac{1}{\varphi'(t)} \frac{d}{dt} \int_0^t \Psi'(s)(\varphi(t) - \varphi(s))^{p-1} m(s) ds.$$

Set $H(t) = \int_0^t \Psi'(s)(\varphi(t) - \varphi(s))^{p-1} m(s) ds$,then

$$\begin{aligned} H(t_1 + h) - H(t_1) &= \int_0^{t_1} [(\varphi(t+h) - \varphi(s))^{p-1} - (\varphi(t_1) - \varphi(s))^{p-1}] m(s) ds \\ &+ \int_0^{t_1+h} \varphi'(s)(\varphi(t_1+h) - \varphi(s))^{p-1} m(s) ds = I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^{t_1} [(\varphi(t+h) - \varphi(s))^{p-1} - (\varphi(t_1) - \varphi(s))^{p-1}] m(s) ds, \\ I_2 &= \int_0^{t_1+h} \varphi'(s)(\varphi(t_1+h) - \varphi(s))^{p-1} m(s) ds. \end{aligned}$$

Since $t_1 + h > t_1$, $p - 1 < 0$ and φ increasing, then we have

$$(\varphi(t_1+h) - \varphi(s))^{p-1} < (\varphi(t_1) - \varphi(s))^{p-1}.$$

This, coupled with the fact that $m(t) < 0, 0 < t < t_1$, implies that $I_1 \geq 0$.
For I_2 using (3.5), we obtain $I_2 \geq \frac{-k(t_1)}{p}(\varphi(t_1+h) - \varphi(t_1))^p$. Then,

$$H(t_1+h) - H(t_1) + \frac{k(t_1)}{p}(\varphi(t_1+h) - \varphi(t_1))^p \geq 0.$$

Then dividing through by h and taking limits as $h \to 0$, we have

$$\lim_{h \to 0} \left\{ \frac{H(t_1+h) - H(t_1)}{h} + \frac{k(t_1)}{p} \left(\frac{(\varphi(t_1+h) - \varphi(t_1))}{h} \right)^p h^{p-1} \right\} \geq 0.$$

Since $p \in (0,1)$, we conclude that $\frac{dH(t_1)}{dt} \geq 0$.
Which implies that $D^{q;\varphi}(t_1) \geq 0$.

Theorem 4 *Assume that hypothesis $(\boldsymbol{H_0})$ holds. Suppose that there exists functions $y, z \in C_p([0,T],\mathbb{R})$, such that*

$$D^{\beta;\varphi} \left(\frac{y(t)}{f(t, y(t))} \right) \leq g(t, y(t)),\ \forall t \in J, \tag{3.6}$$

and

$$D^{\beta;\varphi} \left(\frac{z(t)}{f(t, z(t))} \right) \geq g(t, z(t)),\ \forall t \in J, \tag{3.7}$$

with one of the inequalities being strict.
Then $y^0 < z^0$, where $y^0 = t^{1-\beta} y(t)|_{t=0}$ and $z^0 = t^{1-\beta} z(t)|_{t=0}$ implies
$y(t) < z(t)$, for all $t \in J$.

Proof 4 *Assume that the claim is false. Since* $y^0 < z^0$, $t^{\beta-1}y(t)$ *and* $t^{\beta-1}z(t)$ *are continuous functions, then there exists* t_1 *such that* $0 < t_1 \leq T$ *with* $y(t_1) = z(t_1)$ *and* $y(t) < z(t)$, $0 \leq t < t_1$.
Define

$$Y(t) = \frac{y(t)}{f(t, y(t))} \text{ and } Z(t) = \frac{z(t)}{f(t, z(t))}.$$

Then we have $Y(t_1) = Z(t_1)$, *and by* $\boldsymbol{(H_0)}$ *we get* $Y(t) < Z(t)$.
Setting $m(t) = Y(t) - Z(t), 0 \leq t < t_1$, *then we have* $m(t) < 0$, *and* $m(t_1) = 0$ *with* $m \in C_p([0, T], \mathbb{R})$. *By Lemma 2, we have*

$$g(t_1, y(t_1)) > D^{\beta;\varphi}(Y(t_1)) > D^{\beta;\varphi}(Z(t_1)) \geq g(t_1, z(t_1)).$$

This is a contraction with $y(t_1) = z(t_1)$, *thus the conclusion of the theorem holds.*

Theorem 5 *Assume that hypothesis* $\boldsymbol{(H_0)}$ *holds and* a, b, c *are real constants with* $a + b \neq 0$. *Suppose that there exists functions* $y, z \in C_p([0, T], \mathbb{R})$, *such that*

$$D^{\beta;\varphi}\left(\frac{y(t)}{f(t, y(t))}\right) \leq g(t, y(t)),\ \forall t \in J, \tag{3.8}$$

and

$$D^{\beta;\varphi}\left(\frac{z(t)}{f(t, z(t))}\right) \geq g(t, z(t)),\ \forall t \in J, \tag{3.9}$$

with one of the inequalities being strict. If $a > 0, b < 0$ *and* $y(T) < z(T)$, *then*

$$a\frac{y(0)}{f(0, y(0))} + b\frac{y(T)}{f(T, y(T))} < a\frac{z(0)}{f(0, z(0))} + b\frac{z(T)}{f(T, z(T))},$$

implies $y(t) < z(t)$, *for all* $t \in J$.

Proof 5 *We have*

$$a\left(\frac{y(0)}{f(0, y(0))} - \frac{z(0)}{f(0, z(0))}\right) < b\left(\frac{z(T)}{f(T, z(T))} - \frac{y(T)}{f(T, y(T))}\right),$$

since $a > 0, b < 0$ *and* $y(T) < z(T)$ *by* $\boldsymbol{(H_0)}$, *we have* $y(0) < z(0)$.
Hence the application of Theorem 4 yields that $y(t) < z(t)$.

Theorem 6 *Assume that the conditions of Theorem 4 hold. Suppose that there exists a real number* $M > 0$ *such that*

$$g(t, x_1) - g(t, x_2) \leq M\left(\frac{x_1}{f(t, x_1)} - \frac{x_2}{f(t, x_2)}\right), t \in J \,\text{and}\, x_1, x_2 \in \mathbb{R} \,\text{with}\, x_1 \geq x_2.$$

Then

$$a\frac{y(0)}{f(0, y(0))} + b\frac{y(T)}{f(T, y(T))} < a\frac{z(0)}{f(0, z(0))} + b\frac{z(T)}{f(T, z(T))},$$

implies, provided $\frac{1}{\Gamma(p)}(\varphi(T) - \varphi(0))^{p-1} \geq M, p \in (0, 1)$

$$y(t) < z(t) \,\text{for all}\, t \in J.$$

Proof 6 *We set*

$$\frac{z_\varepsilon(t)}{f(t, z_\varepsilon(t))} = \frac{z(t)}{f(t, z(t))} + \varepsilon,\ \varepsilon > 0,$$

and let

$$Z_\varepsilon(t) = \frac{z_\varepsilon(t)}{f(t, z_\varepsilon(t))} \text{ and } Z(t) = \frac{z(t)}{f(t, z(t))},\ t \in J.$$

So, we have

$$Z_\varepsilon(t) > Z(t) \Rightarrow z_\varepsilon(t) > z(t).$$

Since

$$g(t,x_1) - g(t,x_2) \leq M\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right) \text{ and } D^{\beta;\varphi}\left(\frac{y(t)}{f(t,y(t))}\right) \leq g(t,y(t)).$$

one has

$$\begin{aligned} D^{\beta;\varphi}(Z_\varepsilon(t)) &= D^{\beta;\varphi}(Z(t)) + D^{\beta;\varphi}(\varepsilon) \\ &= D^{\beta;\varphi}(Z(t)) + \frac{\varepsilon}{\Gamma(p)}(\varphi(T)-\varphi(0))^{p-1},\ p \in (0,1) \\ &\geq g(t,z_\varepsilon(t)) + \varepsilon\left(\frac{1}{\Gamma(p)}(\varphi(T)-\varphi(0))^{p-1} - M\right) \\ &> g(t,z_\varepsilon(t)), \end{aligned}$$

provided $\frac{1}{\Gamma(p)}(\varphi(T)-\varphi(0))^{p-1} \geq M$. *Also, we have* $z_\varepsilon(0) > z(0) \geq y(0)$.
Hence, the application of Theorem 5 yields that $y(t) < z_\varepsilon(t)$, *for all* $t \in J$.
Taking the limits as $\varepsilon \to 0$, *we have* $y(t) \leq z(t)$, *for all* $t \in J$.

4. An Application

In this section we give an example to illustrate our main result.
Consider the following hybrid fractional differential equation:

$$\begin{cases} {}^{C}D_{0^+}^{\frac{5}{2},t}\left(\dfrac{x(t)}{\Phi(t,x(t))}\right) = \varphi(t,x(t)), \qquad t \in \Delta = [0,1], \\ \dfrac{x(0)}{\sqrt{\frac{1}{10}|x(0)|+1}} + \dfrac{x(1)}{\sqrt{\frac{1}{10}|x(1)|+1}} = 0. \end{cases} \tag{4.1}$$

where $\beta = \frac{3}{2}$, $T=1$, $a=b=1$, $c=0$,
$\varphi(t) = t$, $\varphi(t,x(t)) = \dfrac{\sqrt{|x(t)|+e^{-|x(t)|}}}{t^2+2t+3}\sin^2(x(t))$, and
$\Phi(t,x(t)) = \sqrt{\dfrac{1}{10}|x(t)|+1}$.
It is clear that the assumption (H_0) is satisfied.
To prove the assumption (H_2), let $t \in \Delta$ and $x, y \in \mathcal{C}(\Delta,\mathbb{R})$, then we have

$$|\Phi(t,x(t)) - \Phi(t,y(t))| = \left|\sqrt{\frac{1}{10}|x(t)|+1} - \sqrt{\frac{1}{10}|x(t)|+1}\right|,$$

$$|\Phi(t,x(t)) - \Phi(t,y(t))| \leq \frac{1}{10}\frac{\left|x(t)-y(t)\right|}{\sqrt{\frac{1}{10}|x(t)|+1} + \sqrt{\frac{1}{10}|x(t)|+1}},$$

$$|\Phi(t,x(t)) - \Phi(t,y(t))| \leq \frac{1}{10}\left|x(t)-y(t)\right|.$$

Thus, the assumption (H_1) holds true with $L = \frac{1}{10}$.
It remains to verify the assumption (H_2). Let $t \in \Delta$ and $u \in \mathbb{R}$, then we have

$$|\varphi(t,x(t))| = \left|\frac{\sqrt{|x(t)|+1}}{t^2+2t+3}\right|,$$

$$|\varphi(t,x(t))| \leq \frac{1}{3}\sqrt{|x(t)|} + \frac{1}{3},$$

Wich implies that the assumption (H_2) is verified with $P = Q = \frac{1}{3}$ and $p = \frac{1}{2}$.
Thus, as a consequence of Theorem 1 we can deduce that the problem (4.1) has a solution on $[0,1]$. To guarantee the uniqueness of this solution, it is enaugh to take the constants c_1 and c_2 such that

$$\frac{|c_1| + |c_2|}{10} + \frac{1}{3\Gamma(5/2)} < 1.$$

5. Conclusion

In the present paper, we gave the definition of solutions for nonlocal fractional hybrid boundary value problem by using the $\varphi-$Caputo fractional derivative of order $\beta \in (2,3)$. In addition, by employing Krasnoselskii fixed point theorem, the existence result of solutions for this problem is discussed. Finally, as application, a nontrivial example is presented to illustrate our theoretical results.

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Conflict of interest

The authors declare that they have no conflict of interest.

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