Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.67822

Combinatorial sequences resulting from the inputs of powers of triangular matrices

Imène Touaibia* and Miloud Mihoubi

ABSTRACT: In this paper, we give combinatorial interpretations for symmetric functions by the successive powers of triangular matrices. Several interpretations and properties on generalized r-Stirling numbers of the second kind may be deduced.

Key Words: Symmetric functions, generalized r-Stirling numbers, combinatorial sequences, matrix interpretations, recurrence relations.

Contents

1. Introduction

Are combinatorics and matrix theory really different subjects? question that dates back to the 1992s when the author [3] denied the difference between these two subjects and affirmed that there exists a mutually beneficial relationship between them and called it *combinatorial matrix theory* (CMT). After this assertion, different works and applications of these two fields have been applied especially in combinatorics, such the study of the triangular matrices, a special type of matrices which occurs specially in linear differential equations, statistics and in several combinatorial sequences which may be considered as power inputs of upper triangular matrices [4,11,13]. Our contribution fits into the CMT and links essentially the matrix powers and the complete symmetric functions. The main results are about the some combinatorial sequences by using the successive powers of matrices.
For the next, we use the notation

$$\delta_{(m\leq n)} = \begin{cases} 1 & \text{if } m \leq n \\ 0 & \text{if } m > n \end{cases} \quad \text{and} \quad \delta_{(n)} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

For later uses, let us giving some elementary properties on the upper triangular matrices. Indeed, let $(d_{r,k}; 1 \leq r \leq k)$ be a sequence of complex numbers such that $d_{r,r} \neq 0$, $r \geq 1$, and let L be an invertible square matrix of finite or infinite order defined by its (r,k)-entry

$$[L]_{r,k} = d_{r,k}\delta_{(r\leq k)}, \quad k, r \geq 1. \tag{1.1}$$

* Corresponding author

Submitted April 08, 2023. Published January 02, 2025
2010 *Mathematics Subject Classification:* 15A16, 11B75, 03E05

By meaning

$$\left[L^{0}\right]_{r,k}=\delta_{(k-r)}\quad\text{and}\quad\left[L^{1}\right]_{r,k}=d_{r,k}\delta_{(r\leq k)},\tag{1.2}$$

it is easy to see, for $s\geq 2$, that

$$\left[L^{s}\right]_{r,k}=\sum_{r\leq j_1\leq\cdots\leq j_{s-1}\leq k}d_{r,j_1}d_{j_1,j_2}\cdots d_{j_{s-1},k}.\tag{1.3}$$

Also, if we define a sequence of real or complex numbers $L_{r,k}(s)$ by

$$L_{r,k}(s):=\left[L^{s}\right]_{r,k},\quad k\geq r\geq 1,\quad s\in\mathbb{Z},\tag{1.4}$$

it follows from the equality $L^{s+t}=L^{s}L^{t}$ that

$$L_{r,k}(s+t)=\sum_{j=r}^{k}L_{r,j}(s)\,L_{j,k}(t),\quad k\geq r\geq 1,\quad s,t\in\mathbb{Z}.\tag{1.5}$$

In particular, when $s=1$ or $t=1$ we get

$$L_{r,k}(s+1)=\sum_{j=r}^{k}d_{j,k}L_{r,j}(s)\quad\text{and}\quad L_{r,k}(s+1)=\sum_{j=r}^{k}d_{r,j}L_{j,k}(s).\tag{1.6}$$

Furthermore, for any non-negative integer m, if we set

$$\left[\Delta L^{s}\right]_{r,k}=\Delta L_{r,k}(s):=L_{r,k}(s+1)-L_{r,k}(s),$$

it follows

$$\Delta^{2}L^{s}=L^{s+2}-2L^{s+1}+L^{s}=L^{s}(L-I)^{2}\quad\text{and}\quad\Delta^{m}L^{s}=L^{s}(L-I)^{m},$$

and since $\Delta^{m}L_{r,k}(s)=\left[\Delta^{m}L^{s}\right]_{r,k}$, then, we can state the following lemma.

Lemma 1.1 *If $d_{r,r}\geq 1$ and $d_{r,k}\geq 0$ when $k>r\geq 1$ then $(L_{r,k}(s);\ s\geq 0)$ is an infinite convex sequence. In particular*

$$L_{r,k}(s+2)-2L_{r,k}(s+1)+L_{r,k}(s)\geq 0,\quad s\geq 0.\tag{1.7}$$

In this paper, we try to give some properties of the sequence $(L_{r,k}(s))$ for special cases. For a given sequence (a_n) with $a_n\neq 0$, $n\geq 1$, for the case when $[L]_{r,k}=a_{r+1}\delta_{(k-r)}+a_r-a_{r+1}$, $k\geq r\geq 1$, one can verify that the sequence $(L_{r,k}(s))$ is given explicitly by

$$L_{r,k}(s)=a_{r+1}^{s}\delta_{(k-r)}+a_{r}^{s}-a_{r+1}^{s},\quad s\in\mathbb{Z},\quad k>r\geq 1.$$

But for several cases the sequence $(L_{r,k}(s))$ can present a combinatorial sequence on a finite set, such the cases $d_{r,k}=a_r$ and $d_{r,k}=a_k$. The following selected cases will be studied below

$$d_{r,k}=a_r,\quad d_{r,k}=a_r+\cdots+a_k,\quad d_{r,k}=a_r+b_k\text{ and }d_{r,k}=a_r\cdots a_k.$$

We also give some applications for the case when $d_{r,k}=a_r$, see section 3. Furthermore, if there exists a linear recurrence relation in few terms independent of r and k involving the sequence $(L_{r,k}(s))$, it can be served to determine the inverse matrix L^{-1} of L, see the last section. For any sequence (b_n) such that $b_n\neq 0$ for all $n\geq 1$, we note that other cases can be deduced and give similarly results on using the following lemma.

Lemma 1.2 *Let $[\mathcal{L}]_{r,k}=\frac{b_k}{b_r}[L]_{r,k}$. Then*

$$\left[\mathcal{L}^{s}\right]_{r,k}=\frac{b_k}{b_r}\left[L^{s}\right]_{r,k},\ s\geq 1.$$

2. Some properties of $[L^s]_{r,k}$ when $[L]_{r,k} = a_r \delta_{(r \le k)}$

Let $(a_j; j \ge 1)$ be a sequence of real numbers with $a_j \ne 0$, $j \ge 1$, and let

$$U_{r,k}(s) := \begin{cases} \delta_{(k-r)} & \text{if } s = 0, \\ a_r \delta_{(r \le k)} & \text{if } s = 1, \\ a_r \sum_{r \le j_1 \le \cdots \le j_{s-1} \le k} a_{j_1} \cdots a_{j_{s-1}} & \text{if } s \ge 2. \end{cases} \tag{2.1}$$

From the above, it follows

$$U_{r,k}(s) = [U^s]_{r,k} \quad \text{with} \quad [U]_{r,k} = a_r \delta_{(r \le k)}. \tag{2.2}$$

The (r,k) entry of U^{-1} is

$$\left[U^{-1}\right]_{r,k} = \frac{1}{a_r} \delta_{(k-r)} - \frac{1}{a_{r+1}} \delta_{(k-r-1)} \tag{2.3}$$

and we set $U^{-s} := \left(U^{-1}\right)^s$ and

$$U_{r,k}(-s) := \left[U^{-s}\right]_{r,k}, \quad s \ge 0. \tag{2.4}$$

From (1.6) we deduce the following corollary.

Corollary 2.1 *For $k \ge r \ge 1$ and $s, t \in \mathbb{Z}$, we get*

$$U_{r,k}(s+1) = \sum_{j=r}^{k} a_j U_{r,j}(s) \quad \text{and} \quad U_{r,k}(s+1) = a_r \sum_{j=r}^{k} U_{j,k}(s). \tag{2.5}$$

Corollary 2.2 *For $s \in \mathbb{Z}$, the sequence $(U_r(k,s))$ satisfies*

$$U_{r,k}(s-1) = \frac{1}{a_k}\left(U_{r,k}(s) - U_{r,k-1}(s)\right), \quad k-1 \ge r \ge 1, \tag{2.6}$$

$$U_{r,k}(s-1) = \frac{U_{r,k}(s)}{a_r} - \frac{U_{r+1,k}(s)}{a_{r+1}}, \quad k \ge r \ge 1. \tag{2.7}$$

Furtheremore, for $s \ge 1$, if $a_j > 0$ for all $j \ge 1$, we have

$$U_{r,k+1}(s) \ge U_{r,k}(s) \quad \text{and} \quad \frac{U_{r,k}(s)}{a_r} \ge \frac{U_{r+1,k}(s)}{a_{r+1}}, \quad k \ge r \ge 1. \tag{2.8}$$

Proof: From Corollary 2.1 it follows

$$\begin{aligned} U_{r,k}(s) &= \sum_{j=r}^{k-1} a_j U_{r,j}(s-1) + a_k U_{r,k}(s-1) \\ &= U_{r,k-1}(s) + a_k U_{r,k}(s-1), \\ U_{r,k}(s) &= a_r \sum_{j=r+1}^{k} U_{j,k}(s-1) + a_r U_{r,k}(s-1) \\ &= \frac{a_r}{a_{r+1}} U_{r+1,k}(s) + a_r U_{r,k}(s-1), \end{aligned}$$

from which follow the relations (2.6) and (2.7). For $s \ge 1$, if $a_j > 0$ for all $j \ge 1$, it follows that $U_{r,k}(s-1) \ge 0$. This shows, using (2.6) and (2.7), the two inequalities. □

From Corollary 2.2 we may state the following corollary.

Corollary 2.3 *For $s \ge 1$ and $a_j > 0$ for all $j \ge 1$, if the sequence $(a_j; j \ge 1)$ increase, then the sequences*

$$\left(\frac{U_{r,k}(s)}{a_r}; r \ge 1\right), \quad (U_{r,k}(s);\ k \ge r)$$

are convex sequences.

For a given sequence of real numbers $\mathbf{b}=(b_j;j\geq 1)$, let $(h_s;s\geq 0)$ be the complete symmetric function defined by

$$h_0(b_1,\ldots,b_k)=1,\quad h_s(b_1,\ldots,b_k)=\sum_{1\leq j_1\leq\cdots\leq j_s\leq k} b_{j_1}\cdots b_{j_s},\qquad s\geq 1, \tag{2.9}$$

and have ordinary generating function to be

$$\sum_{s\geq 0}h_s(b_1,\ldots,b_k)\,t^s=\prod_{j=1}^{k}(1-b_jt)^{-1}. \tag{2.10}$$

Many properties of these numbers have been discussed in combinatorics. Some particular cases have combinatorial interpretations related to the partitions of a set or a multiset. Between these numbers, we find the binomial coefficients, the q-binomial coefficients, the Stirling, r-Stirling [2], the r-Jacobi-Stirling [6, Sec. 1.4], p-Stirling numbers of the second kind [12] and the numbers studied in [9].

Proposition 2.1 *For $k\geq r\geq 1$ we have*

$$U_{r,k}(s) = a_r h_{s-1}(a_r,\ldots,a_k)\,\delta_{(r\leq k)},\quad s\geq 1, \tag{2.11}$$

$$U_{r,k}(-s) = (-1)^{k-r}\frac{a_r}{a_r\cdots a_k}h_{s+r-k}\left(a_r^{-1},\ldots,a_k^{-1}\right)\delta_{(k-r\leq s)}. \tag{2.12}$$

Proof: From the definition of the sequence $(U_{r,k}(s);\ s\geq 0)$ it follows

$$\sum_{s\geq 0}U_{r,k}(s)\,t^s=\delta_{(k-r)}+\frac{a_rt}{(1-a_rt)\cdots(1-a_kt)}$$

and this gives (2.11). For (2.12), one can verify for $k\geq r\geq 1$ that we have

$$\begin{aligned}
\sum_{s\geq 0}U_{r,k}(-s)\,t^s &= \sum_{s\geq 0}\left[U^{-s}\right]_{r,k}t^s\\
&= \left[\left(I-U^{-1}t\right)^{-1}\right]_{r,k}\\
&= -\frac{a_rt^{k-r}}{(t-a_k)\cdots(t-a_r)}\\
&= (-1)^{k-r}\frac{a_rt^{k-r}}{a_k\cdots a_r}\frac{1}{\left(1-a_r^{-1}t\right)\cdots\left(1-a_k^{-1}t\right)}\\
&= (-1)^{k-r}\frac{a_rt^{k-r}}{a_k\cdots a_r}\sum_{s\geq 0}h_s\left(a_r^{-1},\ldots,a_k^{-1}\right)t^s\\
&= (-1)^{k-r}\frac{a_r}{a_k\cdots a_r}\sum_{s\geq k-r}h_{s+r-k}\left(a_r^{-1},\ldots,a_k^{-1}\right)t^s,
\end{aligned}$$

from which follows the identity (2.12). □

3. Applications to generalized Stirling numbers

We present in this section the following particular cases

$$a_k=k,\quad a_k=1+q+\cdots+q^{k-1}\ \text{ and }\ a_k=k\,(k+z)\,.$$

3.1. Case $a_k = k$

For $a_k = k$ in Corollary 2.1 we get

$$\frac{U_{r,k}(s)}{r} = h_{s-1}(r, \ldots, k) = \left\{ \begin{matrix} k+s-1 \\ k \end{matrix} \right\}_r, \quad k \geq r \geq 1, \; s \geq 1,$$

where $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r$ is the (n,k)-th r-Stirling numbers of the second kind [2, Th. 8].
So, these numbers can be interpreted as

$$\left\{ \begin{matrix} k+s \\ k \end{matrix} \right\}_r = \frac{1}{r} \left[U^{s+1} \right]_{r,k}, \quad \text{with} \quad [U]_{r,k} = r\delta_{(r \leq k)}, \quad k \geq r \geq 1, \; s \geq 0.$$

For $k \geq r \geq 1$, $s \geq 0$, Corollaries 2.1 and 2.2 state that

$$\begin{aligned} \left\{ \begin{matrix} k+s+1 \\ k \end{matrix} \right\}_r &= \sum_{j=r}^{k} j \left\{ \begin{matrix} j+s \\ j \end{matrix} \right\}_r, \\ \left\{ \begin{matrix} k+s+1 \\ k \end{matrix} \right\}_r &= \sum_{j=r}^{k} j \left\{ \begin{matrix} k+s \\ k \end{matrix} \right\}_j, \\ \left\{ \begin{matrix} k+s+1 \\ k \end{matrix} \right\}_r &= \left\{ \begin{matrix} k+s \\ k-1 \end{matrix} \right\}_r + k \left\{ \begin{matrix} k+s \\ k \end{matrix} \right\}_r, \\ \left\{ \begin{matrix} k+s+1 \\ k \end{matrix} \right\}_{r+1} &= \left\{ \begin{matrix} k+s+1 \\ k \end{matrix} \right\}_r - r \left\{ \begin{matrix} k+s \\ k \end{matrix} \right\}_r. \end{aligned}$$

Also, from Corollary 2.3 and Lemma 1.1, it follows that, for $s \geq 0$, the sequences

$$\left(\left\{ \begin{matrix} k+s \\ k \end{matrix} \right\}_r ; r \geq 1 \right) \quad \text{and} \quad \left(\left\{ \begin{matrix} k+s \\ k \end{matrix} \right\}_r ; k \geq r \right)$$

are convex, and the sequence $\left(\left\{ \begin{smallmatrix} k+s \\ k \end{smallmatrix} \right\}_r ; s \geq 0 \right)$ is an infinite convex sequence.

3.2. Case $a_k = 1 + q + \cdots + q^{k-1}$

For $a_k = [k]_q := 1 + q + \cdots + q^{k-1}$ in Corollary 2.1 we get

$$\frac{U_{r,k}(s)}{[r]_q} = h_{s-1}\left([r]_q, \ldots, [k]_q \right), \quad k \geq r \geq 1, \; s \geq 1.$$

From the definition of the q-Stirling numbers of the second kind [7]

$$S_q(k+s, k) = h_s\left([1]_q, \ldots, [k]_q \right)$$

and we define a q-analogue to the r-Stirling numbers of the second kind by

$$S_{r,q}(k+s, k) = h_s\left([r]_q, \ldots, [k]_q \right).$$

This definition shows that $\frac{U_{r,k}(s)}{[r]_q} = S_{r,q}(k+s-1, k)$. These numbers can be interpreted as

$$S_{r,q}(k+s, k) = \frac{1}{[r]_q} \left[U^{s+1} \right]_{r,k}, \text{ with } [U]_{r,k} = [r]_q \, \delta_{(r \leq k)}, \quad k \geq r \geq 1, \; s \geq 0.$$

For $k\geq r\geq 1,\ s\geq 0$, Corollaries 2.1 and 2.2, state that

$$\begin{aligned} S_{r,q}(k+s+1,k) &= \sum_{j=r}^{k}[j]_q\, S_{r,q}(j+s,j),\\ S_{r,q}(k+s+1,k) &= \sum_{j=r}^{k}[j]_q\, S_{j,q}(k+s,k),\\ S_{r,q}(k+s+1,k) &= S_{r,q}(k+s,k-1)+[k]_q\, S_{r,q}(k+s,k),\\ S_{r+1,q}(k+s+1,k) &= S_{r,q}(k+s+1,k)-[k]_q\, S_{r,q}(k+s,k). \end{aligned}$$

Also, from Corollary 2.3 and Lemma 1.1, it follows that, for $s\geq 0$ and $q>0$, the sequences

$$\left(S_{r,q}(k+s,k)\,;r\geq 1\right)\quad\text{and}\quad\left(S_{r,q}(k+s,k)\,;k\geq r\right)$$

are convex, and the sequence $\left(S_{r,q}(k+s,k)\,;s\geq 0\right)$ is an infinite convex sequence.

3.3. Case $a_k=k(k+z)$

For $a_k=k(k+z)$ in Corollary 2.1 we get

$$\frac{U_{r,k}(s)}{r(r+z)}=h_{s-1}\left(r(r+z),\ldots,k(k+z)\right)=\left\{ {k+s-1 \atop k} \right\}_{r,z},\quad k\geq r\geq 1,\ s\geq 1,$$

where $\left\{ {n \atop k} \right\}_{r,z}$ is the (n,k)-th r-Jacobi-Stirling numbers of the second kind, see [5,6,8,9]. These numbers can be interpreted as

$$\left\{ {k+s \atop k} \right\}_{r,z}=\frac{1}{r(r+z)}\left[U^{s+1}\right]_{r,k},\ \text{with}\ [U]_{r,k}=r(r+z)\,\delta_{(r\leq k)},\ k\geq r\geq 1,\ s\geq 0.$$

These numbers satisfy for $k\geq r\geq 1,\ s\geq 0$ the relations

$$\begin{aligned} \left\{ {k+s+1 \atop k} \right\}_{r,z} &= \sum_{j=r}^{k} j(j+z)\left\{ {j+s \atop j} \right\}_{r,z},\\ \left\{ {k+s+1 \atop k} \right\}_{r,z} &= \sum_{j=r}^{k} j(j+z)\left\{ {k+s \atop k} \right\}_{j,z},\\ \left\{ {k+s+1 \atop k} \right\}_{r,z} &= \left\{ {k+s \atop k-1} \right\}_{r,z}+k(k+z)\left\{ {k+s \atop k} \right\}_{r,z},\\ \left\{ {k+s+1 \atop k} \right\}_{r+1,z} &= \left\{ {k+s+1 \atop k} \right\}_{r,z}-r(r+z)\left\{ {k+s \atop k} \right\}_{r,z}. \end{aligned}$$

Also, from Corollary 2.3 and Lemma 1.1, it follows that, for $s\geq 0$, if $z\geq -2$, the sequences

$$\left(\left\{ {k+s \atop k} \right\}_{r,z};r\geq 1\right)\quad\text{and}\quad\left(\left\{ {k+s \atop k} \right\}_{r,z};k\geq r\right)$$

are convex, and for $z\geq 0$, the sequence $\left(\left\{ {k+s \atop k} \right\}_{r,z};s\geq 0\right)$ is an infinite convex sequence.

Remark 3.1 *One can also obtain similar relations as the above cases for the case $a_k=q^k$ on the q-binomial coefficients, see [1,10], and for the case $a_k=k^p$ on the p-Stirling numbers, see [12].*

4. Some properties of $[L^s]_{r,k}$ when $[L]_{r,k} = a_r \cdots a_k \delta_{(r \le k)}$

Let $(a_j ; j \ge 1)$ be a sequence of real numbers with $a_j \ne 0$, $j \ge 1$, and let $R_{r,k}(s)$ be the sequence of complete symmetric functions defined by

$$R_{r,k}(s) := \begin{cases} \delta_{(k-r)} & \text{if } s = 0, \\ a_r \cdots a_k \delta_{(r \le k)} & \text{if } s = 1, \\ \sum_{r \le j_1 \le \cdots \le j_{s-1} \le k} (a_r \cdots a_{j_1}) \cdots (a_{j_{s-1}} \cdots a_k) & \text{if } s \ge 2. \end{cases} \tag{4.1}$$

From the first section, it follows

$$R_{r,k}(s) = [R^s]_{r,k} \quad \text{with} \quad [R]_{r,k} = (a_r \cdots a_k)\, \delta_{(r \le k)}. \tag{4.2}$$

The (r,k) entry of R^{-1} is

$$\left[R^{-1}\right]_{r,k} = \frac{1}{a_r} \delta_{(k-r)} - \delta_{(k-r-1)} \tag{4.3}$$

and we set $R^{-s} := \left(R^{-1}\right)^s$ and let

$$R_{r,k}(-s) := \left[R^{-s}\right]_{r,k}, \quad s \ge 0. \tag{4.4}$$

Now, we can deduce some recurrence relations with proofs based only on easy matrices's product. From (1.6) we deduce the following corollary.

Corollary 4.1 *Then, for $k \ge r \ge 1$ and $s \in \mathbb{Z}$, we get*

$$R_{r,k}(s+1) = \sum_{j=r}^{k} a_j \cdots a_k R_{r,j}(s) \quad \textit{and} \quad R_{r,k}(s+1) = \sum_{j=r}^{k} a_r \cdots a_j R_{j,k}(s). \tag{4.5}$$

Corollary 4.2 *For $s \in \mathbb{Z}$, we have*

$$R_{r,k}(s+1) - a_k R_{r,k-1}(s+1) = a_k R_{r,k}(s), \tag{4.6}$$

$$R_{r,k}(s+1) - a_r R_{r+1,k}(s+1) = a_r R_{r,k}(s). \tag{4.7}$$

Furthermore, for $s \ge 1$, if $a_j > 0$ for all $j \ge 1$, we have

$$R_{r,k+1}(s) \ge a_k R_{r,k}(s) \quad \textit{and} \quad R_{r+1,k}(s) \ge \frac{1}{a_r} R_{r,k}(s), \quad k \ge r \ge 1. \tag{4.8}$$

Proof: Let $s, t \in \mathbb{Z}$. From Corollary 4.1 it follows

$$\begin{aligned} R_{r,k}(s+1) &= \sum_{j=r}^{k} a_j \cdots a_k R_{r,j}(s) \\ &= a_k \sum_{j=r}^{k-1} a_j a_{j+1} \cdots a_{k-1} R_{r,j}(s) + a_k R_{r,k}(s) \\ &= a_k R_{r,k-1}(s+1) + a_k R_{r,k}(s) \quad \text{and} \\ R_{r,k}(s+1) &= \sum_{j=r}^{k} a_r a_{r+1} \cdots a_j R_{j,k}(s) \\ &= a_r \sum_{j=r+1}^{k} a_{r+1} \cdots a_j R_{j,k}(s) + a_r R_{r,k}(s) \\ &= a_r R_{r+1,k}(s+1) + a_r R_{r,k}(s). \end{aligned}$$

For $s \ge 1$, if $a_j > 0$ for all $j \ge 1$, it follows that $R_{r,k}(s) \ge 0$ which gives, using (4.6) and (4.7), the inequalities (4.8). □

Proposition 4.1 *For $k \geq r \geq 1$ we have*

$$R_{r,k}(s) = \frac{a_r \cdots a_k}{a_r} U_{r,k}(s), \quad s \geq 1, \tag{4.9}$$

$$R_{r,k}(-s) = \frac{a_r}{a_k \cdots a_r} U_{r,k}(-s), \quad s \geq 0. \tag{4.10}$$

Proof: For (4.9), since

$$R_{r,k}(1) = a_r \cdots a_k \delta_{(k-r)} = \frac{a_1 \cdots a_k}{a_1 \cdots a_r} a_r \delta_{(k-r)} = \frac{b_k}{b_r} U_{r,k}(1), \quad \text{with } b_k = a_1 \cdots a_k,$$

then, it follows from Lemma 1.2 that $R_{r,k}(s) = \frac{b_k}{b_r} U_{r,k}(s) = \frac{a_r \cdots a_k}{a_r} U_{r,k}(s)$.
For (4.10), by the generating function of the sequence $(R_{r,k}(-s);\ s \geq 0)$, one can verify for $k \geq r \geq 1$ that we have

$$\begin{aligned}
\sum_{s \geq 0} R_{r,k}(-s) t^s &= \left[\left(I - R^{-1} t \right)^{-1} \right]_{r,k} \\
&= -\frac{a_r \cdots a_k t^{k-r}}{(t - a_r) \cdots (t - a_k)} \\
&= (-1)^{k-r} \frac{t^{k-r}}{\left(1 - \frac{t}{a_r}\right) \cdots \left(1 - \frac{t}{a_k}\right)} \\
&= (-1)^{k-r} \sum_{s \geq k-r} h_{s+r-k}\left(a_r^{-1}, \ldots, a_k^{-1}\right) t^s.
\end{aligned}$$

So, use (2.11) and (2.12) to complete the proof. □

5. Some properties of $[L^s]_{r,k}$ when $[L]_{r,k} = \delta_{(r \leq k)} \sum_{j=r}^{k} a_j$

Let $(a_j; j \geq 1)$ be a sequence of real numbers with $a_j \neq 0$, $j \geq 1$, and let $V_r(k, s)$ be the sequence of complete symmetric functions defined by

$$V_{r,k}(s) := \begin{cases} \delta_{(k-r)} & \text{if } s = 0, \\ (a_r + \cdots + a_k) \delta_{(r \leq k)} & \text{if } s = 1, \\ \sum\limits_{r \leq j_1 \leq \cdots \leq j_{s-1} \leq k} (a_r + \cdots + a_{j_1}) \cdots (a_{j_{s-1}} + \cdots + a_k) & \text{if } s \geq 2. \end{cases} \tag{5.1}$$

From the first section, it follows

$$V_{r,k}(s) = [V^s]_{r,k} \quad \text{with} \quad [V]_{r,k} = (a_r + \cdots + a_k) \delta_{(r \leq k)}. \tag{5.2}$$

The (r, k) entry of V^{-1} is

$$\left[V^{-1}\right]_{r,k} = \frac{\delta_{(k-r)}}{a_r} - \left(\frac{1}{a_r} + \frac{1}{a_{r+1}} \right) \delta_{(k-r-1)} + \frac{\delta_{(k-r-2)}}{a_{r+1}} \tag{5.3}$$

and we set $V^{-s} := \left(V^{-1}\right)^s$ and let

$$V_{r,k}(-s) := \left[V^{-s}\right]_{r,k}, \quad s \geq 0. \tag{5.4}$$

Now, we can deduce some recurrence relations with proofs based only on easy matrices's product. From (1.6) we deduce the following corollary.

Corollary 5.1 *Then, for $k \geq r \geq 1$ and $s \in \mathbb{Z}$, we get*

$$V_{r,k}(s+1) = \sum_{j=r}^{k} (a_j + \cdots + a_k) V_{r,j}(s) \text{ and} \tag{5.5}$$

$$V_{r,k}(s+1) = \sum_{j=r}^{k} (a_r + \cdots + a_j) V_{j,k}(s). \tag{5.6}$$

Corollary 5.2 *Let $s \in \mathbb{Z}$ and let*

$$V^{*}_{r,k}(s) = \frac{1}{a_k}(V_{r,k}(s) - V_{r,k-1}(s)) \quad \text{and} \quad V^{**}_{r,k}(s) = \frac{1}{a_r}(V_{r+1,k}(s) - V_{r,k}(s)). \tag{5.7}$$

Then, we have

$$V^{*}_{r,k+1}(s) - V^{*}_{r,k}(s) = V_{r,k+1}(s-1), \tag{5.8}$$

$$V^{**}_{r+1,k}(s) - V^{**}_{r,k}(s) = V_{r,k}(s-1). \tag{5.9}$$

Furthermore, for $s \geq 1$, if $a_j > 0$ for all $j \geq 1$, we have

$$V^{*}_{r,k+1}(s) \geq V^{*}_{r,k}(s) \quad \text{and} \quad V^{**}_{r+1,k}(s) \geq V^{**}_{r,k}(s), \quad k \geq r \geq 1. \tag{5.10}$$

Proof: Let $s \in \mathbb{Z}$. From (1.5) we get

$$V_{r,k}(s+t) = \sum_{j=r}^{k} V_{r,j}(t) V_{j,k}(s). \tag{5.11}$$

Then, for $s = -1$ in (5.11), it follows from Corollary 5.1 that

$$\begin{aligned} V_{r,k+1}(s-1) &= \sum_{j=r}^{k+1} V_{r,j}(s) V_{j,k+1}(-1) \\ &= \frac{1}{a_{k+1}} V_{r,k+1}(s) - \left(\frac{1}{a_k} + \frac{1}{a_{k+1}}\right) V_{r,k}(s) + \frac{1}{a_k} V_{r,k-1}(s), \end{aligned}$$

and for $t = -1$ it follows from Corollary 5.1 that

$$\begin{aligned} V_{r,k}(t-1) &= \sum_{j=r}^{k} V_{r,j}(-1) V_{j,k}(t) \\ &= \frac{1}{a_r} V_{r,k}(t) - \left(\frac{1}{a_r} + \frac{1}{a_{r+1}}\right) V_{r+1,k}(t) + \frac{1}{a_{r+1}} V_{r+2,k}(t). \end{aligned}$$

For $s \geq 1$, if $a_j > 0$ for all $j \geq 1$, it follows that $V_{r,k}(s-1) \geq 0$ which gives (5.10). □

So, we may state the following corollary.

Corollary 5.3 *For $s \geq 1$, if $a_j > 0$ for all $j \geq 1$, and the sequence $(a_j; j \geq 1)$ increase, then the following sequences are convex,*

$$(V_{r,k}(s);\ k \geq 1), \quad (V_{r,k}(s);\ r \geq 1), \quad \left(V^{*}_{r,k}(s);\ k \geq 1\right), \ \left(V^{**}_{r,k}(s);\ r \geq 1\right).$$

Remark 5.1 *The case when $[L]_{r,k} = (a_r - a_{k+1}) \delta_{(r \leq k)}$, with $a_i \neq a_j$ for all $i \neq j$, we can write*

$$a_r - a_{k+1} = (a_r - a_{r+1}) + (a_{r+1} - a_{r+2}) + \cdots + (a_k - a_{k+1}),$$

and by setting $a_j - a_{j+1} := b_j$, we find the above case $[L]_{r,k} = \delta_{(r \leq k)} \sum_{j=r}^{k} b_j$.

6. Some properties of $[L^s]_{r,k}$ when $[L]_{r,k} = (a_r + b_k)\,\delta_{(r\leq k)}$

Let $\mathbf{a} = (a_j)_{j\geq 1}$ and $\mathbf{b} = (b_j)_{j\geq 1}$ be two sequences of real numbers and let $L_r(k,s)$ be the sequence of complete symmetric functions defined by

$$T_{r,k}(s) := \begin{cases} \delta_{(k-r)} & \text{if } s = 0, \\ (a_r + b_k)\,\delta_{(r\leq k)} & \text{if } s = 1, \\ \sum\limits_{r\leq j_1\leq \cdots \leq j_{s-1}\leq k} (a_r + b_{j_1}) \cdots (a_{j_{s-1}} + b_k) & \text{if } s \geq 2. \end{cases} \tag{6.1}$$

From the first section, it follows

$$T_{r,k}(s) = [T^s]_{r,k} \quad \text{with} \quad [T]_{r,k} = (a_r + b_k)\,\delta_{(r\leq k)}. \tag{6.2}$$

Upon using the above results, we can now deduce some recurrence relations with proofs based only on easy matrices's product.

Lemma 6.1 *Let $a_i + b_j \neq 0$, $1 \leq i \leq j$. For $k \geq r \geq 1$, $s \in \mathbb{Z}$, there hold*

$$T_{r,k}(s) = T_{r,k-1}(s) + (a_k + b_k)\,T_{r,k}(s-1) + (b_k - b_{k-1}) \sum_{j=r}^{k-1} T_{r,j}(s-1), \tag{6.3}$$

$$T_{r+1,k}(s) = T_{r,k}(s) - (a_{r+1} + b_r)\,T_{r,k}(s-1) + (a_{r+1} - a_r) \sum_{j=r}^{k} T_{j,k}(s-1). \tag{6.4}$$

Proof: From (1.6), we get

$$\begin{aligned} T_{r,k}(s) &= \sum_{j=r}^{k} (a_j + b_k)\,T_{r,j}(s-1) \\ &= \sum_{j=r}^{k-1} (a_j + b_{k-1})\,T_{r,j}(s-1) + (b_k - b_{k-1}) \sum_{j=r}^{k-1} T_{r,j}(s-1) \\ &\quad + (a_k + b_k)\,T_{r,k}(s-1) \\ &= T_{r,k-1}(s) + (a_k + b_k)\,T_{r,k}(s-1) + (b_k - b_{k-1}) \sum_{j=r}^{k-1} T_{r,j}(s-1), \\ T_{r+1,k}(s) &= \sum_{j=r+1}^{k} (a_{r+1} + b_j)\,T_{j,k}(s-1) \\ &= a_{r+1} \sum_{j=r}^{k} T_{j,k}(s-1) + \sum_{j=r}^{k} b_j T_{j,k}(s-1) - (a_{r+1} + b_r)\,T_{r,k}(s-1) \\ &= (a_{r+1} - a_r) \sum_{j=r}^{k} T_{j,k}(s-1) + \sum_{j=r}^{k} (a_r + b_j)\,T_{j,k}(s-1) \\ &\quad - (a_{r+1} + b_r)\,T_{r,k}(s-1) \\ &= T_{r,k}(s) - (a_{r+1} + b_r)\,T_{r,k}(s-1) + (a_{r+1} - a_r) \sum_{j=r}^{k} T_{j,k}(s-1). \end{aligned}$$

□

The relations of the last Corollary prove that the numbers $T_{r,k}(s)$ satisfy two recurrence relations given by the following corollary.

Corollary 6.1 *Let* $(a_i+b_j)(a_{i+1}-a_i)(b_{j+1}-b_j)\neq 0,\ 1\leq i\leq j.$
Then, for $k\geq r\geq 1$ *and* $s\in\mathbb{Z}$ *there hold*

$$\begin{aligned} T^{*}_{r,k+1}(s) &= T^{*}_{r,k}(s)+T_{r,k}(s-1), &(6.5)\\ T^{**}_{r+1,k}(s) &= T^{**}_{r,k}(s)+T_{r,k}(s-1), &(6.6)\end{aligned}$$

where

$$\begin{aligned}(b_k-b_{k-1})\,T^{*}_{r,k}(s) &:= T_{r,k}(s)-T_{r,k-1}(s)-(a_k+b_k)\,T_{r,k}(s-1),\\ (a_r-a_{r-1})\,T^{**}_{r,k}(s) &:= T_{r,k}(s)-T_{r-1,k}(s)+(a_r+b_{r-1})\,T_{r-1,k}(s-1).\end{aligned}$$

Furthermore, for $s\geq 1$*, if* $a_i+b_j>0$ *for all* $i,j\geq 1$*, we have*

$$T^{*}_{r,k+1}(s)\geq T^{*}_{r,k}(s)\quad\text{and}\quad T^{**}_{r+1,k}(s)\geq T^{**}_{r,k}(s),\qquad k\geq r\geq 1. \tag{6.7}$$

Proof: From (6.3) and (6.4) we have
$(b_k-b_{k-1})\sum_{j=r}^{k-1}T_{r,j}(s-1)=T_{r,k}(s)-T_{r,k-1}(s)-(a_k+b_k)\,T_{r,k}(s-1)$ and
$(a_{r+1}-a_r)\sum_{j=r}^{k}T_{j,k}(s-1)=T_{r+1,k}(s)-T_{r,k}(s)+(a_{r+1}+b_r)\,T_{r,k}(s-1).$
The desired relations follow by using the last expressions in the equalities
$\sum_{j=r}^{k}T_{r,j}(s-1)=\sum_{j=r}^{k-1}T_{r,j}(s-1)+T_{r,k}(s-1)$ and
$\sum_{j=r}^{k}T_{j,k}(s-1)=\sum_{j=r+1}^{k}T_{j,k}(s-1)+T_{r,k}(s-1).$
For $s\geq 1$, if $a_i+b_j>0$ for all $i,j\geq 1$, it follows that $T_{r,k}(s-1)\geq 0$ which gives (6.7). □

So, we may state the following corollary.

Corollary 6.2 *For* $s\geq 1$*, if* $a_j>0$ *for all* $j\geq 1$*, and the sequence* $(a_j;j\geq 1)$ *increase, then the following sequences are convex,*

$$(T_{r,k}(s);\ k\geq 1),\quad (T_{r,k}(s);\ r\geq 1),\quad \left(T^{*}_{r,k}(s);\ k\geq 1\right),\ \left(T^{**}_{r,k}(s);\ r\geq 1\right).$$

The last relations can be used to determine the inverse matrix of T as follows.

Corollary 6.3 *Let* $(a_i+b_j)(a_{i+1}-a_i)(b_{j+1}-b_j)\neq 0,\ 1\leq i\leq j.$
For $k\geq r\geq 1$*, there holds*

$$\left[T^{-1}\right]_{r,k}=T_{r,k}(-1)=\begin{cases}\frac{1}{a_r+b_r} & k=r,\\ -\frac{(a_r+b_{r+1})}{(a_r+b_r)(a_{r+1}+b_{r+1})} & k=r+1,\\ -\frac{(b_{r+2}-b_{r+1})(a_{r+1}-a_r)}{(a_r+b_r)(a_{r+1}+b_{r+1})(a_{r+2}+b_{r+2})} & k=r+2,\\ \frac{b_k-b_{k-1}}{(a_r+b_r)(b_r-b_{r-1})}\prod_{j=r}^{k-1}\frac{a_j+b_{j-1}}{a_{j+1}+b_{j+1}} & k\geq r+3.\end{cases}$$

Proof: By the fact that $T_{r,k}(0)=\delta_{(k-r)}$, then for $s=0$ in the relation (6.5) we get

$$\begin{aligned}T_{r,r+1}(-1) &= \frac{(b_{r+1}-b_k)(a_r+b_{r-1})\,T_r(r,-1)-(b_{r+1}-b_{r-1})}{(b_r-b_{r-1})(a_{r+1}+b_{r+1})},\\ T_{r,r+2}(-1) &= \frac{(b_{r+2}-b_{r+1})(a_{r+1}+b_r)\,T_r(r+1,-1)+(b_{r+2}-b_{r+1})}{(b_{r+1}-b_r)(a_{r+2}+b_{r+2})},\\ T_{r,k+1}(-1) &= \frac{(b_{k+1}-b_k)(a_k+b_{k-1})\,T_r(k,-1)}{(b_k-b_{k-1})(a_{k+1}+b_{k+1})},\quad k\geq r+2.\end{aligned}$$

So, to obtain $T_{r,r+1}(-1)$ set $k=r$ and use $T_{r,r}(-1)=1/(a_r+b_r)$, to obtain $T_{r,r+2}(-1)$ set $k=r+1$ and use $T_{r,r+1}(-1)$, and for $k\geq r+2$, the last recurrence relation gives the desired expression. □

References

1. G.E. Andrews, *Theory of Partitions*, Addison-Wesley, Reading, MA, 1976.
2. A.Z. Broder, The r-Stirling numbers, Discrete Math., 49, 241–259, (1984).
3. R.A. Brualdi, The symbiotic relationship of combinatorics and matrix theory. Linear algebra and its applications, 162, 65-105, (1992).
4. J. Engbers, D. Galvin, and C. Smyth, Restricted Stirling and Lah number matrices and their inverses. J. Combin. Theory Ser. A, 161, 271-298, (2019).
5. W.N. Everitt, K.H. Kwon, L.L. Littlejohn, R. Wellman, and G.J. Yoon, Jacobi-Stirling numbers, Jacobi polynomials, and the left-definite analysis of the classical Jacobi differential expression. J. Comput. Appl. Math., 208, 29–56, (2007).
6. Y. Gelineau, Etude combinatoires des nombres de Jacobi-Stirling et d'Entringer, Doctorat thesis, University of Claude Bernard-Lyon 1 (2010).
7. H.W. Gould, The q-Stirling numbers of first and second kinds, Duke Math. J., 28(2), 281–289, (1961).
8. M. Mihoubi, and A. Rahim, The r-Jacobi-Stirling of the second kind, Miskolc Math. Notes, 17(2), 947–955, (2016).
9. P. Mongelli, Combinatorial interpretations of particular evaluations of complete and elementary symmetric functions, Electron. J. Comb., 19, #P60, (2012).
10. H. Rademacher, *Lectures on Elementary Number Theory*, Blaisdell, New York, 1964.
11. M.Z. Spivey, and A.M. Zimmer, Symmetric polynomials, Pascal matrices, and Stirling matrices., Linear Algebra Appl., 428, 1127–1134, (2008).
12. Y. Sun, Two classes of p-Stirling numbers, Discrete Math., 306, 2801–2805, (2006).
13. Y.-Z. Yang, and C. Micek, Generalized Pascal functional matrix and its applications, Linear Algebra Appl., 423, 230–245, (2007).

Imène Touaibia,
RECITS Laboratory,
Faculty of Mathematics,
University of Sciences and Technology Houari Boumediene (USTHB),
P.O. Box 32, El Alia, 16111 Algiers, Algeria.
E-mail address: itouaibia@usthb.dz, ttouaibia@outlook.com

and

Miloud Mihoubi,
RECITS Laboratory,
Faculty of Mathematics,
University of Sciences and Technology Houari Boumediene (USTHB),
P.O. Box 32, El Alia, 16111 Algiers, Algeria.
E-mail address: mmihoubi@usthb.dz, miloudmihoubi@gmail.com