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Capacity solution for a nonlocal Thermistor problem in Solobev Spaces

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ABSTRACT: In this work, we consider a general case of the thermistor problem, in which three nonlinear terms are added to the parabolic equation governing the model. The first and second nonlinearities are present in the source term. While the third nonlinearity is considered in the evolution term. We prove the existence of a weak solution via the Schauder fixed point theorem. Consequently, we establish the existence of a capacity solution to the considered problem in Sobolev spaces.

Key Words: Existence, capacity solution, nonlinear parabolic equation, Thermistor problem, Sobolev spaces.

Contents

1. Introduction

A thermistor may be found in a broad variety of applications that help form our modern environment. Thermistors have been around since the nineteenth century, during the conclusion of the industrial revolution. Michael Faraday, a British scientist and chemist, is credited with inventing the first NTC thermistor. Faraday is today one of the most recognized physicists due to his contributions in electrochemistry and electromagnetic induction and the invention of the thermistor. He provided a paper on the behaviour (semiconducting) of Ag2S (silver sulfide) in 1833. There are two types of thermistor: PTC and NTC, which are Positive and Negative Temperature Coefficient, respectively.

A thermistor is a circuit component used as a current limiter or as a temperature sensor. It's usually a tiny cylinder made of a ceramic substance whose electrical conductivity is highly dependent on temperature and is often employed in electronic circuits for temperature protection, control, and compensation. On the other hand, the "Thermistor Problem" is a mathematical model consisting of a nonlinear parabolic equation for the temperature coupled with an elliptic equation for the quasi-static development of the electric potential [3,27,11]. The model is expressed as follows:

$$\begin{cases} v_t = \nabla \cdot (\kappa(v)\nabla v) + \rho(v) \left|\nabla\psi\right|^2, \\ \nabla \cdot (\rho(v)\nabla\psi) = 0. \end{cases} \tag{1.1}$$

Where $\rho(v)$ represents the electrical conductivity, which is supposed to be a positive function for realistic reasons. The conductor's temperature is v, while the electrical potential is ψ. The thermistor equations regulate the heat created by an electrical current traveling through a conductor device. Precisely, the first equation depicts heat diffusion, whereas the second equation represents electrical charge conservation. The thermal conductivity and electrical conductivity are represented by the functions $\kappa(v)$ and σ respectively. Their exact shapes are defined by the specific physical purpose in mind. The coupling partially transforms into a parabolic system. with the boundary and initial conditions. For more detail

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see [21,29,10]. The following non-local model is considered as a generalization of the problem appearing in the paper of A. A. Lacey [23,24,30].

$$\begin{cases} \dfrac{\partial b(v)}{\partial t} - \Delta_p v = \dfrac{\lambda f(v)}{(\int_\Omega f(v)dx)^2}, & \text{in } Q, \\ v(.,0) = v_0, & \text{in } \Omega, \\ v = 0, & \text{on } \Gamma \times]0,M[, \end{cases} \tag{1.2}$$

where $f(v)$ is the electrical resistance of the conductor and $\dfrac{f(v)}{(\int_\Omega f(v)dx)^2}$ represents the non-local term of (1.2). $Q := \Omega \times [0,M]$ where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, Γ is the boundary of Ω, and M is a positive constant. In the former, electrical conductivity reduces as temperature rises, whereas in the latter, it rises in terms of the temperature. PTC thermistors can be found in a variety of applications, including switches and electric surge protection devices. The operation of a PTC electric surge device is as follows: When the circuit's current is suddenly increased, the device heats up, causing a dramatic decline in its electrical conductivity, effectively shutting off the circuit. When the surge has passed, the gadget cools down, its conductivity rises, and the circuit reverts to normal operation [5]. However, it was discovered that the quick temperature increase might create large thermal stresses, which can compromise the device's integrity resulting in fractures and device failure, for more details see [23,20]. Because of the nonlinearities present in the electrical conductivity, Joule heating, and viscous heating terms, the material constitutive behavior was believed to be linear. Knowing this, it is not surprising that many researchers are presently giving the research of thermistor problems a lot of attention [28,22,12,1,18,16,8,17,33,25,19].

The aim of this paper is to prove the existence of a capacity solution in the sense of Definition 2.1 to problem (1.2), which is a generalization of the problem appearing in [15]. Our motivation is stimulated by the presence of several applications. On the other hand, b is a nonlinear function that may grow faster than any function at infinity ($b(v) = e^{1+e^v}$, for example).

The rest of this paper is organized as follows. In Section 2, we start by introducing the concept of capacity solutions as well as some basic notations. While in Section 3, we develop the proof of the existence of a capacity solution. Finally, in Section 4, we give some conclusions and perspectives.

2. Preliminaries

Let B be the area of Ω and I the current such that $\lambda = I^2/B^2$ and Δ_p is defined by

$$\Delta_p v = \operatorname{div}(|\nabla v|^{p-2} \nabla v), \quad \forall p \geq 2.$$

We further specify the terms in (1.2). Throughout this paper, we assume that

$(H1)$ $v_0 \in L^\infty(\Omega)$.

$(H2)$ $b : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz continuous increasing function such that $b(0) = 0$, $b' \geq K$, where $K > 0$.

$(H3)$ f is a Lipshitz function and there exists a positive constant σ such that

$$\sigma \leq f(r), \text{ for all } r \in \mathbb{R}. \tag{2.1}$$

Now, we defined the following space where p' is the conjugate of p in the Hölder's sense

$$W = \left\{ v \in L^p\left(0,M; W_0^{1,p}(\Omega)\right) : \frac{\partial v}{\partial t} \in L^{p'}\left(0,M; W^{-1,p'}(\Omega)\right) \right\}.$$

We equip this space by the following norm

$$\|v\|_W = \|v\|_{L^p\left(0,M;W_0^{1,p}(\Omega)\right)} + \left\| \frac{\partial v}{\partial t} \right\|_{L^{p'}\left(0,M;W^{-1,p'}(\Omega)\right)}.$$

$(W, \|.\|_W)$ is a Banach space.

In the sequel, we will need the following two Lemmas:

Lemma 2.1 *(See [32]) Assume that θ is a non-negative, absolutely continuous function, satisfying the following inequality:*

$$\theta'(s) \leq h\theta(s) + g(s), \quad \text{for } \ s \geq s_0,$$

where h and g are two non-negative integrable functions on $[0, M]$. Then, for each $s \in [0, M]$,

$$\theta(s) \leq \exp\left(\int_0^s h(\tau)d\tau\right) \cdot \left[\theta(0) + \int_0^s g(\tau)d\tau\right].$$

Lemma 2.2 *(See [4]) Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. F $1 \leq p, q \leq \infty$, let*

$$W = \left\{ v \in L^p\left([0, M]; X_0\right) \mid \frac{\partial v}{\partial t} \in L^q\left([0, M]; X_1\right) \right\}.$$

(1) If $p < \infty$ then the embedding of W into $L^p([0, M]; X)$ is compact.

(2) If $p = \infty$ and $q > 1$ then the embedding of W into $C([0, M]; X)$ is compact.

Throughout this paper, we will use the standard reference [26] for the studies in Sobolev spaces. The concept of a capacity solution, which provides more regularity than other types of solutions, is now being discussed. The concept of a capacity solution was initially introduced by Xu [34].

Definition 2.1 v is called to be a capacity solution of the problem (1.2), if the following conditions are satisfied

- $v \in W$.
- v verify the following equation

$$\frac{\partial b(v)}{\partial t} - \Delta_p v = \frac{\lambda f(v)}{(\int_\Omega f(v)dx)^2}, \quad \text{in } \ Q.$$

- $v\left(.,0\right) = v_0 \in L^\infty\left(\Omega\right).$

3. An existence result

We devote this section to introducing the main theorem of this paper and proving the existence of a weak solution to problem (1.2).

Theorem 3.1 *Under the assumptions $(H1) - (H3)$, problem (1.2) admits a capacity solution.*

The following result shows the existence of weak solutions to problem (1.2). Which we will use in the proof of Theorem 3.1.

Theorem 3.2 *Under Hypotheses $(H1) - (H3)$, there exists a weak solution v for problem (1.2), that is*

$$v \in L^p\left(0, M; W_0^{1,p}(\Omega)\right), \quad \frac{\partial b(v)}{\partial t} \in L^{p'}\left(0, M; W^{-1,p'}(\Omega)\right).$$

$$v(\cdot, 0) = v_0, \quad \text{in } \ \Omega,$$

and

$$\int_0^M \langle \frac{\partial b(v)}{\partial t}, \psi \rangle dt - \int_Q |\nabla v|^{p-2} \nabla v \nabla \psi dt\ dx = \int_Q \frac{\lambda f(v)}{(\int_\Omega f(v)dx)^2} \psi dt\ dx, \tag{3.1}$$

for all $\psi \in L^p\left(0, M; W_0^{1,p}(\Omega)\right) \cap L^\infty(Q)$, for each $s \in [0, M]$. Where the duality product is define by $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{W^{-1,p'}(\Omega), W^{1,p}(\Omega)}$.

Proof: In order to get the existence of weak solution, we will apply Schauder's fixed point theorem [9]. Multiplying the first equation of the problem (1.2) by
$\psi \in L^p\left(0, M; W_0^{1,p}(\Omega)\right)$ and integrating, we get

$$\int_0^M \langle \frac{\partial b(v)}{\partial t}, \psi \rangle dt - \int_Q \Delta_p v \nabla \psi dt\ dx = \int_Q \frac{\lambda f(v)}{(\int_\Omega f(v) dx)^2} \psi dt\ dx. \tag{3.2}$$

From Green's formula and by using the boundary condition, we obtain

$$\int_0^M \int_\Omega \operatorname{div}(|\nabla v|^{p-2} \nabla v) \psi dt\ dx = -\int_0^M \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \psi dt\ dx. \tag{3.3}$$

We use (3.2) and (3.3), we obtain

$$\begin{cases} \int_0^M \langle \frac{\partial b(v)}{\partial t}, \psi \rangle dt + \int_0^M \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \psi dt\ dx = \lambda \int_Q \frac{s f(w)}{(\int_\Omega f(w) dx)^2} \psi dt\ dx, \\ \psi \in L^p\left(0, M; W_0^{1,p}(\Omega)\right), \\ v(\cdot, 0) = v_0 \in L^2(\Omega), \\ v \in L^p\left(0, M; W_0^{1,p}(\Omega)\right). \end{cases} \tag{3.4}$$

For a fixed $w \in L^p\left(0, M; W_0^{1,p}(\Omega)\right)$ and from [35], we get the existence of a bounded weak solution $v \in L^p\left(0, M; W_0^{1,p}(\Omega)\right)$ for (3.4) and there exists a positive constant C_0 such that

$$\int_0^M \int_\Omega |\nabla v|^p\ dt\ dx \leq C_0. \tag{3.5}$$

Now, let us prove that there exists a positive constant C_1 such that

$$\left\| \frac{\partial v}{\partial t} \right\|_{L^{p'}\left(0, M; W^{-1,p'}(\Omega)\right)} \leq C_1.$$

To this end, proceeding as in [14], multiplying the first equation in (1.2) by $\frac{\partial v}{\partial t}$ and using Green's formula, we get

$$\int_\Omega \frac{\partial b(v)}{\partial t} \frac{\partial v}{\partial t} dx + \frac{1}{p} \frac{\partial}{\partial t} \int_\Omega |\nabla v|^p\ dx = \int_\Omega \frac{\lambda f(v)}{\left(\int_\Omega f(v) dx\right)^2} \frac{\partial v}{\partial t} dx.$$

$$\int_\Omega b'(v) \left(\frac{\partial v}{\partial t}\right)^2 dx + \frac{1}{p} \frac{d}{dt} \int_\Omega |\nabla v|^p\ dx = \lambda \int_\Omega \frac{d}{dt} \int_0^v g(\tau) d\tau\ dx,$$

where $g(s) = \dfrac{f(s)}{\left(\int_\Omega f(s) dx\right)^2}$. In virtue of $b'(v) > K$

$$K \int_\Omega \left(\frac{\partial v}{\partial t}\right)^2 dx + \frac{1}{p} \cdot \frac{d}{dt} \int_\Omega |\nabla v|^p\ dx \leq \lambda \frac{d}{dt} \int_\Omega \int_0^v g(\tau) d\tau\ dx. \tag{3.6}$$

Putting $\int_0^\tau g(r) dr := G(\tau)$ and integrating (3.6), we get

$$K \int_0^M \int_\Omega \left(\frac{\partial v}{\partial t}\right)^2 dt\ dx + \frac{1}{p} \left[\int_\Omega |\nabla v|^p\ dx\right]_0^M$$
$$\leq \lambda \int_\Omega [G(v(\cdot, M)) - G(v(\cdot, 0))] dx.$$

Then we get

$$K\int_0^M\int_\Omega\left(\frac{\partial v}{\partial t}\right)^2 dt\ dx+\frac{1}{p}\int_\Omega|\nabla v(\cdot,M)|^p\,dx$$
$$\le\frac{1}{p}\int_\Omega|\nabla v(\cdot,0)|^p\,dx+\lambda\int_\Omega[G(v(\cdot,M))-G(v(\cdot,0))]dx.$$

By using (3.5) and the fact that $\frac{1}{p}\int_\Omega|\nabla v(\cdot,M)|^p\,dx\ge 0$, we obtain

$$K\int_0^M\int_\Omega\left(\frac{\partial v}{\partial t}\right)^2 dt\ dx\le C_0+\lambda\int_\Omega G(v(\cdot,M))dx-\lambda\int_\Omega G(v(\cdot,0))dx.$$

Since v is bounded in $L^p\left(0,M,W_0^1(\Omega)\right)\hookrightarrow L^p(Q)$, furthermore since G is an increasing function then there exists a positive constant C_2 such that

$$\int_0^M\int_\Omega\left(\frac{\partial v}{\partial t}\right)^2 dt\ dx\le C_2. \tag{3.7}$$

Indeed, $v_0\in L^\infty(\Omega)$ then $G\left(v_0(\cdot)\right)$ is bounded. Since $L^p\left(0,M;W_0^{1,p}(\Omega)\right)\hookrightarrow L^p\left(0,M;L^p(\Omega)\right)=L^p\left(Q\right)$ also v is bounded in $L^p\left(0,M;W_0^{1,p}(\Omega)\right)$ then v is bounded in $L^p(Q)\hookrightarrow L^1(Q)$ hence $|v(x,t)|\le C_3$ (Constant) then $G(v(x,t))\le G(C_3)$.
Let us define the following operator $A:L^p(Q)\longrightarrow L^p(Q)$, where $A(s,w)=v$, for all $s\in[0,M]$ where $v\in L^p(0,M;W_0^{1,p}(\Omega))$ being the solution for the problem (3.4). On the other hand, by using the inequality (3.7) and the following embedding $L^2\left(0,M;L^2(\Omega)\right)\hookrightarrow L^{p'}\left(0,M;W^{1,p'}(\Omega)\right)$, we obtain

$$\left\|\frac{\partial v}{\partial t}\right\|_{L^{p'}\left(0,M;W^{-1,p'}(\Omega)\right)}\le C_1. \tag{3.8}$$

From (3.5) and (3.8), we get that for large enough radius $R>0$, $A(B_R)\subset B_R$ where

$$B_R=\left\{w\in L^p(Q)/\|w\|_{L^p(Q)}\le R\right\}\subset L^p(Q).$$

Furthermore, $A(s,w)=v\in W$ with $\|A(s,w)\|_W\le C$. Since the embedding $W\hookrightarrow L^p(Q)$ is compact then A is a compact operator.
To achieve the proof, it suffices to show that A is a continuous operator. Thus let $(w_n)\subset B_M$ be a sequence such that

$$w_n\longrightarrow w \text{ strongly in } L^p(Q)$$

and consider the corresponding function to w_n, that is $v_n=A(s,w_n)$,

$$F_n:=\frac{\lambda f(w_n)}{(\int_\Omega f(w_n)dx)^2}\ \text{and}\ F:=\frac{\lambda\cdot f(w)}{(\int_\Omega f(w)dx)^2}.$$

Then $v_n\longrightarrow v=A(s,w)$ strongly in $L^p(Q)$. Indeed, we subtract the equation verify by v_n with the same equation verify by v and taking the following function as a test function for small $\mu>0$,

$$\psi_\mu(z)=\min\{1,\max\{\frac{z}{\mu},0\}\},\ \text{for all } z\in\mathbb{R}.$$

We get

$$\begin{aligned}&\int_0^s\int_\Omega\frac{\partial}{\partial t}\left(b(v_n)-b(v)\right)\psi_\mu(v_n-v)dt\ dx-\int_0^s\int_\Omega\left(\Delta_p v_n-\Delta_p v\right)\psi_\mu(v_n-v)dt\ dx\\&=\int_0^s\int_\Omega\frac{\lambda f(w_n)}{(\int_\Omega f(w_n)dx)^2}\psi_\mu(v_n-v)dt\ dx-\int_0^s\int_\Omega\frac{\lambda f(w)}{(\int_\Omega f(w)dx)^2}\psi_\mu(v_n-v)dt\ dx.\end{aligned} \tag{3.9}$$

By Green's formula, we get that

$$\begin{aligned}&\int_0^s\int_\Omega(\Delta_p v_n-\Delta_p v)\cdot\psi_\mu(v_n-v)dt\ dx\\&=-\int_0^s\int_\Omega\left[|\nabla v_n|^{p-2}\nabla v_n-|\nabla v|^{p-2}\nabla v\right]\nabla(v_n-v)\cdot\psi'_\mu(v_n-v)dt\ dx.\end{aligned}\tag{3.10}$$

Recall from Tartar's inequality (See [13]), we have

$$\left[|e|^{p-2}e-|c|^{p-2}c\right]\cdot(e-c)\geq C(p)\,|e-c|^p\,,\ \text{if } p\geq 2,\tag{3.11}$$

where $C(p)=2^{2-p}$ when $p\geq 2$ and $C(p)=p-1$ when $1<p<2$. we get that

$$\int_0^s\int_\Omega(\Delta_p v_n-\Delta_p v)\cdot\psi_\mu(v_n-v)dt\ dx\leq 0.\tag{3.12}$$

Using (3.12) and (3.9), to obtain

$$\begin{aligned}&\int_0^s\int_\Omega\frac{\partial}{\partial t}(b(v_n)-b(v))\cdot\psi_\mu(v_n-v)dt\ dx\\&\leq\lambda\int_0^s\int_\Omega\left(\frac{f(w_n)}{(\int_\Omega f(w_n)dx)^2}-\frac{f(w)}{(\int_\Omega f(w)dx)^2}\right)\psi_\mu(v_n-v)dt\ dx.\end{aligned}$$

On the other hand, we have

$$\int_0^s\int_\Omega\frac{\partial}{\partial t}(b(v_n)-b(v))\cdot\psi_\mu(v_n-v)dt\ dx\longrightarrow\int_0^s\int_\Omega\frac{\partial}{\partial t}(b(v_n)-b(v))\cdot\chi_{\{v_n-v>0\}}dt\ dx,$$

when μ tends to zero. On the other hand, we have

$$\int_\Omega(b(v_n)-b(v))^+dx=\int_0^s\int_\Omega\frac{\partial}{\partial t}(b(v_n)-b(v))\cdot\chi_{\{v_n-v>0\}}dt\ dx.$$

It yields

$$0\leq\int_\Omega(b(v_n)-b(v))^+dx\leq\lambda\int_0^s\int_\Omega(F_n-F)\cdot\chi_{\{v_n-v>0\}}dt\ dx.$$

Since, we have $F_n\longrightarrow F$, when n tends to infinity. Then thanks to the dominated convergence Theorem and using the continuity of the function b, we get result for each $v_n\geq v$. Conversely, we suppose that $v_n\leq v$ so, we obtain $0\leq v-v_n$. And we proceed by the same method then, we get result. Finally we obtain the continuity of the operator A. □

Proof of the main result

In order to prove our main result, we apply Theorem 3.2 to obtain the second condition of Definition 2.1. For the first condition of the same definition obtained by using (3.5) and (3.8). For the regularity of the solution v, we may use (3.5), (3.8) and Aubin's Lemma [31], we get that $v\in C\left([0,T];L^1(\Omega)\right)$. By accumulating all of the previous results, the proof of Theorem 3.1 is completed.

Remark 3.1 An example of a field satisfying hypothesis (H3) is

$$x\mapsto\sqrt{1+x^2}.\tag{3.13}$$

4. Conclusion and perspectives

In this paper, we showed the existence of a capacity solution to a nonlocal thermistor problem in Sobolev spaces. The existence of a weak solution to problem (3.4) is guaranteed in [7,6] as well, but under strong hypotheses on f and the initial data. A very difficult issue is determining the uniqueness of capacity solutions to problem (1.2). This is due to the fact that all uniqueness results for the thermistor problem rely on certain restrictive hypotheses on the data. The previous result presented in Theorem 3.1 only establishes the existence of a capacity solution in the case of $p \geq 2$. In the case where $2 > p \geq 1$, proving the existence of a capacity solution for (1.2) is a challenging task. Because, we have the lack of the embedding between some spaces. In the future, we will study the existence and regularity of the global attractor for the considered problem.

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