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Existence of Nodal Solutions for weighted Elliptic Problem Involving Exponential Growth

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ABSTRACT: In this note, we establish the existence of nodal solutions for a logarithmic weighted elliptic problem in the unit ball B of $\mathbb{R}^N$, $N \geq 3$. The non-linearity is assumed to have exponential growth in view of Trudinger-Moser type inequalities. Our method is based on the constrained minimization in Nehari manifold coupled with the quantitative deformation lemma and degree theory.

Key Words: N-Laplacian operator, Weighted Sobolev space, changing-sign solutions, critical exponential growth.

Contents

1. Introduction

This paper we study the following Schrödinger problem:

$$(P_\lambda) \quad \begin{cases} -\nabla.(\omega(x)|\nabla\varphi|^{N-2}\nabla\varphi) + |\varphi|^{N-2}\varphi &= \lambda\, f(x,\varphi) \quad \text{in} \quad B \\ \varphi &= 0 \quad \text{on} \quad \partial B, \end{cases}$$

where λ is a positive parameter, B is the unit open ball in $\mathbb{R}^N$, $N \geq 3$ and the logarithmic weight function

$$\omega(x) = \Big(1 - \log|x|\Big)^{\beta(N-1)}, \quad \beta \in [0,1). \tag{1.1}$$

The non-linearity $f(x,t)$ behaves like $\exp\{\alpha t^{\frac{N}{(N-1)(1-\beta)}}\}$ as $t \to +\infty$, for some $\alpha > 0$ and $\beta \in [0,1)$.

Over the last decade, many researchers have been explored the limiting case of the Sobolev embedding, known as the Trudinger–Moser case. More precisely, let Ω be a domain of $\mathbb{R}^N$ who is smooth bounded. Denote $W_0^{1,N}(\Omega) = \text{closure}\{u \in C_0^\infty(\Omega) \mid \int_\Omega |\nabla u|^N dx < \infty\}$, equipped with the norm $\|u\|_{W_0^{1,N}(\Omega)} = \big(\int_\Omega |\nabla u|^N dx\big)^{\frac{1}{N}}$. The last space is a limit case for the Sobolev embedding Theorem, which yields $W_0^{1,N}(\Omega) \hookrightarrow L^p(\Omega)$ for all $1 \leq p < \infty$, but we can verify by easy examples that $W_0^{1,N}(\Omega) \nsubseteq L^\infty(\Omega)$. The problem arising is to find a growth maximal function $\Phi : \mathbb{R} \to \mathbb{R}^+$ such that $\int_\Omega \Phi(u)dx < \infty$ for $u \in W_0^{1,N}(\Omega)$ with $\|u\|_{W_0^{1,N}(\Omega)} \leq 1$. Trudinger [20] was answered to this equation and proved that the maximal growth is given by $\Phi(t) = e^{|t|^{\frac{N}{N-1}}}$. This result was improved by Moser [17]. More precisely, he showed that for all $u \in W_0^{1,N}(\Omega)$, $\exp(\alpha|u|^{\frac{N}{N-1}}) \in L^1(\Omega)$, $\forall\, \alpha > 0$ and

$$\sup_{\|u\|_{W_0^{1,N}(\Omega)} \leq 1} \int_\Omega e^{\alpha|u|^{\frac{N}{N-1}}} dx < C(N) \quad \Longleftrightarrow \quad \alpha \leq \alpha_N := N\omega_{N-1}^{\frac{1}{N-1}}, \tag{1.2}$$

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ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$. The constant α_N is sharp in the sense that for $\alpha > \alpha_N$ the supremum in (1.2) is infinite. An inequality as (1.2) are now called Trudinger-Moser type inequalities and we refer the reader to [11,10,6,15,16].
A mass of literature are focused on the study of the influence of weights on limiting inequalities of Trudinger-Moser type, we refer to [1,5] for the influence of power weight in the integral term on the maximal growth and [3,4] for the effect of weights in the Sobolev norm. Kufner [13] established weighted Sobolev spaces and introduced the embedding theory for such weighted Sobolev spaces with general weight functions. If $\omega \in L^1(\Omega)$ be a nonnegative function.
Consider

$$W_0^{1,N}(\Omega,\omega) = \text{closure}\{u \in C_0^\infty(\Omega) \;\mid\; \int_\Omega \omega(x)|\nabla u|^N dx < \infty\}. \tag{1.3}$$

the weighted Sobolev space.
In the case that ω is the logarithmic function, the weighted Sobolev spaces of form (1.3) have a specially sense because they concern limiting case of such embedding. On the other hand, we will need to restrict attention to radial functions to get performing results. Hence, let us consider

$$W^1_{0,rad}(B,\omega) = closure\{u \in C^\infty_{0,rad}(\Omega) \;\mid\; \int_\Omega \omega(x)|\nabla u|^N \,dx < \infty\},$$

the weighted Sobolev space of radial functions endowed with the norm

$$\|u\|_{W^1_{0,rad}(B,\omega)} := |\nabla u|_{N,\omega} = \Big(\int_\Omega \omega(x)|\nabla u|^N \,dx\Big)^{\frac{1}{N}}, \tag{1.4}$$

when w be given by (1.1). The first result about the Trudinger-Moser inequalities on Sobolev space with logarithmic weights was established by Calanchi and Ruf, see [3], where they studied the case when $N = 2$ with Sobolev norm of logarithmic type. In light of the follow exponential inequalities.

Theorem 1.1 *[4] Consider $\omega(x)$ given by (1.1) and $\beta \in [0,1)$ then*

$$\int_B e^{|\varphi|^\gamma} dx < +\infty, \quad \forall \;\; \varphi \in \mathbf{X}, \quad \textit{if and only if} \;\; \gamma \leq \gamma_{N,\beta} = \frac{N}{(N-1)(1-\beta)} = \frac{N'}{1-\beta} \tag{1.5}$$

and

$$\sup_{\substack{\varphi\in\mathbf{X}\\ \|\varphi\|\leq 1}} \int_B e^{\alpha|\varphi|^{\gamma_{N,\beta}}} dx < +\infty \quad\Leftrightarrow\quad \alpha \leq \alpha_{N,\beta} = N[\omega_{N-1}^{\frac{1}{N-1}}(1-\beta)]^{\frac{1}{1-\beta}}, \tag{1.6}$$

where N, N' are Hölder conjugate and ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$.

Denote $\gamma := \gamma_{N,\beta} = \dfrac{N'}{1-\beta}$, according to inequalities (1.5) and (1.6), f has subcritical growth at ∞ if

$$\lim_{s\to+\infty} \frac{|f(x,s)|}{e^{\alpha s^\gamma}} = 0, \quad \text{for all} \;\; \alpha > 0 \tag{1.7}$$

and f has critical growth at $+\infty$ if there exists some $\alpha_0 > 0$,

$$\lim_{s\to+\infty} \frac{|f(x,s)|}{e^{\alpha s^\gamma}} = 0, \quad \forall\; \alpha > \alpha_0 \;\text{ and } \quad \lim_{s\to+\infty} \frac{|f(x,s)|}{e^{\alpha s^\gamma}} = \infty, \;\; \forall\; \alpha > \alpha_0. \tag{1.8}$$

We assume that the non-linearity $f(x,t)$verifying:

(V_1) $f : B \times \mathbb{R} \to \mathbb{R}$ is C^1 and radial in x.

(V_2) For each $x \in B$, $t \mapsto \dfrac{f(x,t)}{|t|^{N-1}}$ is increasing for $t \in \mathbb{R} \setminus \{0\}$.

(V_3) There exists $\theta > N$ such that we have $0 < \theta F(x,t) \leq tf(x,t), \forall (x,t) \in \quad B \times \mathbb{R} \setminus \{0\}$ where $F(x,t) = \int_0^t f(x,s)ds.$

(V_4) $\lim_{t \to 0} \frac{|f(x,t)|}{|t|^{N-1}} = 0.$

As an example of non-linearity the function $(x,t) \to |t|^{N-1}t + |t|^N t \exp(\alpha |t|^\gamma)$ verifies the assumptions above. A function φ is called a weak solution of the problem (P_λ) if $\varphi \in \mathbf{X}$ and

$$\int_B w(x)|\nabla \varphi|^{N-2}\nabla \varphi.\nabla \psi dx + \int_B |\varphi|^{N-2}\varphi \psi \ dx = \lambda \int_B f(x,\varphi)\psi dx, \quad \forall \ \psi \in \mathbf{X}.$$

The energy functional corresponding to problem (P_λ) is defined by

$$\mathcal{J}_\lambda(\varphi) = \frac{1}{N}\|\varphi\|^N - \lambda \int_B F(x,\varphi)dx, \tag{1.9}$$

where $\psi \in \mathbf{X}$ and $F(x,t) = \int_0^t f(x,s)ds$. Since there exist $a, \ C > 0$ positive constants and there exists $t_1 > 1$ such for that $|f(x,t)| \leq Ce^{a \ t^\gamma}, \qquad \forall |t| > t_1$, the function $\mathcal{J}_\lambda$ is well defined and of class C^1 whenever the non-linearity $f(x,t)$ is critical or subcritical at ∞. and we have

$$\langle \mathcal{J}'_\lambda(\varphi), \psi \rangle = \mathcal{J}'_\lambda(\varphi)\psi = \int_B \omega(x)|\nabla \varphi|^{N-2}\nabla \varphi.\nabla \psi dx + \int_B |\varphi|^{N-2}\varphi \psi \ dx - \lambda \int_B f(x,\varphi)\psi dx, \quad \forall \ \psi \in \mathbf{X}.$$

We denote by $\varphi^- = \min\{\varphi(x), 0\}$ and $\varphi^+ = \max\{\varphi(x), 0\}$ then we consider the Nehari manifold as $\mathcal{N}_\lambda := \{\varphi \in \mathbf{X} : \langle \mathcal{J}'_\lambda(u), \varphi^+ \rangle = \langle \mathcal{J}'_\lambda(\varphi), \varphi^- \rangle = 0, \varphi^+ \neq 0, \varphi^- \neq 0\}$,
Then we have the useful equalities $\mathcal{J}_\lambda(\varphi) = \mathcal{J}_\lambda(\varphi^+) + \mathcal{J}_\lambda(\varphi^-)$,

$$\langle \mathcal{J}'_\lambda(\varphi), \varphi^+ \rangle = \langle \mathcal{J}'_\lambda(\varphi^+), \varphi^+ \rangle \ \text{ and } \ \langle \mathcal{J}'_\lambda(\varphi), \varphi^- \rangle = \langle \mathcal{J}'_\lambda(\varphi^-), \varphi^- \rangle.$$

We announce now the so called least energy sign-changing solution and nodal solutions of problem (P_λ).
$\upsilon \in \mathbf{X}$ is called nodal solution of (P_λ) if υ is a solution of problem (P_λ) and $\upsilon^\pm \neq 0$ a.e in B.
$\upsilon \in \mathbf{X}$ is called least energy sign-changing solution of (P_λ) if υ is a sign-changing solution of (P_λ) and

$$\mathcal{J}_\lambda(\upsilon) = \inf\{\mathcal{J}_\lambda(\varphi) : \mathcal{J}'_\lambda(\varphi) = 0, \varphi^\pm \neq 0 \ \text{ a.e in } \ B\}$$

Motivated by many works cited above, in this paper we are going to minimize of the energy functional $\mathcal{J}_\lambda$ over the following constraint, $c_\lambda = \inf_{\varphi \in \mathcal{N}_\lambda} \mathcal{J}_\lambda(\varphi)$ The authors in [12] have been developed the Nehari manifold method to tackle a range of problems and to find solutions of problems with a variational structure, see [11]. Despite the wealth of research in this area, nodal solutions to the N-weighted Lapalace equation with critical exponential nonlinearity on the weighted Sobolev space $\mathbf{X}$ still not solved.
The main contributions of this paper :

Theorem 1.2 *We consider a function f which has a subcritical growth at $+\infty$, also (V_1), (V_2), (V_3), and (V_4) are satisfied. We say that (P_λ) has a least energy nodal (sign-changing) radial solution $\upsilon \in \mathcal{N}_\lambda$ if $\lambda > 0$.*

On the other hand, in the case of critical growth nonlinearity, we get.

Theorem 1.3 *Suppose that $f(x,t)$ has a critical growth at $+\infty$ for some α_0 and (V_1), (V_2), (V_3) and (V_4) are verified. Then, there exist $\lambda^* > 0$ such that for $\lambda > \lambda^*$, problem (P_λ) has a least energy nodal (sign-changing) radial solution $\upsilon \in \mathcal{N}_\lambda$.*

To this end this paper is organised as follows. After introduc- ing definitions about the compactness analysis in section 2 propose some mathematical preliminaries . We will investigate some technical lemmas in section 3. we prove our main result in section 4 which is concern the subcritical case. Moreover, section 5 is concerned a more difficult case which is critical case. To investigates this problem, We will used a concentration compactness result of Lions type to prove Theorem 1.3.

2. Preliminaries for the compactness analysis

In the present section, we mention some useful lemmas . We begin by the radial Lemma.

Lemma 2.1 *[4] Consider a radially symmetric function φ in $C_0^1(B)$. Then, we have*

$$|\varphi(x)| \leq \frac{|\log(|x|)|^{\frac{1-\beta}{N'}}}{\omega_{N-1}^{\frac{1}{N}}(1-\beta)^{\frac{1}{N'}}}\|\varphi\|,$$

where ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$.

It follows that the embedding $\mathbf{X} \hookrightarrow L^q(B)$ is continuous for all $q \geq 1$, and that there exists a constant $C > 0$ such that $\|\varphi\|_{N'q} \leq C\|\varphi\|$, for all $u \in \mathbf{X}$. Moreover, the embedding $\mathbf{X} \hookrightarrow L^q(B)$ is compact for all $q \geq N$.

We need a lions type result [14] about an improved Trudinger-Moser inequality when we deal with weakly convergent sequences.

Lemma 2.2 *We consider $\{\varphi_k\}_k$ as sequence in $\mathbf{X}$. Suppose that $\|\varphi_k\| = 1$, $\varphi_k \rightharpoonup u$ weakly in $\mathbf{X}$, $\varphi_k(x) \to \varphi(x)$ a.e $x \in \mathcal{B}$, $\nabla\varphi_k(x) \to \nabla\varphi(x)$ a.e $x \in B$ and $\varphi \not\equiv 0$. So $\sup_k \int_B e^{p\ \alpha_{N,\beta}|\varphi_k|^\gamma} dx < +\infty$ for all $1 < p < U$ where U is defined by:* $U = \begin{cases} \dfrac{1}{(1-\|\varphi\|^N)^{\frac{\gamma}{N}}} & \text{if } \|\varphi\| < 1 \\ +\infty & \text{if } \|\varphi\| = 1 \end{cases}$

Proof. See [8].

Lemma 2.3 *[11] Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded domain and $f : \overline{\Omega} \times \mathbb{R}$ be a continuous function. Let $\{\varphi_n\}_n$ be a sequence in $L^1(\Omega)$ converging to φ in $L^1(\Omega)$. Assume that $f(x, \varphi_n)$ and $f(x, \varphi)$ are also in $L^1(\Omega)$. If $\int_\Omega |f(x, \varphi_n)\varphi_n| dx \leq C$, where C is a positive constant, then $f(x, \varphi_n) \to f(x, \varphi)$ in $L^1(\Omega)$.*

3. Some technical lemmas

Suppose that the function f verifies the conditions (V_1) to (V_4). Let $\varphi \in \mathbf{X}$ with $\varphi^\pm \not\equiv 0$ a.e. in the ball B, and we denote the function $\Upsilon_\varphi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and mapping $L_\varphi : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ as

$$\Upsilon_\varphi(p, q) = \mathcal{J}_\lambda(p\varphi^+ + q\varphi^-), \tag{3.1}$$

and

$$L_\varphi(p, q) = \big(\langle \mathcal{J}'_\lambda(p\varphi^+ + q\varphi^-), p\varphi^+\rangle, \langle \mathcal{J}'_\lambda(p\varphi^+ + q\varphi^-, q\varphi^-\rangle\big) \tag{3.2}$$

Lemma 3.1 *(i) For each $\varphi \in \mathbf{X}$ with $\varphi^+ \neq 0$ and $\varphi^- \neq 0$, there exists an unique couple $(p_\varphi, q_\varphi) \in (0,\infty) \times (0,\infty)$ such that $p_\varphi\varphi^+ + q_\varphi\varphi^- \in \mathcal{N}_\lambda$. So, the set $\mathcal{N}_\lambda$ is nonempty.*

(ii) For all $p, q \geq 0$ with $(p, q) \neq (p_\varphi, q_\varphi)$, we have $\mathcal{J}_\lambda(p\varphi^+ + q\varphi^-) < \mathcal{J}_\lambda(p_\varphi\varphi^+ + q_\varphi\varphi^-)$.

Proof.(i)
f is critical or subcritial , and from (V_1) to (V_4), for all $\varepsilon > 0$, there exists a positive constant $C_1 = C_1(\epsilon)$ such that

$$f(x, t)t \leq \varepsilon|t|^N + C_1|t|^s \exp(\alpha|t|^\gamma) \text{ for all } \alpha > \alpha_0, s > N. \tag{3.3}$$

In the sequel, $\varphi \in \mathbf{X}$ fixed with $\varphi^+ \neq 0$ and $\varphi^- \neq 0$. Using (3.3), for all $\varepsilon > 0$, we get

$$\begin{aligned}\langle \mathcal{J}'_\lambda(p\varphi^+ + q\varphi^-), p\varphi^+\rangle &= \langle \mathcal{J}'_\lambda(p\varphi^+), p\varphi^+\rangle \\ &= \|p\varphi^+\|^N - \lambda\int_B f(x, p\varphi^+)p\varphi^+ dx \\ &\geq \|p\varphi^+\|^N - \lambda\epsilon\int_B |p\varphi^+|^N dx - \lambda C_1\int_B |p\varphi^+|^s \exp(\alpha p|\varphi^+|^\gamma) dx.\end{aligned}$$

Apply the Hölder inequality, with $a, a' > 1$ such that $\dfrac{1}{a} + \dfrac{1}{a'} = 1$, and Lemma 2.1, yields to

$$\langle \mathcal{J}'_\lambda(p\varphi^+ + q\varphi^-), p\varphi^+\rangle \geq \|p\varphi^+\|^N - \lambda\epsilon C_2\|p\varphi^+\|^N - \lambda C_1\left(\int_B |p\varphi^+|^{a's}dx\right)^{\frac{1}{a'}}\left(\int_B \exp(\alpha p a|\varphi^+|^\gamma)dx\right)^{\frac{1}{a}}$$
$$\geq (1-\epsilon C_2 - \lambda\epsilon C_1)\|p\varphi^+\|^N - \lambda C_1\left(\int_B \exp\left(\alpha a\|p\varphi^+\|^\gamma(\frac{|\varphi^+|}{\|\varphi^+\|})^\gamma\right)dx\right)^{\frac{1}{a}} C_3\|p\varphi^+\|^s.$$

By (1.6), we can deduce that the last integral is finite then $p > 0$ is chosen small enough such that $\alpha a\|p\varphi^+\|^\gamma \leq \alpha_{N,\beta}$. Then,

$$\langle \mathcal{J}'_\lambda(p\varphi^+ + q\varphi^-), p\varphi^+\rangle \geq (1-\epsilon C_2 - \lambda\epsilon C_1)\|p\varphi^+\|^N - \lambda C_4\|p\varphi^+\|^s \tag{3.4}$$

holds. Choosing $\epsilon > 0$ such that $1 - \epsilon C_2 - \lambda\epsilon C_1 > 0$ anf for small $p > 0$ and for all $q > 0$ and $s > N$, we get $\langle \mathcal{J}'_\lambda(p\varphi^+ + q\varphi^-), p\varphi^+\rangle > 0$. In the similar way, it can be proved that $\langle \mathcal{J}'_\lambda(p\varphi^+ + q\varphi^-), p\varphi^-\rangle > 0$ for $q > 0$ small enough and all $p > 0$. After that, we can state that there exists $t_1 > 0$ such that

$$\langle \mathcal{J}'_\lambda(t_1\varphi^+ + q\varphi^-), t_1\varphi^+\rangle > 0, \quad \langle \mathcal{J}'_\lambda(p\varphi^+ + t_1\varphi^-), t_1\varphi^-\rangle > 0 \text{ for all } p, q > 0. \tag{3.5}$$

Futhermore, by (V_3), we can deduce that there exists $C_5, C_6 > 0$ such that

$$F(x,t) \geq C_5|t|^\theta - C_6. \tag{3.6}$$

Actually, take $p = t_2^* > t_1$ with t_2^* large enough. So, using (3.3), (3.6), yields to

$$\langle \mathcal{J}'_\lambda(t_2^*\varphi^+ + q\varphi^-), t_2^*\varphi^+\rangle = \langle \mathcal{J}'{}_\lambda(t_2^*\varphi^+), t_2^*u^+\rangle \qquad \leq \|t_2^*\varphi^+\|^N - \lambda\int_B C_5|t_2^*\varphi^+|^\theta dx + \lambda C_6|B| \leq 0,$$

for $q \in [t_1, t_2^*]$. Also, we can choose $q = t_2^* > t_1$ with t_2^* large enough and then

$$\langle \mathcal{J}'_\lambda(t_2^*u^+ + t_2^*u^-), t_2^*\varphi^+\rangle < 0 \text{ holds for } p \in [t_1, t_2^*].$$

Therefore, if $t_2 > t_2^*$, then we obtain that

$$\mathcal{J}'_\lambda(t_2\varphi^+ + q\varphi^-), t_2\varphi^+\rangle < 0 \quad \text{ and } \quad \langle \mathcal{J}'_\lambda(p\varphi^+ + t_2\varphi^-), t_2\varphi^-\rangle < 0 \text{ for all } p, q \in [t_1, t_2]. \tag{3.7}$$

linking between (3.5) and (3.7) with Miranda's Theorem [2], there exists at least a couple of points $(p_\varphi, q_\varphi) \in (0,\infty) \times (0,\infty)$ such that $L_\varphi(p_\varphi, q_\varphi) = (0,0)$, i.e, $p_\varphi\varphi^+ + q_\varphi\varphi^- \in \mathcal{N}_\lambda$.
Now we will show the uniqueness of the couple (p_u, q_u). Roughly speaking, it is sufficient to prove that if $\varphi \in \mathcal{N}_\lambda$ and $p_0\varphi^+ + q_0\varphi^- \in \mathcal{N}_\lambda$ with $p_0 > 0$ and $q_0 > 0$, then $(p_0, q_0) = (1,1)$. Let us assume that $\varphi \in \mathcal{N}_\lambda$ and $p_0\varphi^+ + q_0\varphi^- \in \mathcal{N}_\lambda$. We will get then $\langle \mathcal{J}'_\lambda(p_0\varphi^+ + q_0\varphi^-), p_0\varphi^+\rangle = 0$, $\langle \mathcal{J}'_\lambda(p_0\varphi^+ + q_0\varphi^-), p_0\varphi^-\rangle = 0$, and $\langle \mathcal{J}'_\lambda(\varphi), \varphi^\pm\rangle = 0$, that is,

$$\|p_0\varphi^+\|^N = \lambda\int_B f(x, p_0\varphi^+)p_0\varphi^+dx. \tag{3.8}$$

$$\|q_0\varphi^-\|^N = \lambda\int_B f(x, q_0\varphi^-)q_0\varphi^-dx. \tag{3.9}$$

$$\|\varphi^+\|^N = \lambda\int_B f(x, \varphi^+)\varphi^+dx. \tag{3.10}$$

$$\|\varphi^-\|^N = \lambda\int_B f(x, \varphi^-)\varphi^-dx. \tag{3.11}$$

Combining (3.8) and (3.10), we deduce that $0 = \lambda\int_B \frac{f(x,p_0\varphi^+)p_0\varphi^+}{p_0^N}dx - \lambda\int_B f(x,\varphi^+)\varphi^+dx$. It follows from (V_4) that $t \mapsto \frac{f(x,t)}{t^{N-1}}$ is increasing for $t > 0$, that $p_0 = 1$. We can also show, using (V_4), (3.9) and (3.11), that $q_0 = 1$, who will achieve which the proof of (i).

(ii) To cope with the proof of (ii), it is sufficient to show that (p_φ, q_φ) is unique maximum point of $\Upsilon_\varphi \in [0,\infty)\times[0,\infty)$. By (3.7), (3.8) and $\theta > N$, we get

$$\Upsilon_\varphi(p,q) \le \frac{p^N}{N}\|\varphi^+\|^N + \frac{q^N}{N}\|\varphi^-\|^N - \lambda C_5 p^\theta \int_B |u^+|^\theta dx - \lambda C_5 q^\theta \int_B |\varphi^-|^\theta dx + \lambda C_6 |B|.$$

which provides that $\lim\limits_{|(p,q)|\to\infty} \Upsilon_\varphi(p,q) = -\infty$. From which, it suffices to remark that the maximum point of Υ_φ cannot be assigned on the boundary of $[0,\infty)\times[0,\infty)$. We are looking to use the contradiction method and suppose that $(0,q)$ with $q \ge 0$ is a maximum point of Υ_u. Hence from (3.5), we get

$$p\frac{d}{dp}[\mathcal{J}_\lambda(p\varphi^+ + q\varphi^-)] = \langle \mathcal{J}'_\lambda(p\varphi^+), p\varphi^+\rangle > 0,$$

for small $p > 0$, which means that Υ_φ is increasing with respect to p if $p > 0$ is small enough. This gives a contradiction. We can similarly deduce that Υ_φ can not realize its global maximum on $(p,0)$ with $p \ge 0$.

Lemma 3.2 *For any $\varphi \in \mathbf{X}$ with $\varphi^+ \ne 0$ and $\varphi^- \ne 0$, such that $\langle \mathcal{J}'_\lambda(p\varphi^+, p\varphi^+\rangle \le 0$, the unique maximum point (p_φ, q_φ) of Υ_φ on $[0,\infty)\times[0,\infty)$ belongs to $(0,1]\times(0,1]$.*

Proof. Here we just prove that $0 < p_\varphi \le 1$. . Since $p_\varphi\varphi^+, q_\varphi\varphi^- \in \mathcal{N}_\lambda$, we have that

$$\|p_\varphi\varphi^+\|^N = \lambda \int_B f(x, p_\varphi\varphi^+)\varphi^+ dx. \tag{3.12}$$

Moreover, by $\langle \mathcal{J}'_\lambda(p\varphi^+, p\varphi^+\rangle \le 0$, we have that

$$\|\varphi^+\|^N \le \lambda \int_B f(x, \varphi^+)\varphi^+ dx. \tag{3.13}$$

Combining (3.12) and (3.13), it follows that

$$\int_B f(x,\varphi^+)\varphi^+ dx \ge \int_B \frac{f(x, p_\varphi\varphi^+)p_\varphi\varphi^+}{p_\varphi^N} dx. \tag{3.14}$$

Now, we suppose, by contradiction, that $p_\varphi > 1$. By (V_3), $t \mapsto \dfrac{f(x,t)}{t^{N-1}}$ is increasing for $t > 0$, which contradicts inequality (3.14).Therefore, $0 < p_\varphi \le 1$. The proof of $0 < q_\varphi \le 1$ is similar.

Lemma 3.3 *For all $u \in \mathcal{N}_\lambda$,*

- (i) $\mathcal{J}_\lambda(\varphi) \ge (\frac{1}{N} - \frac{1}{\theta})\|\varphi\|^N$.
- (ii) *there exists $\kappa > 0$ such that* $\|\varphi^+\|, \|\varphi^-\| \ge \kappa$;

Proof. (i) Given $u \in \mathcal{N}_\lambda$, by the definition of $\mathcal{N}_\lambda$ and (V_3) we obtain

$$\begin{aligned}\mathcal{J}_\lambda(\varphi) &= \mathcal{J}_\lambda(\varphi) - \frac{1}{\theta}\langle \mathcal{J}'_\lambda(\varphi), \varphi\rangle \\ &= \frac{1}{N}\|\varphi_n\|^N + \lambda\Big(\int_B \frac{1}{\theta} f(x,\varphi)u - F(x,\varphi)dx\Big) \ge \Big(\frac{1}{N} - \frac{1}{\theta}\Big)\|\varphi\|^N.\end{aligned}$$

Lemma 3.3 implies that $\mathcal{J}_\lambda(\varphi) > 0$ for all $\varphi \in \mathcal{N}_\lambda$. As a consequence, $\mathcal{J}_\lambda$ is bounded by below in $\mathcal{N}_\lambda$, and therefore $c_\lambda := \inf\limits_{\varphi\in\mathcal{N}_\lambda} \mathcal{J}_\lambda(\varphi)$ is well-defined.

The following lemma deals with the asymptotic property of c_λ.

(ii) We use contradiction method. Our proof start by suggest that there exists a sequence $\{\varphi_n^+\} \subset \mathcal{N}_\lambda$ for which $\varphi_n^+ \to 0$ in $\mathbf{X}$. Since $\{\varphi_n\} \subset \mathcal{N}_\lambda$, then $\langle \mathcal{J}'_\lambda(\varphi_n), \varphi_n^+\rangle = 0$. Hence, it follows using (3.3), (3.4) and the radial Lemma 2.1 that

$$\begin{aligned}\|\varphi_n^+\|^N &= \lambda\int_B f(x,\varphi_n^+)\varphi_n^+dx \le \epsilon\lambda\int_B|\varphi_n^+|^N dx + \lambda C_1\int_B|\varphi_n^+|^s\exp(\alpha|\varphi_n^+|^\gamma)dx \\ &\le \epsilon\lambda C_6\|\varphi_n^+\|^N + \lambda C_1\int_B|\varphi_n^+|^s\exp(\alpha|\varphi_n^+|^\gamma)dx.\end{aligned} \tag{3.15}$$

Let $a>1$ with $\frac{1}{a}+\frac{1}{a'}=1$. Since $\varphi_n^+\to 0$ in **X**, for n large enough, yields to $\|\varphi_n^+\|\le(\frac{\alpha_{N,\beta}}{\alpha a})^{\frac{1}{\gamma}}$. according to Hölder inequality, (1.6) and again the radial Lemma 2.1, we get

$$\begin{aligned}\int_B|\varphi_n^+|^s\exp(\alpha|\varphi_n^+|^\gamma)dx &\le \left(\int_B|\varphi_n^+|^{sa'}dx\right)^{\frac{1}{a'}}\left(\int_B\exp\left(\alpha a\|\varphi^+\|^\gamma(\frac{|\varphi^+|}{\|\varphi^+\|})^\gamma\right)dx\right)^{\frac{1}{a}} \\ &\le C_7\left(\int_B|\varphi_n^+|^{sa'}dx\right)^{\frac{1}{a'}}\le C_8\|\varphi_n^+\|^s.\end{aligned}$$

With aid of (3.15) with the last inequality, we obtain

$$\|\varphi_n^+\|^N\le\lambda\epsilon C_6\|\varphi_n^+\|^N+\lambda C_8\|\varphi_n^+\|^s. \tag{3.16}$$

For $\epsilon>0$ such that $1-\lambda\epsilon C_6>0$. Since $N<s$, then (3.16) contradicts the fact that $\varphi_n^+\to 0$ in **X**.

Lemma 3.4 *Let* $c_\lambda=\inf_{\varphi\in\mathcal{N}_\lambda}\mathcal{J}_\lambda(\varphi)$, *then* $\lim_{\lambda\to\infty}c_\lambda=0$.

Proof. Let us Fix $\varphi\in\mathbf{X}$ with $\varphi^\pm=0$. Then, by Lemma 3.1, there exists a point pair (p_λ,q_λ) such that $p_\lambda\varphi^++q_\lambda\varphi^-\in\mathcal{N}_\lambda$ for each $\lambda>0$. Let $\mathcal{T}_\varphi$ be the set defined by

$$\mathcal{T}_u:=\{(p_\lambda,q_\lambda)\in[0,\infty)\times[0,\infty):L_u(p_\lambda,q_\lambda)=(0,0),\lambda>0\},$$

where L_φ is given by (3.2).
Since $p_\lambda\varphi^++q_\lambda\varphi^-\in\mathcal{N}_\lambda$, by assumption (V_2), (3.7) and (3.8), we have

$$\begin{aligned}p_\lambda^N\|\varphi^+\|^N+q_\lambda^N\|\varphi^-\|^N &= \lambda\int_B f(x,p_\lambda\varphi^++q_\lambda\varphi^-)(p_\lambda\varphi^++q_\lambda\varphi^-)dx \\ &\ge \lambda\theta C_5p_\lambda^\theta\int_B|\varphi^+|^\theta dx+\lambda\theta C_5q_\lambda^\theta\int_B|\varphi^-|^\theta dx-\lambda\theta C_6|B|.\end{aligned}$$

Since $\theta>N$, the set $\mathcal{T}_\varphi$ is bounded. Therefore, if $\{\lambda_n\}\subset(0,\infty)$ satisfies $\lambda_n\to\infty$ as $n\to\infty$, then up to subsequence, there exists $\bar{p},\bar{q}>0$, such that $p_{\lambda_n}\to\bar{p}$ and $q_{\lambda_n}\to\bar{q}$.
We suppose that $\bar{p}=\bar{q}=0$. As before, we argue contradiction and suppose that $\bar{p}>0$ and $\bar{q}>0$. For each $n\in\mathbb{N}$, $p_{\lambda_n}\varphi^++q_{\lambda_n}\varphi^-\in\mathcal{N}_{\lambda_n}$. So,

$$\|p_{\lambda_n}\varphi^++q_{\lambda_n}\varphi^-\|^N=\lambda_n\int_B f(p_{\lambda_n}\varphi^++q_{\lambda_n}\varphi^-)(p_{\lambda_n}\varphi^++q_{\lambda_n}\varphi^-)dx. \tag{3.17}$$

It should be mentioned that $p_{\lambda_n}\varphi^+\to\bar{p}\varphi^+$ and $q_{\lambda_n}\varphi^-\to\bar{q}\varphi^-$ in **X**.

On the other hand, $\lambda_n\to 0$ as $n\to\infty$ and $\{p_{\lambda_n}\varphi^++q_{\lambda_n}\varphi^-\}$ is bounded in **X**. Also, from (3.17),we have

$$\int_B|\nabla(\bar{p}\varphi^++\bar{q}\varphi^-)|^N dx=\left(\lim_{n\to\infty}\lambda_n\right)\lim_{n\to\infty}\int_B f(p_{\lambda_n}\varphi^++q_{\lambda_n}\varphi^-)(p_{\lambda_n}\varphi^++q_{\lambda_n}\varphi^-)dx,$$

which give a contradiction.
Then, $\bar{p}=\bar{q}=0$, so, $p_{\lambda_n}\to 0$ and $q_{\lambda_n}\to 0$ as $n\to\infty$. By (V_2) and (3.17), we get

$$0\le c_\lambda=\inf_{\mathcal{N}_\lambda}\mathcal{J}_\lambda(\varphi)\le\mathcal{J}_\lambda(p_{\lambda_n}\varphi^++q_{\lambda_n}\varphi^-)\to 0.$$

We deduce, $c_\lambda\to 0$ as $\lambda\to\infty$.

Lemma 3.5 *If $\varphi_0 \in \mathcal{N}_\lambda$ satisfies $\mathcal{J}_\lambda(\varphi_0) = c_\lambda$, then $\mathcal{J}'_\lambda(\varphi_0) = 0$.*

Proof. we argue by contradiction. suppose that $\mathcal{J}'_\lambda(\varphi_0) \neq 0$. By the continuity of $\mathcal{J}'_\lambda$, there exists $\iota, \delta \geq 0$ such that

$$\|\mathcal{J}'_\lambda(v)\|_{\mathbf{X}^*} \geq \iota \text{ for all } \|v - \varphi_0\| \leq 3\delta. \tag{3.18}$$

Choose $\tau \in (0, \min\{\frac{1}{4}, \frac{\delta}{4\|\varphi_0\|}\})$. Let $D = (1-\tau, 1+\tau) \times (1-\tau, 1+\tau)$ and define $g : D \to \mathbf{X}$, by

$$g(\rho, \vartheta) = \rho\varphi_0^+ + \vartheta\varphi_0^-, (\rho, \vartheta) \in D.$$

By virtue of $\varphi_0 \in \mathcal{N}_\lambda$, $\mathcal{J}_\lambda(\varphi_0) = c_\lambda$ and Lemma 3.1, it is straightforward to verify that

$$\bar{c_\lambda} := \max_{\partial D} \mathcal{J}_\lambda \circ g < c_\lambda. \tag{3.19}$$

Let $\epsilon := \min\{\frac{c_\lambda - \bar{c_\lambda}}{3}, \frac{\iota\delta}{8}\}$, $S_r := B(u_0, r), r \geq 0$ and $\mathcal{J}_\lambda^a := \mathcal{J}_\lambda^{-1}(]-\infty, a])$. To cope with, using Quantitative Deformation Lemma [[21], Lemma 2.3], there exists a deformation $\eta \in C\left([0,1] \times g(D), \mathbf{X}\right)$ such that:

(1) $\eta(1, v) = v$, if $v \notin \mathcal{J}_\lambda^{-1}([c_\lambda - 2\epsilon, c_\lambda + 2\epsilon]) \cap S_{2\delta}$

(2) $\eta\left(1, \mathcal{J}_\lambda{}^{c_\lambda+\epsilon} \cap S_\delta\right) \subset \mathcal{J}_\lambda{}^{c_\lambda - \epsilon}$,

(3) $\mathcal{J}_\lambda(\eta(1, v)) \leq \mathcal{J}_\lambda(v)$, for all $v \in \mathbf{X}$.

Using lemma 3.1 (ii), we have $\mathcal{J}_\lambda(g(\rho, \vartheta)) \leq c_\lambda$. Futhermore, we have,

$$\|g(s,t) - \varphi_0\| = \|(\rho - 1)\varphi_0^+ + (\vartheta - 1))\varphi_0^-\| \leq |\rho - 1|\|\varphi_0^+\| + |\vartheta - 1|\|\varphi_0^-\| \leq 2\tau\|\varphi_0\|,$$

then $g(\rho, \vartheta) \in S_\delta$ for $(\rho, \vartheta) \in \bar{D}$. nevertheless, it follows from (2) that

$$\max_{(\rho,\vartheta)\in\bar{D}} \mathcal{J}_\lambda(\eta(1, g(\rho, \vartheta))) \leq c_\lambda - \epsilon. \tag{3.20}$$

In the following, we prove that $\eta(1, g(D)) \cap \mathcal{N}_\lambda$ is nonempty. And in this case it contradicts (3.20) due to the definition of c_λ. To do this, we first define

$$\bar{g}(\rho, \vartheta) := \eta(1, g(\rho, \vartheta)),$$

$$\begin{aligned}\Upsilon_0(\rho, \vartheta) &= (\langle \mathcal{J}'_\lambda(g(\rho, \vartheta)), \varphi_0^+\rangle, \langle \mathcal{J}'_\lambda(g(\rho, \vartheta)), \varphi_0^-\rangle) \\ &= (\langle \mathcal{J}'_\lambda(\rho\varphi_0^+ + \vartheta\varphi_0^-), \varphi_0^+\rangle, \langle \mathcal{J}'_\lambda(\rho\varphi_0^+ + \vartheta\varphi_0^-), \varphi_0^-\rangle) \\ &:= (I^1_{\varphi_0}(\rho, \vartheta), I^2_{\varphi_0}(\rho, \vartheta))\end{aligned}$$

and

$$\Upsilon_1(\rho, \vartheta) := (\frac{1}{\rho}\langle \mathcal{J}'_\lambda(\bar{g}(\rho, \vartheta)), (\bar{g}(\rho, \vartheta))^+\rangle, \frac{1}{\vartheta}\langle \mathcal{J}'_\lambda(\bar{g}(\rho, \vartheta)), (\bar{g}(\rho, \vartheta))^-\rangle).$$

Moreover, a simple calculation, shows that

$$\begin{aligned}\left.\frac{I^1_{\varphi_0}(\rho, \vartheta)}{\partial\rho}\right|_{(1,1)} &= (N-1)\|\varphi_0^+\|^N - \lambda\int_B f'(x, \varphi_0^+)|\varphi_0^+|^2 dx \\ &= (N-1)\lambda\int_B f(\varphi_0^+)\varphi_0^+ dx - \lambda\int_B f'(x, \varphi_0^+)|\varphi_0^+|^2 dx\end{aligned}$$

and

$$\left.\frac{I^1_{\varphi_0}(\rho, \vartheta)}{\partial\vartheta}\right|_{(1,1)} = 0.$$

In the same manner,

$$\left.\frac{I^2_{\varphi_0}(\rho, \vartheta)}{\partial\rho}\right|_{(1,1)} = 0$$

and

$$\frac{I^2_{\varphi_0}(\rho,\vartheta)}{\partial\vartheta}\bigg|_{(1,1)} = (N-1)\lambda\int_B f(x,\varphi_0^-)\varphi_0^- dx - \lambda\int_B f'(x,\varphi_0^-)|\varphi_0^-|^2 dx.$$

Let

$$\mathrm{J} = \begin{pmatrix} \frac{I^1_{\varphi_0}(\rho,\vartheta)}{\partial\rho}\Big|_{(1,1)} & \frac{I^2_{\varphi_0}(\rho,\vartheta)}{\partial\rho}\Big|_{(1,1)} \\ \frac{I^1_{\varphi_0}(\rho,\vartheta)}{\partial\vartheta}\Big|_{(1,1)} & \frac{I^2_{\varphi_0}(\rho,\vartheta)}{\partial\vartheta}\Big|_{(1,1)} \end{pmatrix}.$$

At this point, $\det \mathrm{J} \neq 0$. Moreover, the point $(0,1)$ is the isolated zero which is unique of the C^1 function Υ_0. According to the Brouwer's degree in $\mathbb{R}^2$, we can obtain $\deg(\Upsilon_0, D, 0) = 1$.
It follows from (3.20) that $g(\rho,\vartheta) = \bar{g}(\rho,\vartheta)$ on ∂D. For the boundary dependence of Brouwer's degree (see [[9], Theorem 4.5]), we get $\deg(\Upsilon_1, D, 0) = \deg(\Upsilon_0, D, 0) = 1$. Furthermore, there exists some $(\overline{\rho},\overline{\vartheta}) \in D$ such that

$$\eta(1, g(\overline{\rho},\overline{\vartheta}) \in \mathcal{N}_\lambda.$$

Lemma 3.6 *If υ is a least energy sign-changing solution of problem (P_λ), then υ has exactly two nodal domains.*

Proof. Assume by contradiction that $\upsilon = \upsilon_1 + \upsilon_2 + \upsilon_3$ satisfies

$$\upsilon_i \neq 0, i = 1,2,3, \upsilon_1 \geq 0, \upsilon_2 \leq 0, \text{ a.e. in } B$$
$$B_1 \cap B_2 = \emptyset, B_1 := \{x \in B : \upsilon_1(x) > 0\}, B_2 := \{x \in B : \upsilon_2(x) < 0\}$$

$$\upsilon_1\Big|_{B\setminus B_1\cup B_2} = \upsilon_2\Big|_{B\setminus B_2\cup B_1} = \upsilon_3\Big|_{B_1\cup B_2} = 0,$$

and

$$\langle \mathcal{J}'_\lambda(\upsilon), \upsilon_i\rangle = 0 \text{ for } i = 1,2,3. \tag{3.21}$$

Let $\nu = \upsilon_1 + \upsilon_2$ and it is easy to see that $\nu^+ = \upsilon_1$, $\nu^- = \upsilon_2$ and $\nu^\pm \neq 0$. From Lemma 3.1, it follows that there exists a unique couple $(p_\nu, q_\nu) \in [0,\infty) \times [0,\infty)$ such that $p_\nu\upsilon_1 + q_\nu\upsilon_2 \in \mathcal{N}_\lambda$. So, $\mathcal{J}_\lambda(p_\nu\upsilon_1 + q_\nu\upsilon_2) \geq c_\lambda$. Moreover, using (3.21), we obtain that $\langle \mathcal{J}'_\lambda(\nu), \nu^\pm\rangle = 0$. Then, by Lemma 3.2, we have $0 < p_\nu, q_\nu \leq 1$.

Now, combining (3.21), (V_3) and (V_4), we have that

$$\begin{aligned} 0 = &\frac{1}{\theta}\langle \mathcal{J}'_\lambda(\upsilon), \upsilon_3\rangle = \frac{1}{\theta}\langle \mathcal{J}'_\lambda(\upsilon_3), \upsilon_3\rangle \\ < &\mathcal{J}_\lambda(\upsilon_3), \end{aligned}$$

and

$$\begin{aligned} c_\lambda \leq &\mathcal{J}_\lambda(p_\nu\upsilon_1 + q_\nu\upsilon_2) \\ = &\mathcal{J}_\lambda(p_\nu\upsilon_1 + q_\nu\upsilon_2) - \frac{1}{\theta}\langle \mathcal{J}'_\lambda p_\nu\upsilon_1 + q_\nu\upsilon_2), p_\nu\upsilon_1 + q_\nu\upsilon_2\rangle \\ &= (\frac{1}{N} - \frac{1}{\theta})p_\nu^N\|\upsilon_1\|^N + (\frac{1}{N} - \frac{1}{\theta})q_\nu^N\|\upsilon_2\|^N \\ &+ \lambda\int_B[\frac{1}{\theta}f(x, p_\nu\upsilon_1)(p_\nu\upsilon_1) - F(x, p_\nu\upsilon_2)]dx + \lambda\int_B[\frac{1}{\theta}f(x, q_\nu\upsilon_1)(p_\nu\upsilon_2) - F(x, q_\nu\upsilon_2)]dx \\ &\leq \mathcal{J}_\lambda(\upsilon_1 + \upsilon_2) - \frac{1}{\theta}\langle \mathcal{J}'_\lambda\upsilon_1 + \upsilon_2), \upsilon_1 + \upsilon_2\rangle \\ &= \mathcal{J}_\lambda(\upsilon_1 + \upsilon_2) + \frac{1}{\theta}\langle \mathcal{J}'_\lambda(\upsilon), \upsilon_3\rangle \\ &< \mathcal{J}_\lambda(\upsilon_1 + \upsilon_2) + \mathcal{J}_\lambda(\upsilon_3) = \mathcal{J}_\lambda(\upsilon) = c_\lambda, \end{aligned}$$

which is a contradiction. Therefore, $\upsilon_3 = 0$ and υ has exactly two nodal domains.

4. The subcritical case

We write some useful Lemmas which are proved in [8].

Lemma 4.1 *If $\{\varphi_n\} \subset \mathcal{N}_\lambda$ is a minimizing sequence for c_λ, then there exists some $\varphi \in \mathbf{X}$ such that*

$$\int_B f(x,\varphi_n^\pm)\varphi_n^\pm dx \to \int_B f(x,\varphi^\pm)\varphi^\pm dx,$$

and

$$\int_B F(x,\varphi_n^\pm)dx \to \int_B F(x,\varphi^\pm)dx.$$

Lemma 4.2 *There exists some $\upsilon \in \mathcal{N}_\lambda$ such that $\mathcal{J}_\lambda(\upsilon) = c_\lambda$.*

Proof of Theorem 1.2. From Lemma 3.5, Lemma 3.6 and Lemma 4.2, we deduce that υ is a least energy sign-changing solution form problem (P_λ) with exactly tow nodal domains.

5. The critical case

Lemma 5.1 *There exists $\lambda^* > 0$ such that if $\lambda \geq \lambda^*$, and $\{\varphi_n\} \subset \mathcal{N}_\lambda$ is a minimizing sequence for c_λ, then there exists some $\varphi \in \mathcal{N}_\lambda$ such that $\mathcal{J}_\lambda(\varphi) = c_\lambda$.*

Proof. Let $\{\varphi_n\} \subset \mathcal{N}_\lambda$ be a sequence such that $\lim_{n\to\infty} \mathcal{J}_\lambda(\varphi_n) = c_\lambda$. We have

$$\mathcal{J}_\lambda(\varphi_n) \to c_\lambda \text{ and } \langle \mathcal{J}'_\lambda(\varphi_n), u\rangle \to 0, \forall u \in \mathbf{X}.$$

That is

$$\mathcal{J}_\lambda(\varphi_n) = \frac{1}{N}\|\varphi_n\|^N - \int_B F(x,\varphi_n)dx \to c_\lambda, \quad n \to +\infty \tag{5.1}$$

and

$$|\langle \mathcal{J}'_\lambda(\varphi_n), u\rangle| = \Big|\int_B \omega(x)|\nabla\varphi_n|^{N-2}\nabla\varphi_n.\nabla u dx + \int_B |\varphi|^{N-2}\varphi u \, dx - \int_B f(x,\varphi_n)u dx\Big| \leq \varepsilon_n\|u\|, \tag{5.2}$$

for all $u \in \mathbf{X}$, where $\varepsilon_n \to 0$, as $n \to +\infty$.
According to lemma 3.3, φ_n is bounded in $\mathbf{X}$. Moroever, we have from (5.2) and (V_2), that

$$0 < \int_B f(x,\varphi_n)\varphi_n \leq C \tag{5.3}$$

and

$$0 < \int_B F(x,\varphi_n) \leq C.$$

Since by Lemma 3.2, we have

$$f(x,\varphi_n) \to f(x,\varphi) \text{ in } L^1(B) \text{ as } n \to +\infty, \tag{5.4}$$

then, it follows from (V_2) and the generalized Lebesgue dominated convergence Theorem that

$$F(x,\varphi_n) \to F(x,\varphi) \text{ in } L^1(B) \text{ as } n \to +\infty. \tag{5.5}$$

Arguing as Lemma 4.2, we have that, up to a subsequence,

$$\begin{aligned} &\varphi_n \rightharpoonup \varphi \text{ in } \mathbf{X}, \\ &\varphi_n \to \varphi \text{ in } L^t(B) \text{ for } t \in [1,\infty), \\ &\varphi_n \to \varphi \text{ a.e. in } B. \end{aligned} \tag{5.6}$$

$$\begin{aligned} &\varphi_n^\pm \rightharpoonup \varphi^\pm \text{ in } \mathbf{X}, \\ &\varphi_n^\pm \to \varphi^\pm \text{ in } L^t(B) \text{ for } t \in [1,\infty), \\ &\varphi_n^\pm \to \varphi^\pm \text{ a.e. in } B \text{ for some } \varphi \in \mathbf{X}. \end{aligned} \tag{5.7}$$

According to lemma 3.4, there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, yields to $c_\lambda < \frac{1}{N}(\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}$. In the sequel, the results that are valid for φ_n and φ, are also valid for $\varphi_n^{\pm}$ and $\varphi^{\pm}$. After that , we are going to care about the following.

Claim 1. $\nabla\varphi_n(x) \to \nabla\varphi(x)$ *a.e.* in B and φ is a solution of the problem (P_λ). See [7] and [8] for the proof.

Claim 2. $\varphi^+ \neq 0$ and $\varphi^- \neq 0$. We argue by contradiction, $\varphi^+ = 0$. Then, $\int_B F(x,\varphi_n)dx \to 0$ and consequently we get

$$\frac{1}{N}\|\varphi_n\|^N \to c_\lambda < \frac{1}{N}(\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}. \tag{5.8}$$

To tackle the proof, we claim that there exists $q > 1$ such that

$$\int_B |f(x,\varphi_n)|^q dx \leq C. \tag{5.9}$$

By (5.2), we have $\left|\|\varphi_n\|^N - \int_B f(x,\varphi_n)\varphi_n dx\right| \leq C\varepsilon_n$. So

$$\|\varphi_n\|^N \leq C\varepsilon_n + \Big(\int_B |f(x,\varphi_n)|^q dx\Big)^{\frac{1}{q}}\Big(\int_B |\varphi_n|^{q'}dx\Big)^{\frac{1}{q'}},$$

where q' is the conjugate of q. Since (φ_n) converge to 0 in $L^{q'}(B)$ $\lim_{n\to+\infty}\|\varphi_n\|^N = 0$. Using Lemma 3.3, this result cannot occur. Now for the proof of the claim (5.9), since f has critical growth, for every $\varepsilon > 0$ and $q > 1$ there exists $t_\varepsilon > 0$ and $C > 0$ such that for all $|t| \geq t_\varepsilon$, we have $|f(x,t)|^q \leq Ce^{\alpha_0(\varepsilon+1)t^\gamma}$. Consequently,

$$\begin{aligned}\int_B |f(x,\varphi_n)|^q dx &= \int_{\{|\varphi_n|\leq t_\varepsilon\}} |f(x,\varphi_n)|^q dx + \int_{\{|\varphi_n|>t_\varepsilon\}} |f(x,\varphi_n)|^q dx\\ &\leq \omega_{N-1}\max_{B\times[-t_\varepsilon,t_\varepsilon]}|f(x,t)|^q + C\int_B e^{\alpha_0(\varepsilon+1)|\varphi_n|^\gamma}dx.\end{aligned}$$

Since $Nc_\lambda < (\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}$, there exists $\eta \in (0,\frac{1}{N})$ such that $Nc_\lambda = (1-\eta)(\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}$. On the other hand, $\|\varphi_n\|^\gamma \to (Nc_\lambda)^{\frac{\gamma}{N}}$, so there exists $n_\eta > 0$ such that for all $n \geq n_\eta$, we get $\|\varphi_n\|^\gamma \leq (1-\eta)\frac{\alpha_{N,\beta}}{\alpha_0}$. Therefore, $\alpha_0(1+\varepsilon)(\frac{|\varphi_n|}{\|\varphi_n\|})^\gamma\|\varphi_n\|^\gamma \leq (1+\varepsilon)(1-\eta)\alpha_{N,\beta}$. We choose $\varepsilon > 0$ small enough to get

$$\alpha_0(1+\varepsilon)\|\varphi_n\|^\gamma \leq \alpha_{N,\beta}.$$

So, the second integral is uniformly bounded in view of (1.6) and the claim is proved.

Since (φ_n) is bounded, up to a subsequence, we can assume that $\|\varphi_n\| \to \rho > 0$. We affirm that $\mathcal{J}_\lambda(\varphi) = c_\lambda$. Indeed, by (V_2) and claim 2, we have

$$\mathcal{J}_\lambda(\varphi) = \frac{1}{N}\int_B [f(x,\varphi)\varphi - NF(x,\varphi)]dx \geq 0. \tag{5.10}$$

According to the lower semi continuity of the norm and (5.5) , we obtain,

$$\mathcal{J}_\lambda(\varphi) \leq \frac{1}{N}\liminf_{n\to\to\infty}\|\varphi_n\|^N - \int_B F(x,\varphi)dx = c_\lambda.$$

Suppose that $\mathcal{J}_\lambda(\varphi) < c_\lambda$. Then

$$\|\varphi\|^N < \rho^N. \tag{5.11}$$

In addition,

$$\frac{1}{N}\lim_{n\to+\infty}\|\varphi_n\|^N = (c_\lambda + \int_B F(x,\varphi)dx), \tag{5.12}$$

which means that $\rho^N = N\Big(c_\lambda + \int_B F(x,\varphi)dx\Big)$. Set $u_n = \frac{\varphi_n}{\|\varphi_n\|}$ and $u = \frac{\varphi}{\rho}$. We have $\|u_n\| = 1$, $u_n \rightharpoonup u$ in $\mathbf{X}$, $u \not\equiv 0$ and $\|u\| < 1$. It follow by Lemma 2.2 that $\sup_n \displaystyle\int_B e^{p\ \alpha_{N,\beta}|u_n|^\gamma}dx < \infty$, provided $1 < p < \big(1-\|u\|^N\big)^{-\frac{\gamma}{N}}$. By using (5.5) and (5.12), we get: $Nc_\lambda - N\mathcal{J}_\lambda(\varphi) = \rho^N - \|\varphi\|^N$. From (5.10), Lemma 4.1 and the last equality, we obtain

$$\rho^N \le Nc_\lambda + \|\varphi\|^N < (\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}} + \|\varphi\|^N. \tag{5.13}$$

Since $\rho^\gamma = \Big(\frac{\rho^N - \|\varphi\|^N}{1-\|\varphi\|^N}\Big)^{\frac{1}{(N-1)(1-\beta)}}$, we deduce from (5.13) that

$$\rho^\gamma < \Big(\frac{(\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}}{1-\|\varphi\|^N}\Big)^{\frac{1}{(N-1)(1-\beta)}}. \tag{5.14}$$

We have the estimation $\displaystyle\int_B |f(x,\varphi_n)|^q dx < C$. Infact, since f has critical growth, for every $\varepsilon > 0$ and $q > 1$ there exists $t_\varepsilon > 0$ and $C > 0$ such that for all $|t| \ge t_\varepsilon$, we have $|f(x,t)|^q \le Ce^{\alpha_0(\varepsilon+1)t^\gamma}$. So,

$$\begin{aligned}\int_B |f(x,\varphi_n)|^q dx &= \int_{\{|\varphi_n|\le t_\varepsilon\}} |f(x,\varphi_n)|^q dx + \int_{\{|\varphi_n|> t_\varepsilon\}} |f(x,\varphi_n)|^q dx\\ &\le \omega_{N-1} \max_{B\times[-t_\varepsilon,t_\varepsilon]} |f(x,t)|^q + C\int_B e^{\alpha_0(\varepsilon+1)|\varphi_n|^\gamma}dx.\\ &\le C_\varepsilon + C\int_B e^{\alpha_0(1+\varepsilon)\|\varphi_n\|^\gamma \frac{|\varphi_n|^\gamma}{\|\varphi_n\|^\gamma}}dx \le C,\end{aligned}$$

provided $\alpha_0(1+\varepsilon)\|\varphi_n\|^\gamma \le p\ \alpha_{N,\beta}$ and $1 < p < \mathcal{U}(\varphi) = (1-\|\varphi\|^N)^{\frac{-\gamma}{N}}$.

From (5.14), there exists $\delta \in (0,\frac{1}{2})$ such that $\rho^\gamma = (1-2\delta)\Big(\frac{(\frac{\alpha_{N,\beta}}{\alpha_0})^{\frac{N}{\gamma}}}{1-\|\varphi\|^N}\Big)^{\frac{1}{(N-1)(1-\beta)}}$.
Since $\lim_{n\to+\infty}\|\varphi_n\|^\gamma = \rho^\gamma$, then, for n large enough

$$\alpha_0(1+\varepsilon)\|\varphi_n\|^\gamma \le (1+\varepsilon)(1-\delta)\ \ \alpha_{N,\beta}\Big(\frac{1}{1-\|\varphi\|^N}\Big)^{\frac{\gamma}{N}}.$$

We choose $\varepsilon > 0$ small enough such that $(1+\varepsilon)(1-\delta) < 1$ which implies that

$$\alpha_0(1+\varepsilon)\|\varphi_n\|^\gamma < \alpha_{N,\beta}\Big(\frac{1}{1-\|\varphi\|^N}\Big)^{\frac{\gamma}{N}}.$$

Hence, the sequence $(f(x,\varphi_n))$ is bounded in L^q, $q > 1$. By using the Hölder inequality, we write

$$\Big|\int_B f(x,\varphi_n)(\varphi_n-\varphi)dx\Big| \ \le\ C\Big(\int_B |\varphi_n-\varphi|^{q'}dx\Big)^{\frac{1}{q'}} \to 0 \ \ as \ \ n\to+\infty,$$

where $\frac{1}{q}+\frac{1}{q'} = 1$.
Since $\langle \mathcal{J}_\lambda{}'(\varphi_n), (\varphi_n-\varphi)\rangle = o_n(1)$, it follows that

$$\int_B \big(\omega(x)|\nabla\varphi_n|^{N-2}\nabla\varphi_n.(\nabla\varphi_n-\nabla\varphi) + |\varphi|^{N-2}\varphi(\varphi_n-\varphi)\big)dx) \to 0.$$

On the other side,

$$\int_B \big(\omega(x)|\nabla\varphi_n|^{N-2}\nabla\varphi_n.(\nabla\varphi_n-\nabla\varphi) + |\varphi|^{N-2}\varphi(\varphi_n-\varphi)\big)dx = \|\varphi_n\|^N - \int_B \big(\omega(x)|\nabla\varphi_n|^{N-2}\nabla\varphi_n.\nabla\varphi - |\varphi|^N\big)dx\cdot$$

Passing to the limit in the last equality, we get

$$\rho^N - \|\varphi\|^N = 0,$$

therefore $\|\varphi\| = \rho$. This is in contradiction with (5.9). Therefore, $\mathcal{J}_\lambda(\varphi) = c_\lambda$. By Claim 1, $\mathcal{J}'_\lambda(\varphi) = 0$ and by Claim 2, $\varphi \neq 0$.

Proof of Theorem 1.3. From Lemma 3.6 and Lemma 5.1, we deduce that φ is a least energy sign-changing solution for problem (P_λ) with exactly tow nodal domains.

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