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Nonlinear variational inequality in Reflexive Banach space with application in epidemiology

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ABSTRACT: This paper studies some nonlinear variational inequalities with operators defined on reflexive and separable Banach spaces and verifying pseudomonotone properties. This study comes up with some new results, such as the existence and the strong convergence of solutions. All these results are based on the Galerkin method. As an application, the established result is used for an epidemic model by taking the spatio-temporal SIR model, which is defined as a reaction-diffusion system with Signorini boundary condition to generate a variational inequality. In such application, the Signorini boundary condition is considered for infected individuals while the Neumann boundary condition is considered for susceptible and recovered individuals. The paper ends with a numerical analysis of the problem by applying a splitting method and the Uzawa algorithm based on the augmented Lagrangian method. Some tests are included to show the solutions.

Key Words: SIR model, variational inequality, Signorini condition, Uzawa algorithm.

Contents

1. Introduction

The theory of variational inequalities (VI) has its origin in the calculus of variations, and it is a very useful tool in current mathematical research. It was initially developed to deal with the difficulties of equilibrium problems. Then, it has been spread to include problems from economics and game theory [49], finance for example the work of Scrimali [48], and optimization. They were also introduced by Hartman and Stampacchia for the first time in their work [31] as powerful models that help study partial differential equations with inequality like boundary conditions arising in obstacle problems and unilateral contact problems in mechanics [22]. These have been extended and generalized to study a wide class of problems arising control problems, operations research, engineering sciences, etc. [7,6]. We believe that the real importance of VI comes also from the fact that they are equivalent to optimization problems [24], and hence, in such cases, numerical resolution may be possible.

The primary issues in the study of variational inequalities are the existence of solutions and the numerical method for finding solutions. So there have been many articles on these subjects of variational inequalities in the literature up to now [10,27,30,38,22]. The Galerkin method and finite element method are commonly employed in proving the existence and computing solutions of partial differential equations as well as Signorini problems in Sobolev spaces. The Galerkin method for operator equations in separable and reflexive Banach spaces has been extensively investigated in [56]. In the work [36], the authors used the Galerkin technique to establish some results for variational inequalities in separable and reflexive Banach spaces. The purpose of this paper is to study and demonstrate some new results for nonlinear variational inequalities with coercive and pseudomonotone operators in reflexive and separable Banach spaces. These results will be proven with the help of the Galerkin method. The second part deals with an application of the framework in epidemiology situations, which consists of a compartmental SIR model

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generated from partial differential equations with the so-called Signorini boundary condition. The use of this type of condition leads to variational inequality problems, which are introduced for the first time in epidemiology.

The Signorini problems are known as a class of very important variational inequalities, which arise in many practical problems such as the elasticity with unilateral conditions [50,25], the fluid mechanics problems in media with semi-permeable boundaries [22,27], and the electro-paint process [1]. Henceforth, Signorini's conditions are considered for the epidemiological model.

In general, mathematical modeling has a crucial role in various other scientific (physics, biology, earth sciences, chemistry) and technical fields (computer science, electrical engineering). Epidemiology is one of the fields where mathematical modeling is an extremely important theoretical technique and a useful tool to understand the spread of diseases and to develop approaches to control them and predict their behaviour [33,52]. The Kermack-McKendrick and Reed-Frost epidemic models are the framework for compartmental models, which have been applied in a considerable number of mathematical models, such as the Susceptible-Infected-Recovered (SIR) models [12,11], the demographic and migration models established in sociology and demography [14,35], the predator-prey dynamics modelled by the Lotka-Volterra model [12,44] and the stochastic models as given in [29,43,17,45].

All of the models introduced above are primarily based on compartmental analysis and involve systems of ordinary differential equations (ODEs) [20,46,3]. Despite the fact that these models are very crucial in their formulation, analysis, and mathematical and numerical solutions, they are inappropriate to describe the diffusion and dynamics in space. To include the spatial dynamics of epidemics, ODE-based models have been developed to include meta-populations, such as multi-group and multi-patch models describing a population with several sub-populations connected by immigration or emigration [5,40]. The idea here is derived from the definition of regional compartments in accordance with the different zones of the spatial environment, with coupling terms added to the model equations to take into account the flow of species between the different sub-regions [34,42,40,4].

Models driven by partial differential equations (PDEs) have been developed to describe the spatial dynamics of the host population and the spread of the disease. This allows the dynamics to be described both in time and space at all levels [2,9,15]. The majority of these PDE systems use reaction-diffusion equations to model the random movement of the population as a diffusion process. In these researches, the diffusion rates are typically established as constants and are accompanied by zero-flux boundary conditions [20,16]. In a different study [57], the author presents their diffusive model incorporating zero-flux boundary conditions, yet they have introduced spatially dependent parameters to account for the spatial heterogeneity. Additionally, some authors have chosen to replace the zero-flux boundary with a homogeneous Dirichlet condition, which signifies an environment that is hostile for the survival of individuals, as seen in its application to the SVIR Model in [53], Predator-Prey Models in [37,54,21], and in the age-structured SIR epidemic model [19]. Other researchers have explored non-standard approaches. The author [47] implemented nonlocal boundary conditions where the flux at the boundary depends on the average concentration within a specific neighborhood, enabling more complex interactions between the system and its surroundings. In [41], the author investigated models with nonlinear diffusion density, where the diffusion rate itself depends on the concentration, leading to potentially richer dynamics.

In this paper, we introduce a PDE modeling the spread of diseases with the Signorini boundary condition [50,22,27], which leads to the quasi-variational inequality studied in the framework introduced in the first part of this work. Regarding the numerical study of the problem, since the problem is a nonlinear evolutionary quasi-variational inequality, we first proceed with a splitting method to reduce the complexity. Then we use the Newton schema for the resolution of the first time step, and for the half-time step, which is a system of elliptic equations with a Signorini boundary, it is easy to prove that is equivalent to an optimization problem. The augmented Lagrangian takes its place as an efficacy tool to solve the optimization problem by an equivalent with the saddle point problem of the augmented Lagrangian. [18,32,55,51]. The Uzuwa algorithm defined in [28,26] has been used as a numerical method to find the saddle point.

This work is divided into three sections. The first section is a framework providing the result of the existence of solutions to nonlinear variational inequalities in reflexive and separable Banach spaces. The second section is the application of the framework result in an inequality variational resulting from the Signorini condition as a boundary condition on the spatio-temporal SIR model. Furthermore, we provide an augmented Lagrangian method, and we apply the Uzawa algorithm to explicitly compute the solution. The last section is devoted to a numerical illustration with Matlab to validate the theoretical results.

2. The framework

Let's consider $(E, \|.\|)$ as a reflexive and separable Banach space with its dual is called E^*, K is a nonempty closed and convex set in E and $\mathcal{A} : K \longrightarrow E^*$ is a operator. Let us begin with providing some definitions of several properties that will be used in the main result:

- $\mathcal{A}$ is operator pseudomonotone (see [13]) i.e.

$$u_n \rightharpoonup u, \limsup_{n\to\infty} \langle \mathcal{A}u_n, u_n - u\rangle \leq 0 \Rightarrow \langle \mathcal{A}u, u - w\rangle \leq \liminf_{n\to\infty} \langle \mathcal{A}u_n, u_n - w\rangle, \forall w \in K \tag{2.1}$$

- $\mathcal{A}$ is coercive, this means there are some points $w_0 \in K$ such that

$$\frac{\langle \mathcal{A}u - \mathcal{A}w_0, u - w_0\rangle}{\|u - w_0\|} \to +\infty \text{ as } \|u\| \to +\infty. \tag{2.2}$$

- $\mathcal{A}$ is satisfy condition $(S)_+$, for any sequence $(u_n) \subset K$, we have

$$u_n \rightharpoonup u \text{ and } \lim_{n\to\infty} \sup \langle \mathcal{A}u_n - \mathcal{A}u, u_n - u\rangle \leq 0 \quad \text{then } \|u_n - u\| \to 0. \tag{2.3}$$

- $\mathcal{A}$ is demicontinuous i.e. for any sequence $u_n \to u$ strongly, then $\mathcal{A}u_n \rightharpoonup \mathcal{A}u$ in the weak * topology .

Now, let's define the following variational inequality:

$$\begin{cases} \text{Find } u \in K \\ \langle \mathcal{A}u, v - u\rangle \geq \langle f(u), v - u\rangle \quad \forall v \in K, \end{cases} \tag{2.4}$$

where $f \in \mathbb{C}^0(E, E^*)$, and f is bounded in K.
It is easy to see that when $K = E$, then (2.4) is equivalent to the following variational problem:

$$\text{Find } u \in E \quad \mathcal{A}u = f(u).$$

Since E is a separable space, hence there exists an increasing sequence $E_n \subset E_{n+1}$ which is dense in E. For each $n \geq 0$, define the finite-dimensional space E_n

$$E_n = span\{w_1, w_2, ..., w_n\}$$

We consider the following Galerkin variational inequalities corresponding to (2.4) on E_n :

$$\text{Find} \quad u_n \in E_n \cap K : \langle \mathcal{A}u_n - f(u_n), v - u_n\rangle \geq 0 \quad \forall y \in E_n \cap K, n = 1, 2, \ldots \tag{2.5}$$

Theorem 2.1 *Suppose that K is a closed convex set in E, $\mathcal{A} : K \to E^*$ is pseudomonotone, coercive, bounded, and satisfied the $(S)_+$ condition (2.3). Then, the following claims are true :*

1. *For each $n \geq 1$, the Galerkin variational inequality (2.5) has a solution u_n and the sequence (u_n) is bounded.*
2. *There exists a subsequence (u_{n_k}) which converges strongly to a solution u of variational inequality (2.4).*
3. *If (2.4) has a unique solution, then the total sequence (u_n) converges strongly to u.*

Proof: We can suppose that $w_0 = 0 \in K$ without losing generality. If not, we may take the set $\tilde{K} = K - w_0$, the sequence $\tilde{E}_n$ equal to $E_n - w_0$, and define $\tilde{\mathcal{A}}u$ as $\mathcal{A}(u + w_0)$, where $u \in \tilde{K}$.

1. Define the operator $\mathcal{F}u = \mathcal{A}u - f(u)$, it follows

$$\frac{\langle \mathcal{F}u - \mathcal{F}0, u\rangle}{\|u\|} = \frac{\langle \mathcal{A}u - \mathcal{A}(0), u\rangle}{\|u\|} - \frac{\langle f(u) - f(0), u\rangle}{\|u\|}$$

Since $\mathcal{A}$ is a coercive operator, then there is $R > \|w_0\|$ such that

$$\frac{\langle \mathcal{A}u - \mathcal{A}(0), u\rangle}{\|u\|} \geq M \quad \text{for} \quad \|u\| \geq R,$$

that gives the following result

$$\frac{\langle \mathcal{F}u - \mathcal{F}0, u\rangle}{\|u\|} \geq M - \|f(u) - f(0)\| \quad \text{for all } \|u\| \geq R.$$

Let us choose a number M such that $M > \max(0, \|f(u) - f(0)\| - \|\mathcal{F}0\|) := \epsilon$. This implies that

$$\langle \mathcal{F}u, u\rangle > (M - \epsilon)\|u\| > 0 \quad \text{for all } \|u\| \geq R. \tag{2.6}$$

Let $j : E_n \to E$ be an injection map and its dual j^*. For each E_n, the mapping $\mathcal{F}_n := j^*\mathcal{F}j : E_n \cap K \cap \bar{B}_E(0, R) \to E_n^*$ is continuous. This is because the continuity of j and j^*, and since $\mathcal{A}$ is pseudomonotone and bounded, $\mathcal{A}$ is demicontinuous (see Proposition 27.7 in [56]) and f is continuous then $\mathcal{F}$ is demicontinuous on K (see Theorem 3.10 in [13]). Here,

$$\bar{B}_E(0, R) = \{x \in E : \|x\| \leq R\}$$

is a closed ball with a radius $R > 0$. Besides, $E_n \cap K \cap \bar{B}_E(0, R)$ is a compact set in E_n, which contains 0, the existence and uniqueness are realized as a result of several standard Theorems of the variational inequality (see [38,39]), which implies that (2.4) has a solution $u_n \in E_n \cap K \cap \bar{B}_E(0, R)$, that is,

$$\begin{aligned} &\langle (j^*\mathcal{F}j)(u_n), z - u_n\rangle \geq 0, && \forall z \in E_n \cap K \cap \bar{B}_E(0, R) \\ \Leftrightarrow \quad &\langle \mathcal{F}u_n, j(z - u_n)\rangle \geq 0, && \forall z \in E_n \cap K \cap \bar{B}_E(0, R) \\ \Leftrightarrow \quad &\langle \mathcal{F}u_n, z - u_n\rangle \geq 0, && \forall z \in E_n \cap K \cap \bar{B}_E(0, R) \end{aligned} \tag{2.7}$$

Now, let verify that $\|u_n\| < R$ for all $n > 0$. In fact, if there is $n_0 > 0$ such that $\|u_{n_0}\| = R$, then from (2.6), we have

$$\langle \mathcal{F}u_{n_0}, 0 - u_{n_0}\rangle = -\langle \mathcal{F}u_{n_0}, u_{n_0} - 0\rangle < 0$$

which contradicts (2.7). Hence, $\|u_n\| < R$. This implies that for any $y \in E_n \cap K$, there exists $t \in (0, 1)$ such that $v_t = u_n + t(y - u_n) \in E_n \cap K \cap \bar{B}_E(0, R)\big)$.
Taking $z = v_t$ in (2.7) guaranteed that

$$\langle \mathcal{F}u_n, y - u_n\rangle \geq 0 \quad \forall y \in E_n \cap K, \tag{2.8}$$

which follows that u_n is a solution of (2.4) and that $\|u_n\| < R$ for all $n > 0$.

2. The sequence (u_n) is bounded in E, then there exists a subsequence (u_{n_k}) such that u_{n_k} convergent weakly to u which belong to K, in fact that the latter is is a closed convex set which imply that it is also a weakly closed convex set.

Since $\mathcal{A}$ is bounded in E^* which means that $\|\mathcal{A}u_{n_k}\| \leq \rho$ for all $k \geq 1$, then there exist a subsequence with the same notation $(\mathcal{A}u_{n_k})$ such that it satisfies the * weak convergence $\mathcal{A}u_{n_k} \rightharpoonup^* w$ with $w \in E^*$.

Note that

$$\overline{\bigcup_{n=1}^{\infty} E_n \cap K} = K.$$

Then, for any $\epsilon > 0$, there exists $u' \in E_{n_1}$ such that $\|u' - u\| < \epsilon$. Hence, $u' \in E_{n_k} \cap K$ for all $k \geq 0$. From (2.8), we have

$$\begin{aligned}\langle \mathcal{A}u_{n_k}, u_{n_k} - u\rangle &= \langle w - \mathcal{A}u_{n_k}, u\rangle + \langle w, u_{n_k} - u\rangle + \langle \mathcal{A}u_{n_k} - w, u_{n_k}\rangle \\ &\leq \langle w - \mathcal{A}u_{n_k}, u\rangle + \langle w, u_{n_k} - u\rangle + \|\mathcal{A}u_{n_k} - w\| \, \|u_{n_k}\| \\ &\leq \langle w - \mathcal{A}u_{n_k}, u\rangle + \langle w, u_{n_k} - u\rangle + R\, \|\mathcal{A}u_{n_k} - w\| \\ &\leq \langle w - \mathcal{A}u_{n_k}, u\rangle + \langle w, u_{n_k} - u\rangle + R \sup_{\xi \in E^*, \|\xi\|=1} \langle \mathcal{A}u_{n_k} - w, \xi\rangle\end{aligned}$$

It follows that

$$\begin{aligned}\limsup_{k\to\infty} \langle \mathcal{A}u_{n_k}, u_{n_k} - u\rangle \leq \limsup_{k\to\infty} (\langle w - \mathcal{A}u_{n_k}, u\rangle &+ \langle w, u_{n_k} - u\rangle \\ &+ R \sup_{\xi \in E^*, \|\xi\|=1} \langle \mathcal{A}u_{n_k} - w, \xi\rangle)\end{aligned}$$

which means

$$\limsup_{k\to\infty} \langle \mathcal{A}u_{n_k}, u_{n_k} - u\rangle \leq 0. \tag{2.9}$$

Using (2.9) yields to

$$\begin{aligned}&\lim\sup_{k\to\infty} \langle \mathcal{A}u_{n_k} - \mathcal{A}u, u_{n_k} - u\rangle \\ &\quad \leq \limsup_{k\to\infty} \langle \mathcal{A}u_{n_k}, u_{n_k} - u\rangle + \limsup_{k\to\infty} \langle -\mathcal{A}u, u_{n_k} - u\rangle \leq 0.\end{aligned} \tag{2.10}$$

Using (2.9) yields to

$$\begin{aligned}&\lim\sup_{k\to\infty} \langle \mathcal{A}u_{n_k} - \mathcal{A}u, u_{n_k} - u\rangle \\ &\quad \leq \limsup_{k\to\infty} \langle \mathcal{A}u_{n_k}, u_{n_k} - u\rangle + \limsup_{k\to\infty} \langle -\mathcal{A}u, u_{n_k} - u\rangle \leq 0.\end{aligned} \tag{2.11}$$

Since $\mathcal{A}$ satisfies the condition $(S)_+$, then the equation (2.11) implies the strong convergence i.e. $\|u_{n_k} - u\| \to 0$ as $k \to \infty$.
By passing to limit in (2.8), u is solution of (2.4) and we have

$$\langle \mathcal{F}u, v - u\rangle \geq 0 \quad \forall v \in K.$$

Due to the weak sequential continuity of the function f, we have the convergence

$$\lim_{n\to\infty} \langle f(u_n) - f(u), v\rangle = 0, \forall v \in K, \tag{2.12}$$

Using the convergence above, the passing to limit in (2.8) leads to u as solution of (2.4) and we have

$$\langle \mathcal{F}u, v - u\rangle \geq 0 \quad \forall v \in K.$$

3. Take any subsequence (u_{m_k}) of (u_n) such that $u_{m_k} \to u_0$. Then, using the same procedure as in the of proof of first points of 2.1, we can prove that $u_{m_k} \to u_0$ strongly and u_0 is a solution of (2.4). Since (2.4) has a unique solution, we have $u_0 = u$. As a result, we have shown that every subsequence converges to an unique limit. Thus, the whole sequence converges.

□

3. Application in epidemiology Problem

We start this section with some useful notations. Let Ω be an open bounded polyhedral domain subset of $\mathbb{R}^d (d = 1, 2, 3)$ with a boundary $\Gamma = \partial\Omega$ divided into two parts such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Each Γ_i has a different boundary condition. We shall denote the Sobolev space by $H^1(\Omega)$.

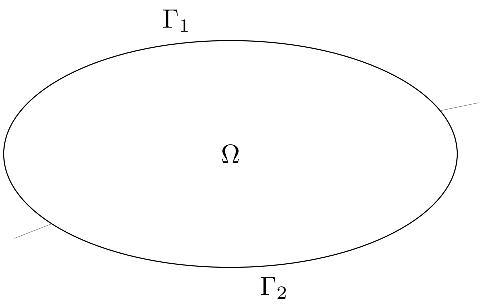

Figure 1: The domain Ω and its boundaries Γ_1 and Γ_2

Before considering a spatially-temporally dependent problem, we define the spatial-temporal domains $\mathcal{Q}$ and Σ_i, respectively, by

$$\mathcal{Q} = (0, T) \times \Omega \text{ and } \Sigma_i = (0, T) \times \Gamma_i \quad i = 1, 2.$$

Define the boundary Γ_1 as a free boundary, which for $x \in \Gamma_1$, it is necessary to distinguish two situations:

- $u(x) > g(x)$: the individual thus has a tendency to enter Ω, the wall Γ_1 prevents this, however, so that the flux is zero at x, i.e. :
$$\frac{\partial u}{\partial \nu}(x) = 0 \tag{3.1}$$
- $u(x) \leq g(x)$: the individual tends to leave Ω, the wall Γ_1 allows this and therefore the flux satisfies
$$\frac{\partial u}{\partial \nu}(x) \leq 0, \tag{3.2}$$
where $\frac{\partial u}{\partial \nu}(x)$ is finite. Hence, u is continuous in the neighbourhood of x, which, if the wall Γ_1 is infinitely thin, implies that $u(x) = g(x)$.

Summing up, the boundary conditions on Γ_1 are :

$$\frac{\partial u}{\partial \nu} \leq 0, \ u \leq g \text{ and } (u - g)\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1 \tag{3.3}$$

There are thus a priori two regions Δ_1 and Δ_2 on Γ_1, for which $u = g(x)$ and $\frac{\partial u}{\partial \nu} = 0$, respectively. But these two regions' positions are unknown a priori, therefore locating them is actually equivalent to resolving the reaction-diffusion SIR model, which will be the direct problem with a free boundary in Γ_1 as follows:

$$\begin{cases} u_t - \operatorname{div}(q\nabla u) = f(u), & \text{in } \mathcal{Q} \\ \frac{\partial u}{\partial \nu} \leq 0, \ u \leq g \text{ and } (u - g)\frac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma_1 \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma_2 \\ u(0, x) = u_0(x) > 0 \end{cases} \tag{3.4}$$

where $u = \left\{ \begin{array}{c} S \\ I \\ R \end{array} \right\}$, $f(u) = \left\{ \begin{array}{c} -\beta SI \\ \beta SI - \gamma I \\ \gamma I \end{array} \right\}$, $g = \left\{ \begin{array}{c} g_1 \\ g_2 \\ g_3 \end{array} \right\}$ and $q = \left\{ \begin{array}{ccc} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{array} \right\}$, where

β is the infection rate, γ is the recovery rate,and for each compartment $q_i \in L^\infty(\Omega)$ presented the diffusion coefficient and g_i is function belong to $L^\infty(\Gamma_1)$ represented the threshold of the Signorini condition.

We shall pose this problem in the form of a time-dependent variational inequality. To this end, we introduce the non empty closed and convex set $\mathfrak{C}$ of $L^2\left(0,T,H^1(\Omega)^3\right)$:

$$\mathfrak{C}=\left\{v\in L^2\left(0,T,H^1(\Omega)^3\right)\mid v_i\leq g_i\ a.e.\ \text{ on }\Gamma_1,\forall\, i=1,2,3\right\}$$

By taking $v-u$ as test function and multiplying with the equation defined in problem (4.1), leads to the weak formulation

$$\int_\Omega u_t.(v-u)+\int_\Omega q\nabla u.\nabla(v-u)-\int_{\Gamma_1}q.\frac{\partial u}{\partial\nu}(v-u)=\int_\Omega f(u)(v-u) \tag{3.5}$$

Introducing the function g in the boundary term $q.\frac{\partial u}{\partial\nu}(v-u)$ and applying the Signorini boundary condition properties yields to

$$\int_{\Gamma_1}q\frac{\partial u}{\partial\nu}(v-u)=\int_{\Gamma_1}q.\frac{\partial u}{\partial\nu}(v-g)+\int_{\Gamma_1}q.\frac{\partial u}{\partial\nu}(g-u)=\int_{\Gamma_1}q.\frac{\partial u}{\partial\nu}(v-g)\geq 0.$$

Using this in the equation (3.5) gives

$$\int_\Omega u_t.(v-u)+\int_\Omega q\nabla u.\nabla(v-u)\geq\int_\Omega f(u)(v-u).$$

By the above notations, the problem (4.1) is equivalent in the weak sense to the following problem:

$$\left\{\begin{array}{l}\text{Find } u\in\mathfrak{C}\text{ such that:}\\ \langle\mathcal{A}u,v-u\rangle\geq\langle f(u),v-u\rangle\quad\forall v\in\mathfrak{C}\end{array}\right. \tag{3.6}$$

where $\mathcal{A}u=u_t-\operatorname{div}(q\nabla u)$.

We will start by proving the uniqueness for problem (3.6). Since $f:L^2(0,T;H^1(\Omega)^3)\to L^2(0,T;L^2(\Omega)^3)$ is locally Lipschitz, it will be used a standard truncation of the source as follows :

$$h(u)=\begin{cases}f(u), & \text{if}\quad\|u\|\leq\delta,\\ f\left(\frac{\delta u}{\|u\|}\right), & \text{if}\quad\|u\|>\delta,\end{cases}$$

where δ is a positive constant. We note here that for each such δ, the function $h:L^2(0,T;H^1(\Omega)^3)\to L^2(0,T;L^2(\Omega)^3)$ are globally Lipschitz continuous.

Consider the new problem associated to g

$$\left\{\begin{array}{l}\text{Find } u\in\mathfrak{C}\text{ such that:}\\ \langle\mathcal{A}u,v-u\rangle\geq\langle h(u),v-u\rangle,\quad\forall v\in\mathfrak{C}.\end{array}\right. \tag{3.7}$$

Let suppose that the problem (3.7) has two solutions u and w. Subtracting the weak formulations of the solutions yields

$$\int_\Omega(\partial_t u-\partial_t w)v+q\int_\Omega(\nabla u-\nabla w)\nabla v\ \leq\int_\Omega(h(u)-h(w))\,v$$

by taking $v=u-w$ as a test function gives

$$\int_\Omega(\partial_t u-\partial_t w)(u-w)+q\int_\Omega(\nabla u-\nabla w)^2\ \leq\int_\Omega(h(u)-h(w))\,(u-w).$$

By Young inequality, we get:

$$\int_\Omega(\partial_t u-\partial_t w)(u-w)+q\int_\Omega(\nabla u-\nabla w)^2\ \leq\frac{\epsilon}{2}\|h(u)-h(w)\|^2+\frac{1}{2\epsilon}\|u-w\|^2$$

$$\int_\Omega(\partial_t u-\partial_t w)(u-w)+q\|\nabla u-\nabla w)\|^2\leq\frac{\epsilon}{2}\|h(u)-h(w)\|^2+\frac{1}{2\epsilon}\|u-w\|^2$$

hence

$$\frac{1}{2}\frac{d}{dt}\|u-w\|^2 + q\|\nabla u - \nabla w\|^2 \;\leq \frac{\epsilon}{2}\|h(u)-h(w)\|^2 + \frac{1}{2\epsilon}\|u-w\|^2$$

thus

$$\frac{1}{2}\frac{d}{dt}\|u-w\|^2 + q\|\nabla u - \nabla w\|^2 \;\leq \frac{\epsilon L^2}{2}\|u-w\|^2 + \frac{1}{2\epsilon}\|u-w\|^2$$

Then

$$\frac{1}{2}\frac{d}{dt}\|u-w\|^2 + \left(q - \frac{\epsilon L^2 C}{2}\right)\|\nabla u - \nabla w\|^2 \;\leq \frac{1}{2\epsilon}\|u-w\|^2$$

Let's choose $\epsilon = \frac{2q}{L^2 C}$, we get by Gronwall's lemma that

$$u - w = 0$$

and then the uniqueness follows.

The solution u_δ satisfies the following energy inequality:

$$\frac{1}{2}\frac{d}{dt}\|u_\delta\|^2 + q\|\nabla u_\delta\|^2 \;\leq \frac{\epsilon L^2}{2}\|u_\delta\|^2 + \frac{1}{2\epsilon}\|u_\delta\|^2.$$

By integration over the time between 0 and t_1

$$\|u_\delta(.,t)\|^2 \leq \|u_0\|^2 + C_\delta \int_0^{t_1} \|u_\delta(.,t)\|^2$$

Then the Gronwall's inequality gives:

$$\|u_\delta(.,t)\|^2 \leq \; e^{C_\delta t_1}\|u_0\|^2. \tag{3.8}$$

Let's now prove the positivity of the solution of our problem, by rewriting u_δ as follow $u_\delta = (\xi)^+ - (\xi)^-$ where $(\xi)^+, (\xi)^- \geq 0$, they are defined by

$$(\xi)^+ := \max(0, u_\delta) \text{ and } (\xi)^- := \max(0, -u_\delta)$$

By using the estimation of solution proven in (3.8), we get

$$\|(\xi^1)^-(.,t)\|^2_{L^2(\Omega)} \leq \|(\xi^1)^-(.,0)\|^2_{L^2(\Omega)} e^{C_\delta t_1}.$$

Since $u_0 > 0$ which means that $(\xi)^-(.,0) := (\xi_0)^- = 0$, last estimation yields to $(\xi)^-(.,t) = 0$ then the positivity of solution is proven. Since the operator $\mathcal{A}$ is verifying the hypotheses of theorem (2.1), that means the existence of an unique non negative solution for inequality variational (3.6).

3.1. Numerical scheme

The numerical method proposed for getting the solution is the splitting method as given for example in [23]. The idea here is to divide the equation (4.1) into two sub-equations, the nonlinear reaction equation

$$\frac{1}{2}\frac{\partial u}{\partial t} = f(u), \quad \text{ in } \mathcal{Q} \tag{3.9}$$

which is solved for the first half of the time step, and the linear diffusion system

$$\frac{1}{2}\frac{\partial u}{\partial t} = \operatorname{div}(q\nabla u), \quad \text{ in } \mathcal{Q} \tag{3.10}$$

with boundaries conditions. It is solved for the second half of the time step. The solution process is outlined below.

Let Δt be the uniform time step, defining the time t as $n\Delta t$ for all $n = 0, 1, 2....N$. Let $u^{n+1/2}$ be the approximate solutions of (3.9) at each point of Ω and time $t_{n+\frac{1}{2}} = (n+\frac{1}{2})\Delta t$ i.e. the solution after the first half time step. The numerical solutions of (3.10) at each point of Ω and time t_n will be denoted u^n.

The solution process, for $n > 0$, begins with the known value u^n the value $u^{n+1/2}$ is usually computed using the forward Euler scheme for equation (3.9)

$$u^{n+\frac{1}{2}} = u^n + 2\Delta t\, f(u^n), \text{ in } \Omega \tag{3.11}$$

where $f(u^n) = \left\{ \begin{array}{c} -\beta S^n I^n \\ \beta S^n I^n - \gamma I^n \\ \gamma I^n \end{array} \right\}$ is the term of reaction at time t_n for all points of the domain Ω.

Using $u^{n+\frac{1}{2}}$ as a second member, the values u^{n+1} are computed as the solution of the next problem

$$\begin{cases} u^{n+1} - 2\Delta t \nabla \cdot (q \nabla u^{n+1}) = u^{n+\frac{1}{2}} & \text{in } \Omega \\ \dfrac{\partial u^{n+1}}{\partial \nu} \leq 0, \; u^{n+1} \leq g, \; (u^{n+1} - g)\frac{\partial u^{n+1}}{\partial \nu} = 0 & \text{on } \Gamma_1 \\ \frac{\partial u^{n+1}}{\partial \nu} = 0 & \text{on } \Gamma_2 \end{cases} \tag{3.12}$$

hence, u^{n+1} is a solution of the the weak formulation of the problem (3.12). For the simplicity of notation, we can call u^{n+1} as u, which is satisfying the following problem

$$\left\{ \begin{array}{l} \text{Find } u \in \mathfrak{C} \text{ such that:} \\ \widetilde{a}(u, v-u) \geq \widetilde{f}(v-u) \quad \forall v \in \mathfrak{C} \end{array} \right. \tag{3.13}$$

where $\widetilde{a}(u,v) = \displaystyle\int_\Omega u\, v + 2\Delta \mathrm{t} \int_\Omega q \nabla u \cdot \nabla v$, $\widetilde{f}(v) = \langle u^{n+\frac{1}{2}}, v \rangle$ and $\langle .,. \rangle$ is duality pairing. It clear that the operator $\widetilde{a}$ verifies the hypotheses of the theorem (2.1), which guaranteed the existence and uniqueness of u. As a result, the equation (3.13) is well-posed.

Let us introduce the functional defined by

$$J(v) = \frac{1}{2}\widetilde{a}(v,v) - \widetilde{f}(v)$$

It is clear to see that (3.13) equivalent to the following constrained (convex) minimization problem

$$\left\{ \begin{array}{l} \text{Find } u \in \mathfrak{C} \text{ such that:} \\ J(u) \leq J(v) \quad \forall v \in \mathfrak{C}. \end{array} \right. \tag{3.14}$$

This kind of optimization problem has been handled by using a wide variety of techniques. for example, the projection and penalty approach. Our recommended approach to solving (3.14) is the augmented Lagrangian formulation

3.2. Augmented Lagrangian formulation

This paragraph provides one of the useful ways for dealing with the minimization problem of the same kind of problem (3.14), known as the augmented Lagrangian-based method(See [18,32,55,51]), it will evolve to generate an algorithm that will help us to obtain the solution to the problem. (3.12). Then, let's introduce some auxiliary helpful tools in the augmented Lagrangian-based method, as the following set:

$$\mathfrak{D} = \left\{ q \in L^2\left(\Gamma_1\right)^3, q_i - g_i \leq 0 \; \forall i = 1,2,3 \text{ on } \Gamma_1 \right\}$$

and its characteristic functional $1_{\mathfrak{D}} : L^2\left(\Gamma_1\right)^3 \longrightarrow \mathbb{R}^3 \cup \{+\infty\}$ defined by

$$1_{\mathfrak{D}}(v) = \left\{ \begin{array}{cc} 0, & v \in \mathfrak{D} \\ +\infty, & \text{otherwise} \end{array} \right.$$

The constrained minimization problem (3.14) can rewrite using the set $\mathfrak{D}$ to be equivalent to an unconstrained minimization problem given as follows :

$$\begin{cases} \text{Find } (u,w) \in H^1(\Omega)^3 \times L^2(\Gamma_1)^3, & \text{such that} \\ J(u) + 1_{\mathfrak{D}}(w) \leq J(v) + 1_{\mathfrak{D}}(q) & \forall (v,q) \in H^1(\Omega)^3 \times L^2(\Gamma_1)^3 \\ u - w = 0 & \text{on } L^2(\Gamma_1)^3 \end{cases} \tag{3.15}$$

The augmented Lagrangian function of the Problem (3.14), $\mathscr{L}_r$ defined over $H^1(\Omega)^3 \times L^2(\Gamma_1)^3 \times L^2(\Gamma_1)^3$ by

$$\mathscr{L}_r(v,q;\mu) = J(v) + 1_{\mathfrak{D}}(q) + \langle \mu, v-q \rangle_{\Gamma_1} + \frac{r}{2}\|v-q\|^2_{L^2(\Gamma_1)} \tag{3.16}$$

Since the functional J is strictly convex and the constraints are linear, a saddle point of $\mathscr{L}_r$ exists and it is the solution to the saddle point problem

$$\begin{cases} \text{Find } (u,w;\lambda) \in H^1(\Omega)^3 \times L^2(\Gamma_1)^3 \times L^2(\Gamma_1)^3 \text{ such that} \\ \mathscr{L}_r(u,w;\mu) \leq \mathscr{L}_r(u,w;\lambda) \leq \mathscr{L}_r(v,q;\lambda) \\ \text{for all } (v,q,\mu) \in H^1(\Omega)^3 \times L^2(\Gamma_1)^3 \times L^2(\Gamma_1)^3 \end{cases} \tag{3.17}$$

Equivalently, $((u,w),\lambda)$ is the solution of the min-max problem

$$\max_{\mu} \min_{(v,q)} \mathscr{L}_r(v,q;\mu) = \min_{(v,q)} \max_{\mu} \mathscr{L}_r(v,q;\mu). \tag{3.18}$$

The standard Uzawa method for augmented Lagrangian may be used to get a saddle point of $\mathscr{L}_r$ [8]. The main difficulty of the standard Uzawa method is the coupling of unknowns u and w, a quite natural procedure consists of using the following Uzawa block relaxation method [28,26]. Concerning the equation

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Algorithm 1 Uzawa algorithm for augmented Lagrangian

Initialization $k = 0$**.** w^{-1} and λ^0 are given.

Iteration $k \geq 0$**. While stopping criterion**
Compute successively u^k, w^k and λ^{k+1} as follows :

1. u^k by:
$$\mathscr{L}_r\left(u^k, w^{k-1}; \lambda^k\right) \leq \mathscr{L}_r\left(v, w^{k-1}; \lambda^k\right), \quad \forall v \in H^1(\Omega)^3. \tag{3.19}$$
2. w^k by :
$$\mathscr{L}_r\left(u^k, w^k; \lambda^k\right) \leq \mathscr{L}_r\left(u^k, q; \lambda^k\right), \quad \forall q \in L^2(\Gamma_1). \tag{3.20}$$
3. Update the Lagrange multiplier
$$\lambda^{k+1} = \lambda^k + r\left(u^k - w^k\right) \tag{3.21}$$

(3.19), the functional $v \mapsto \mathscr{L}_r\left(v, w^{k-1}; \lambda^k\right)$ is Gâteaux-differentiable on $H^1(\Omega)^3$, therefore the solution of (3.19) may be described by the Euler-Lagrange equation

$$\frac{\partial}{\partial v}\mathscr{L}_r\left(u^k, w^{k-1}; \lambda^k\right) \cdot v = 0, \quad \forall v \in H^1(\Omega)^3.$$

A straightforward calculation yields

$$\widetilde{a}\left(u^k, v\right) + r(u^k, v)_{\Gamma_1} = \widetilde{f}(v) + (rw^{k-1} - \lambda^k, v)_{\Gamma_1}, \quad \forall v \in H^1(\Omega)^3. \tag{3.22}$$

For the equation (3.20), over $\mathfrak{D}$ the functional $q \mapsto \mathscr{L}_{\mathrm{r}}\left(u^k, q; \lambda^k\right)$ can be simplified

$$\mathscr{L}_r\left(u^k, q; \lambda^k\right) = \frac{r}{2}\|q\|^2_{\Gamma_1} - \langle \lambda^k + ru^k, q \rangle_{\Gamma_1} + J(u^k) + \frac{r}{2}\left\|u^k\right\|^2_{\Gamma_1} + \langle \lambda^k, u^k \rangle_{\Gamma_1}.$$

Let's introduce the functional $\mathcal{F}$ as following :

$$\mathcal{F}(q) = \frac{r}{2}\|q\|^2_{\Gamma_1} - (\lambda^k + ru^k, q)_{\Gamma_1} + cts,$$

Now, we can rewrite (3.20) using the definition of $\mathcal{F}$, with an equivalent way as given in the following minimization problem

$$\begin{cases} \text{Find } w^k \in L^2\left(\Gamma_1\right)^3, & \text{such that} \\ \mathcal{F}(w^k) \leq \mathcal{F}(q), & \forall\, q \in L^2\left(\Gamma_1\right)^3 \\ w^k - g \leq 0 & \text{on } \Gamma_1 \end{cases} \tag{3.23}$$

Using the saddle-point theory, we may explicitly compute the solution to (3.23). The saddle-point equations are satisfied by the solution of (3.23).

$$r(w^k, q)_{\Gamma_1} - (ru^k + \lambda^k, q)_{\Gamma_1} + (\gamma^k, q)_{\Gamma_1} = 0, \quad \forall q \in L^2\left(\Gamma_1\right)^3, \tag{3.24}$$

$$(\gamma^k, w^k - g)_{\Gamma_1} = 0 \tag{3.25}$$

where $\gamma^k \geq 0$ is the Lagrange multiplier. Since $w^k - g \leq 0$ and $\gamma^k \geq 0$, the condition of the equation (3.25) is equivalent to the statement that γ^k may not vanish at a point of Γ_1 if the corresponding constraint is active, i.e. $w^k - g = 0$.
By few manipulations in equation (3.24), we deduce that

$$w^k = \frac{1}{r}\left(\lambda^k + ru^k - \gamma^k\right).$$

Taking this result and substituting it with the condition (3.25), we obtain

$$\left(\gamma^k, \frac{1}{r}\left(\lambda^k + ru^k - \gamma^k\right) - g\right)_{\Gamma_1} = 0 \tag{3.26}$$

If $\gamma^k > 0$, we must have

$$\lambda^k_c + ru^k_n - \gamma^k - rg = 0.$$

Then, the Lagrange multiplier can find by

$$\gamma^k = \max\left(0, \lambda^k_c + r\left(u^k_n - g\right)\right). \tag{3.27}$$

Substituting (3.27) into (3.26), we get the solution of (3.20)

$$w^k = u^k + \frac{1}{r}\left[\lambda^k - \left(\lambda^k + r\left(u^k - g\right)\right)^+\right], \tag{3.28}$$

where $x^+ = \max(0, x)$.
The new algorithm is generated by substituting the result (3.22),(3.28) in algorithm (1) and it is given as follows:

Algorithm 2 The updated Uzawa algorithm

Initialization $k = 0$. w^{-1} and λ^0 are given.

Iteration $k \geq 0$. **While stopping criterion**
Compute successively u^k, w^k and λ^{k+1} as follows :

1. u^k by:
$$\widetilde{a}\left(u^k, v\right) + r(u^k, v)_{\Gamma_1} = \widetilde{f}(v) + (rw^{k-1} - \lambda^k, v)_{\Gamma_1}, \quad \forall v \in V. \tag{3.29}$$
2. w^k by :
$$w^k = u^k + \frac{1}{r}\left[\lambda^k - \left(\lambda^k + r\left(u^k - g\right)\right)^+\right] \tag{3.30}$$
3. Update the Lagrange multiplier
$$\lambda^{k+1} = \lambda^k + r\left(u^k - w^k\right) \tag{3.31}$$

Running the algorithm (2) until the relative error on u^k, w^k is sufficiently "small", i.e.

$$\frac{\left\|u^k - u^{k-1}\right\|_\Omega^2 + \left\|w^k - w^{k-1}\right\|_{\Gamma_1}^2}{\left\|u^k\right\|_\Omega^2 + \left\|w^k\right\|_{\Gamma_1}^2} < \epsilon^2.$$

where ϵ is very small and goes to zero while it will be fixed in the value 10^{-4} for getting good results.

4. Numerical examples

$$\begin{cases} S_t - \triangle S = -\beta SI & \text{in } \mathcal{Q}, \\ I_t - \triangle I = -\beta SI - \gamma I & \text{in } \mathcal{Q}, \\ R_t - \triangle R = \gamma I & \text{in } \mathcal{Q}, \\ \frac{\partial S}{\partial \nu} = 0, \frac{\partial R}{\partial \nu} = 0 & \text{on } \Sigma, \\ \frac{\partial I}{\partial \nu} \leq 0,\ I \leq g_I \text{ and } (I - g_I)\frac{\partial I}{\partial \nu} = 0 & \text{on } \Sigma_1, \\ \frac{\partial I}{\partial \nu} = 0, & \text{on } \Sigma \backslash \Sigma_1, \\ S(x,0) = S_0(x), I(x,0) = I_0(x), R(x,0) = R_0(x) & \text{in } \Omega. \end{cases} \tag{4.1}$$

Consider the domain $\Omega = [0,1] \times [0,1]$ with boundary divided as follows: $\Gamma_1 = \{0\} \times [0,1]$ as a Signorini boundary, and the Neumann boundary will be the rest of the boundary $\Gamma_2 = [0,1] \times \{0,1\} \cup \{1\} \times [0,1]$. Let us define a finite element discretization of a domain Ω using the triangulation with a uniform space step $dx = dy = 1/32$ as given in the figure 2, this discretization has 1024 points and 1922 triangles elements.

Now let us define our parameters as follows. The rate infection $\beta = 0.7\ day^{-1}$, the recovery rate $\gamma = 0.1\ day^{-1}$, time step $dt = 0.05$. The initial conditions are given as follows $S_0 = 9\ individual/km^2, R_0 = 0\ individual/km^2$, and $I_0 = 1\ individual/km^2$ which means that the initial population in Ω is 10240 individuals. The penalty parameter in the algorithm 2 is chosen like $r = 10^{-3}$, the Signorini threshold g_I is a constant function which takes the value $g_I = 2$.

In the case of the susceptible compartment, the observations in all figures indicate that the epidemic is starting and spreading quickly since β is truly significant. For example, in the first 5 days figure 3(a) reveals that the number of susceptible individuals decrease quickly from 9 *individual* in every km^2 to 2 $individual/km^2$. Figures 3(b)-3(c) confirm our observations, proving that all susceptible individuals being infected, and the number of susceptible individuals tend to vanish after 300 days as seen in figure 3(d).

In the case of the infected compartment, it's evident that during the early days of the epidemic, the infected people are beginning to spread since the infection rate β is higher than the recovery rate γ. They

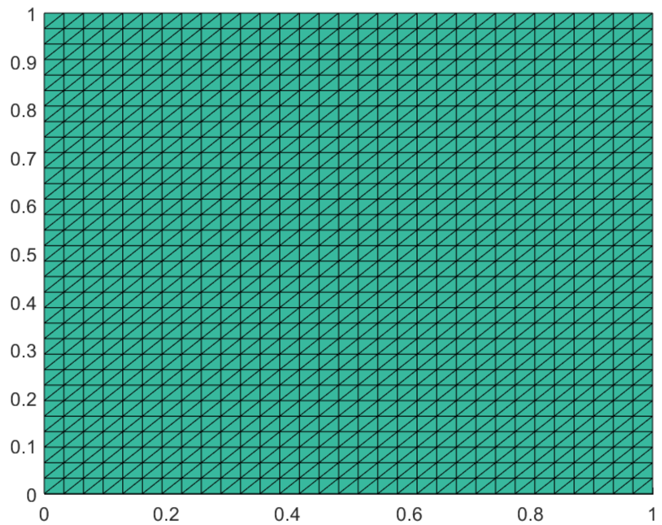


Figure 2: The domain Ω

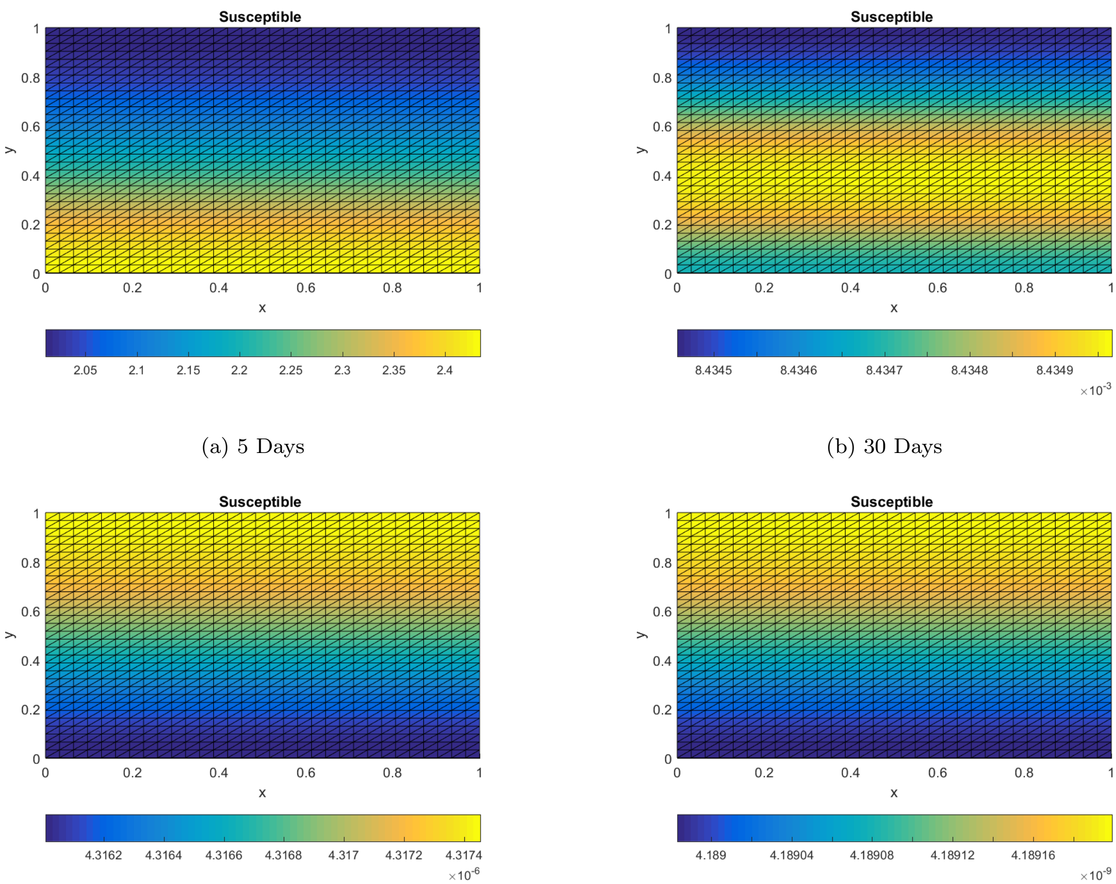
are less than or equal to the threshold g_I in the Signorini boundary Γ_1, as shown in figure 4(a). After reaching the threshold, it is noted that the infected individuals begin to leave the domain Ω, though the average of infected remains significant, as shown in figure 4(b), this will be continued until they become less than the threshold g_I in Γ_1. It occurs at the same time that they start to recover but the other infected individuals are still moving. If this diffusion assists infected people in reaching the threshold at the Signorini boundary, they will be forced to leave the domain, resulting in a rapid decline in the number of infected. These observations are similar to the results illustrated in figures 4(c) - 4(d).

In the case of recovered individuals, since the standard Model SIR is based on the concept of immunity, the results in figure 5(a) confirm it, and it can be said that after 5 days, recovery begins shortly after the onset of the disease, but with a low and slow percentage due to the rate of infection β is bigger than the recovery rate γ, the recovering wave follows the spread of the epidemic in all domain, specifically in the Signorini boundary (see figure 5(b), figure 5(c) illustrates how its importance increases and its value grows. Normally, all the infected individuals who don't leave the domain will be recovered as figure 5(d) proves.

Figure 6(b) clarifies the thoughts and results obtained from the evolution in each compartment, as we can see there is a decrease in the total number of the population after infected individuals leave the domain a result of reaching the threshold g_I. At the same time, the recovery operator becomes more important to save more infected from leaving the domain until the number of infected individuals is disappeared and our system enters stability mode as shown in the figure 6(a).

Conclusion

To sum up this work, we have shown results on the existence of solutions for nonlinear variational inequalities with typical operators define on reflexive and separable Banach spaces and verify pseudomonotone property and the condition $(S)_+$ which gives us the strong convergence, all this is based on Galerkin approach. Also, we introduced the Signorini condition boundary which seems to be the first time included in any epidemiology problem, by adding this type of boundary to the spatio-temporal SIR which is defined as a reaction-diffusion system, and leads to a variational inequalities and give us a chance to validate the results of the framework and open the door for the variational inequalities in this domain, to continue with more contributions in this direction.



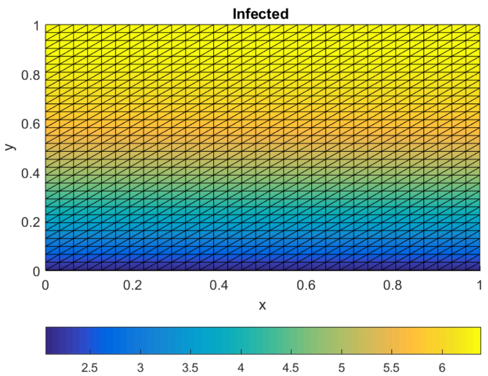
(a) 5 Days

(b) 30 Days

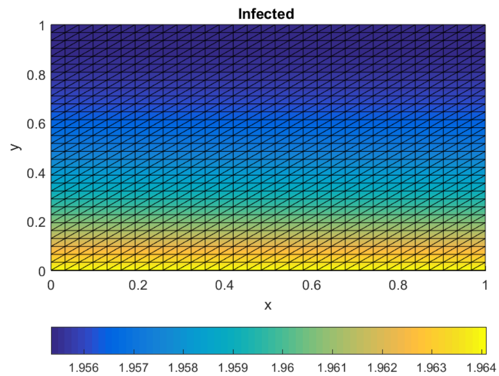
(c) 100 Days

(d) 300 Days

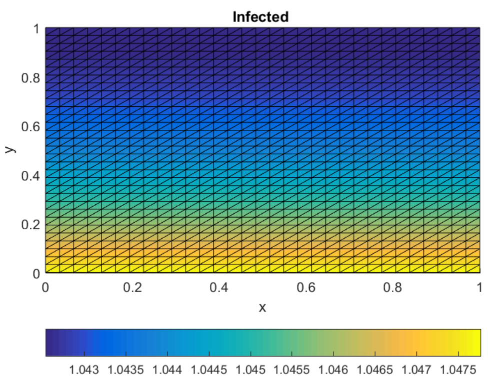
Figure 3: The diffusion of susceptible individuals



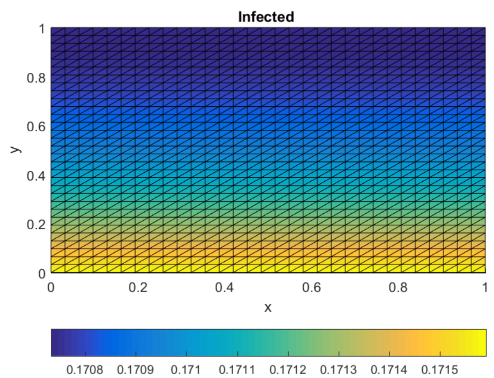
(a) 5 Days



(b) 30 Days

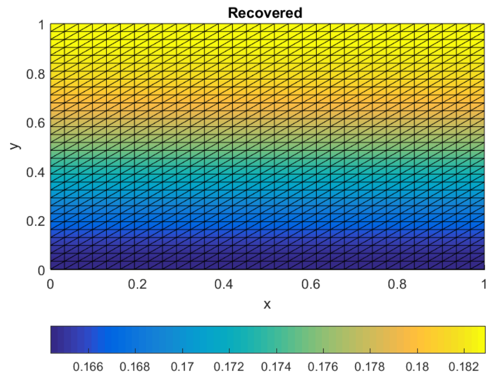


(c) 100 Days

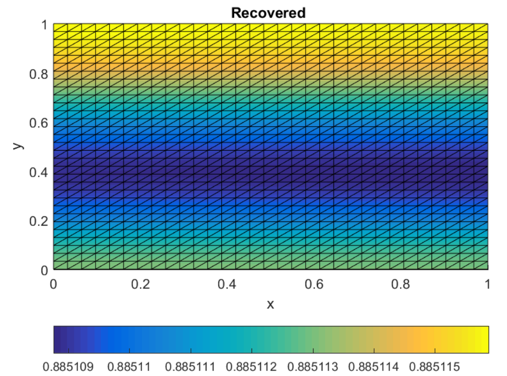


(d) 300 Days

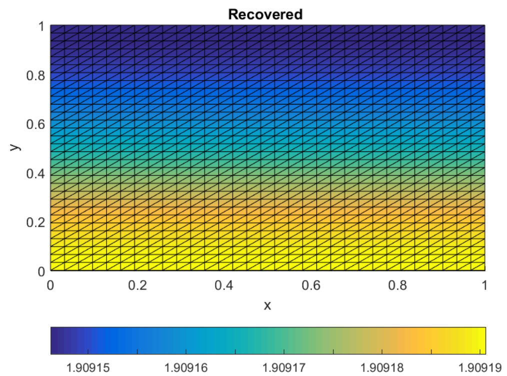
Figure 4: The diffusion of infected individuals



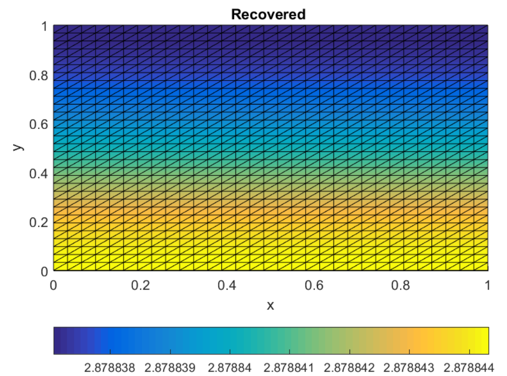
(a) 5 Days



(b) 30 Days

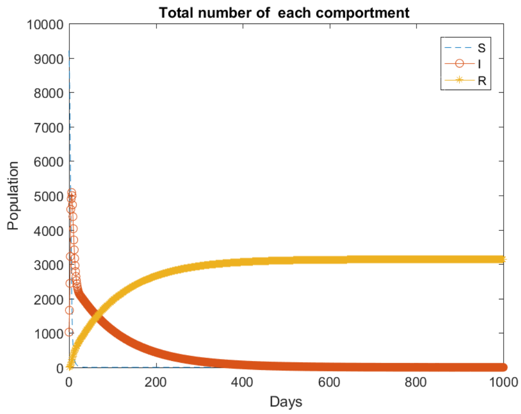


(c) 100 Days

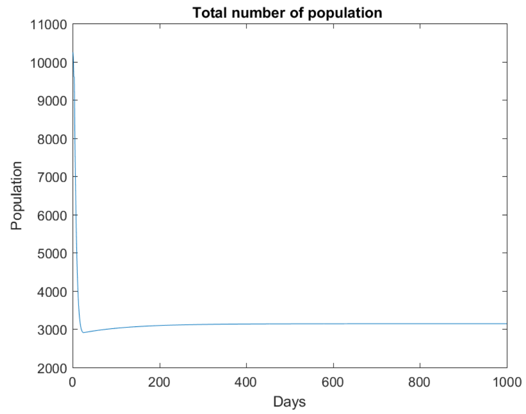


(d) 300 Days

Figure 5: The diffusion of recovered individuals



(a) The total number for each compartment



(b) The total number of population

Figure 6: Variation of the population after 1000 days

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study

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