Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.67863

Existence of Solution of Implicit type Initial Value Fractional Dynamic Equation on Time Scales

Bikash Gogoi, Utpal Kumar Saha *, Bipan Hazarika

ABSTRACT: In this paper we inquire into the existence and uniqueness theorem to the initial value non-linear implicit type fractional dynamic equation by using the newly developed Caputo nabla (∇) fractional derivative operator. The existence of the solution is based on Schauder's fixed point theorem and Banach contraction theorem. One example has been provided to justify our findings.

Keywords: Fractional dynamic equation, Caputo ∇- fractional derivative and Riemann-Liouville ∇-fractional integro-differential equation, Schauder's fixed point theorem, Arzelà-Ascoli theorem on time scale, Banach contraction theorem.

Contents

1. Introduction

The main objectives of this paper is to introduce the topic of dynamic equation which is the unification of differential and difference equation in a single domain called time scale, denoted by $\mathbb{T}$. The topic was first developed by Stefan Hilger (see [6,7]) in the year of 1988 in his Ph.D. work supervised by Bernd Aulbach. The concept of dynamic equation is useful for modeling both continuous and discrete phenomena simultaneously in one domain. For example, the species "Periodical Cicada" lives as moth for 13 years or 17 years and then as mature for 28 days. In this particular case the problem should be considered in the time scale domain. For the given species we need the following time scale $D = \bigcup_{J=0}^{\infty}\big[J(13y+28d), J(13y+28d)+28d\big]$ or $\bigcup_{J=0}^{\infty}\big[J(17y+28d), J(17y+28d)+28d\big]$, where $d =$ days, $y =$ years and $J \in \mathbb{N}\cup\{0\}$. The study of fractional dynamic equation on time scales may find tremendous applications in the field of applied sciences, biological sciences and engineering areas. For further details of time scale we refer the reader to see [1,5,8,10,11,12,21,22] and the references cited therein.

The study of fractional order calculus gives more accurate results than the ordinary calculus in real world problem. The combination of fractional calculus in time scale calculus extends the concept of ordinary calculus and gives a bright direction in the branch of mathematical science. The combination was introduced for the first time in the year 2012 as a dissertation topic of N. R. O. Bastos and his Ph.D supervisor D. F. M. Torres. In the literature an ample amount of works can be found on fractional linear and non-linear dynamic equation involving different operators such as Caputo, Riemann-Liouville, Caputo-Hadamard and many more. For convenience we refer the readers to go through [2,3,13,14,15, 18,19,9,20,23,24,25] and the references cited therein. Since real situations often deal with the nonlinear differential equation, it is worthwhile to study the varieties of nonlinear fractional dynamic equation. Motivated by the above work and the new definition of Caputo ∇- derivative developed in [21], here

* Corresponding author

Submitted April 12, 2023. Published July 11, 2025
2010 *Mathematics Subject Classification:* 26E70, 34N05, 26A33.

we study the implicit type initial value fractional dynamic equation (IVFDE) involving the Caputo ∇-derivative of the type

$$\begin{cases} {}^{C}D_0^{\psi}x(t) = \mathcal{H}\big(t, x(t), {}^{C}D_0^{\psi}x(t)\big), & \psi \in \mathbb{R} \\ x(t)|_{t=0} = x_0, & x_0 \in \mathbb{R}. \end{cases} \tag{1.1}$$

For $t \in \mathcal{J}_T, \mathcal{J}_T = [0,T] \cap \mathbb{T}_{\mathcal{K}}$, $T \in \mathbb{T}^+$ and $\mathcal{H} : \mathcal{J}_T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a left-dense (ld) continuous function, to be discussed later on.

The rest of the paper is organized as follows: In section 2, we have highlighted some auxiliary results related to fractional dynamic equation on time scales. In section 3, we have presented the existence and uniqueness of the IVFDE (1.1). In section 4, we have given an example related to our main findings which make the manuscript easier to understand and the conclusion part of the paper is presented in section 5.

2. Preliminaries

Definition 2.1 [21] Time scale $\mathbb{T}$ is a non empty closed subset of $\mathbb{R}$. For connectedness of $\mathbb{T}$, there are two jump operators called backward and forward jump operator. Let $\rho : \mathbb{T} \to \mathbb{R}$, defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},\ \forall t \in \mathbb{T},$$

then $\rho(t)$ is a backward jump operator and

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},\ \forall t \in \mathbb{T},$$

is a forward jump operator. Later if $t > \inf \mathbb{T}$ then $t \in \mathbb{T}$ is said to be left scattered and left dense if $\rho(t) < t$ and $\rho(t) = t$, respectively. If d is the possible minimum right scattered, then we write $\mathbb{T}_{\mathcal{K}} = \mathbb{T} \setminus d$, else $\mathbb{T}_{\mathcal{K}} = \mathbb{T}$.
Since our manuscript is concerned with the nabla derivative, the results related to the delta derivative is omitted and so is also forward jump operator.

Definition 2.2 [12] Let $v : \mathcal{J}_T \to \mathbb{R}$. If at the left dense point in $\mathcal{J}_T$, v is continuous and right sided limit exists at right dense points of $\mathcal{J}_T$, then v is said to be ld- continuous.
The set $\xi(\mathcal{J}_T, \mathbb{R})$ is used to denote all ld- continuous functions from $\mathcal{J}_T$ to $\mathbb{R}$.

Definition 2.3 [12] Let $x \in \xi(\mathbb{T}, \mathbb{R})$, if $X^{\nabla}(t) = x(t)$, the nabla integral of x is presented by

$$\int_{t_0}^{t} x(v)\nabla v = X(t) - X(t_0),$$

where $t_0 \in \mathbb{T}$.

Definition 2.4 [21] Consider a ld- continuous function $v : \mathbb{T}_{\mathcal{K}} \to \mathbb{R}$, the nabla derivative of v(t) of order $\psi \in \mathbb{R}$ is

$$v^{(\psi)}(t) = \begin{cases} \frac{v(t)-v(\rho(t))}{\mu(t)^{\psi}}, & \text{if } \rho(t) < t \\ \lim_{s\to t} \frac{v(t)-v(s)}{(t-s)^{\psi}}, & \text{if } \rho(t) = t, \end{cases}$$

where μ denotes the backward graininess function. Also for $t \in \mathbb{T}$, value of $\mu(t) = t - \rho(t)$.

Definition 2.5 [21, Riemann-Liouville integro differential equation] Let $x : \mathbb{T}_{\mathcal{K}} \to \mathbb{R}$ be a ld- continuous function, then the Riemann-Liouville integro-differential equation of order $\psi \in \mathbb{R}$ is given by

$${}^{RL}D_a^{-\psi}x(t) = \mathbb{I}_a^{\psi}x(t) = \frac{1}{\Gamma(\psi)}\int_a^t \big(t - \rho(v)\big)^{\psi-1}x(v)\nabla v.$$

For any two ld continuous function $x(t)$ and $y(t)$, the operator $\mathbb{I}^{\psi}$ is linear, that is

$$\mathbb{I}^{\psi}x(t) - \mathbb{I}^{\psi}y(t) = \mathbb{I}^{\psi}\big(x(t) - y(t)\big)$$

Definition 2.6 [21, Higher order ∇- derivative] Assume $\mathcal{X} : \mathbb{T}_{\mathcal{K}} \to \mathbb{R}$ is a ld- continuous function on a time scale $\mathbb{T}$. The second order nabla derivative $\mathcal{X}_{\nabla\nabla} = \mathcal{X}_{\nabla}^{(2)}$ can be defined, provided $\mathcal{X}_{\nabla}$ is differentiable on ${\mathbb{T}_{\mathcal{K}}}^{(2)} = \mathbb{T}_{\mathcal{K}\mathcal{K}}$ with derivative $\mathcal{X}_{\nabla}^{(2)} = (\mathcal{X}_{\nabla})_{\nabla} : {\mathbb{T}_{\mathcal{K}}}^{(2)} \to \mathbb{R}$. Similarly, the n^{th} order nabla derivative we get $\mathcal{X}_{\nabla}^{(n)} : {\mathbb{T}_{\mathcal{K}}}^{n} \to \mathbb{R}$, it is attained by cut out n right scattered left end points from $\mathbb{T}$.

Definition 2.7 [21, Caputo ∇- derivative] Let $\mathcal{X}$ be a ld- continuous function, where $\mathcal{X}_{\nabla}^{(n)}$ exists on ${\mathbb{T}_{\mathcal{K}}}^{n}$. Then the Caputo nabla derivative of order $\psi \in \mathbb{R}$ is defined by

$$^{C}D_{a}^{\psi}\mathcal{X}(t) = \frac{1}{\Gamma(n-\psi)}\int_{a}^{t}(t-\rho(u))^{n-\psi-1}\mathcal{X}_{\nabla}^{n}(u)\nabla(u),$$

where $n = [\psi] + 1$. In general if $\psi \in (0,1)$ then,

$$^{C}D_{a}^{\psi}\mathcal{X}(t) = \frac{1}{\Gamma(1-\psi)}\int_{a}^{t}\big(t-\rho(u)\big)^{-\psi}\mathcal{X}_{\nabla}(u)\nabla u.$$

Definition 2.8 [11] Let $\mathcal{G}$ is a non empty convex and closed subset of a Banach space F and let $x : \mathcal{G} \to F$ is a continuous mapping such that $x(\mathcal{G})$ is relatively compact in F. Then the function x contain a fixed point in $\mathcal{G}$.

Definition 2.9 For $x \in \xi\big(\mathcal{J}_T, \mathbb{R}\big) \cap \xi^1\big(\mathcal{J}_T, \mathbb{R}\big)$, $x(t)$ is a solution of the IVFDE (1.1), if $x(t)$ satisfies the IVFDE (1.1) everywhere for each $t \in \mathcal{J}_T$.
The set $\xi^1\big(\mathcal{J}_T, \mathbb{R}\big)$ is used to denote all ld-continuous nabla derivable function from $\mathcal{J}_T$ to $\mathbb{R}$.

Definition 2.10 [12, Arzelà-Ascoli theorem]Let $\mathcal{D} \subseteq \xi\big(\mathcal{J}_T, \mathbb{R}\big)$. Then $\mathcal{D}$ is said to be relatively compact if and only if $\mathcal{D}$ is bounded and equicontinuous.

Theorem 2.1 *[11] A function $u : \mathcal{X} \to \mathcal{Y}$ is a completely continuous mapping, if $B \subseteq \mathcal{X}$, such that B is bounded then $u(\mathcal{X})$ is a relatively compact in $\mathcal{X}$.*

Proposition 2.1 *[12] Let $\xi\big(\mathcal{J}_T, \mathbb{R}\big)$ is a set of all ld- continuous functions from the time scale domain $\mathcal{J}_T$ to $\mathbb{R}$.*
The set $\xi\big(\mathcal{J}_T, \mathbb{R}\big)$ formed a Banach space equipped by the norm

$$\|x\|_{\xi} = \sup_{t\in\mathcal{J}_T}|x(t)|.$$

3. Main Findings

We can relate the IVFDE (1.1) as a mathematical model to analyze the observed periodic outbreaks of certain infectious diseases. If we involved toxic effect in the population of insect, then the change of population in that particular area can be modeled with the situation, where $x(t)$ is the insect population at a particular time t. ${}^{C}D_{0}^{\psi}x(t)$ is the rate of change of the population with respect to t and $x(0) = x_0$ is the population of insect at an initial time in that area.

For the existence of solution of the IVFDE (1.1), we reduce the equation into a fixed point problem and we claim that the solution IVFDE (1.1) satisfies the Voltera integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\psi)}\int_{0}^{t}\big(t-\rho(v)\big)^{\psi-1}\mathcal{H}\big(t, x(v), {}^{C}D^{\psi}x(v)\big)\nabla v. \tag{3.1}$$

For showing our findings, we need to present the following results.

Proposition 3.1 *[4] Let $r, v \in \mathbb{T}$ such that $r < v$. Consider a ld- continuous function g on $\mathcal{J}_T$, then*

$$\begin{aligned}\int_{\rho(r)}^{v} g(t)\nabla t &= \int_{\rho(r)}^{r} g(t)\nabla t + \int_{r}^{\rho(v)} g(t)\nabla t + \int_{\rho(v)}^{v} g(t)\nabla t \\ &= [r-\rho(r)]g(r) + [v-\rho(v)]g(v) + \int_{r}^{\rho(v)} g(t)\nabla t.\end{aligned}$$

Theorem 3.1 *Let a ld- continuous function g on a time scale interval $[r, v]_T$, and if the function $\mathcal{G}$ is extension of g in real interval $[r, v]$, such that*

$$\mathcal{G}(s) = \begin{cases} g(s), & \text{if } s \in \mathbb{T} \\ g(t), & \text{if } t \in \big(\rho(t), t\big) \notin \mathbb{T}, \end{cases}$$

then

$$\int_r^v g(t)\nabla t \leq \int_r^v \mathcal{G}(t)dt.$$

Proof: Let $a \in [r, v]_T$, such that $\rho(a) < a$. From the Proposition 3.1, we obtain

$$\int_{\rho(a)}^a g(t)\nabla t = [a - \rho(a)]g(a).$$

Since the function $g(t)$ is increasing, so it's extension $\mathcal{G}(t)$ is also increasing. Using the mean value theorem of integration, we get

$$[a - \rho(a)]\mathcal{G}(\rho(a)) \leq \int_{\rho(a)}^a \mathcal{G}(t)dt \leq [a - \rho(a)]\mathcal{G}(a)$$

$$[a - \rho(a)]g(\rho(a)) \leq \int_{\rho(a)}^a \mathcal{G}(t)dt \leq [a - \rho(a)]g(a)$$

and hence,

$$\int_{\rho(a)}^a g(t)\nabla t \leq \int_{\rho(a)}^a \mathcal{G}(t)dt.$$

Now if the interval $[r, v]_T$ has only one left scattered point say $a(r < a)$, then by using the Proposition 3.1 and the above results, we obtain

$$\begin{aligned} \int_r^v g(t)\nabla t &= \int_r^{\rho(a)} g(t)\nabla t + \int_{\rho(a)}^a g(t)\nabla t + \int_a^v g(t)\nabla t \\ &\leq \int_r^{\rho(a)} \mathcal{G}(t)dt + \int_{\rho(a)}^a \mathcal{G}(t)dt + \int_a^v \mathcal{G}(t)dt. \end{aligned}$$

Similarly we can extend the above result for n left scattered point on $[r, v]$, and thus obtain

$$\int_r^v g(t)\nabla t \leq \int_r^v \mathcal{G}(t)dt.$$

□

Lemma 3.1 *Any function $h \in \xi\big(\mathcal{J}_T, \mathbb{R}\big) \cap \xi^1\big(\mathcal{J}_T, \mathbb{R}\big)$ is a solution of (1.1), if $h \in \xi\big(\mathcal{J}_T, \mathbb{R}\big)$ satisfy the following Voltera integral equation*

$$x(t) = x_0 + \frac{1}{\Gamma(\psi)} \int_0^t \big(t - \rho(s)\big)^{\psi - 1} \mathcal{H}\big(s, x(s), {}^C D_0^{\psi} x(s)\big)\nabla s, \tag{3.2}$$

for $\psi \in \mathbb{R}$.

Proof: Consider ${}^C D^{\psi} x(t) = h(t)$ with $x(t)|_{t=0} = x_0$, in view of equation (1.1), one can have

$$x(t) = x_0 + \frac{1}{\Gamma(\psi)} \int_0^t \big(t - \rho(s)\big)^{\psi - 1} h(s)\nabla s. \tag{3.3}$$

Then applying Definition 2.5 in (3.3) we obtain

$$x(t) = x_0 + \mathbb{I}^{\psi} h(t).$$

Moreover, the IVFDE (1.1) is equivalent to

$$h(t) = \mathcal{H}\big(t, x_0 + \mathbb{I}^{\psi} h(t), h(t)\big). \tag{3.4}$$

Later, we create some hypotheses that will be used to support our principal result.

(L1) The mapping $\mathcal{H} : \mathcal{J}_T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is ld- continuous.

(L2) There exist $E_1 \in \xi(\mathcal{J}_T, \mathbb{R})$ and two constants E_2 and E_3 such that $E_2 > 0$ and $0 < E_3 < 1$ satisfying

$$|\mathcal{H}(t, h_1, h_2)| \leq |E_1(t)| + E_2|h_1| + E_3|h_2|,$$

for all $t \in \mathcal{J}_T$ and $(h_1, h_2) \in \mathbb{R} \times \mathbb{R}$.

(L3) There exist two constants $F_1 > 0$ and $0 < F_2 < 1$ such that

$$|\mathcal{H}(t, h_1, h_2) - \mathcal{H}(t, r_1, r_2)| \leq F_1|h_1 - r_1| + F_2|h_2 - r_2|,$$

for all $t \in \mathcal{J}_T$ and $h_i, r_i \in \mathbb{R} \times \mathbb{R}$ for $i = 1, 2$.

□

Proposition 3.2 *For $t \in \mathcal{J}_T$, $T \in \mathbb{R}^+$ and $x \in \xi\big(\mathcal{J}_T, \mathbb{R}\big)$, the operator $\mathbb{I}^{\psi} x(t)$ for $\psi \in \mathbb{R}$ is bounded with respect to the norm defined in the Proposition 2.1.*

Proof: Since, $(t - \rho(v))^{\psi - 1}$ is an increasing monotonic function, so by using the Theorem 3.1 we get

$$\int_0^t (t - \rho(v))^{\psi - 1} \nabla v \leq \int_0^t (t - v)^{\psi - 1} dv. \tag{3.5}$$

Now from the Definition 2.5 we obtain

$$\begin{aligned} \|\mathbb{I}^{\psi} x\|_{\xi} &= \sup_{t \in \mathcal{J}_T} |\mathbb{I}^{\psi} x(t)| \\ &\leq \Big| \frac{1}{\Gamma(\psi)} \int_0^t (t - \rho(v))^{\psi - 1} x(v) \nabla v \Big| \\ &< \frac{1}{\Gamma(\psi)} \int_0^t (t - s)^{\psi - 1} dv \sup_{v \in \mathcal{J}_T} |x(v)| \\ &\leq \frac{T^{\psi}}{\Gamma(\psi + 1)} \|x\|_{\xi}. \end{aligned}$$

Thus we get

$$\|\mathbb{I}^{\psi} x\|_{\xi} \leq \frac{T^{\psi}}{\Gamma(\psi + 1)} \|x\|_{\xi}, \text{ which is bounded.}$$

□

Proposition 3.3 *For any $h(t), r(t) \in \xi\big(\mathcal{J}_T, \mathbb{R}\big), t \in \mathcal{J}_T$, where $T \in \mathbb{R}^+$. Then for $\psi \in \mathbb{R}$ the operator $\mathbb{I}^{\psi}$ is linear with respect to the norm defined in the Proposition 2.1.*

Proof: Using the Definition 2.5 we get

$$\begin{aligned}\|\mathbb{I}^{\psi}h-\mathbb{I}^{\psi}r\|_{\xi} &= \sup_{t\in\mathcal{J}_T}|\mathbb{I}^{\psi}h(t)-\mathbb{I}^{\psi}r(t)|\\ &\le |\mathbb{I}^{\psi}\big(h(t)-r(t)\big)|\\ &\le \frac{T^{\psi}}{\Gamma(\psi+1)}\|h-r\|_{\xi}.\end{aligned}$$

Thus the operator $\mathbb{I}^{\psi}$ is linear. □

For sufficient conditions of the existence of the solution of IVFDE (1.1), we will use the conditions of the Definition 2.8.

Theorem 3.2 *Consider that (L1) – (L3) hold and if*

$$\frac{E_2T^{\psi}}{\Gamma(\psi+1)}+E_3<1,$$

for $t\in\mathcal{J}_T$. then the IVFDE (1.1) must contain a solution.

Proof: Let ${}^{C}D^{\psi}x(t)=h(t)$ for $t\in\mathcal{J}_T$. Using (3.4) we obtain

$$\begin{aligned}|h(t)| =&\big|\mathcal{H}\big(t,x_0+\mathbb{I}^{\psi}h(t),h(t)\big)\big|\\ \le&|E_1(t)|+E_2|x_0|+E_2|\mathbb{I}^{\psi}h(t)|+E_3|h(t)|\\ \le&|E_1(t)|+E_2|x_0|+\frac{E_2T^{\psi}}{\Gamma(\psi+1)}|h(t)|+E_3|h(t)|\\ \le&\frac{|E_1(t)|+E_2|x_0|}{1-\big(\frac{E_2T^{\psi}}{\Gamma(\psi+1)}+E_3\big)}\\ =&\,\gamma,\end{aligned}$$

that is

$$|h(t)|\le\gamma \tag{3.6}$$

where

$$\gamma=\frac{|E_1(t)|+E_2|x_0|}{1-\big(\frac{E_2T^{\psi}}{\Gamma(\psi+1)}+E_3\big)}.$$

Taking the norm of $\xi\big(\mathcal{J}_T,\mathbb{R}\big)$ on both side of (3.6) we get

$$\begin{aligned}\|h\|_{\xi} &\le \frac{\|E_1\|_{\xi}+E_2\|x_0\|_{\xi}}{1-\big(\frac{E_2T^{\psi}}{\Gamma(\psi+1)}+E_3\big)}\\ &=\mathcal{L},\end{aligned} \tag{3.7}$$

where

$$\mathcal{L}=\frac{\|E_1\|_{\xi}+E_2\|x_0\|_{\xi}}{1-\big(\frac{E_2T^{\psi}}{\Gamma(\psi+1)}+E_3\big)}. \tag{3.8}$$

Now consider a set $\mathcal{M}_{\mathcal{L}}=\{h\in\xi\big(\mathcal{J}_T,\mathbb{R}\big):\|h\|_{\xi}\le\mathcal{L}\}\subseteq\xi\big(\mathcal{J}_T,\mathbb{R}\big)$ and a mapping $\mathcal{Z}:\mathcal{M}_{\mathcal{L}}\to\mathcal{M}_{\mathcal{L}}$ such that

$$\mathcal{Z}(h)=\mathcal{H}\big(t,x_0+\mathbb{I}^{\psi}h(t),h(t)\big). \tag{3.9}$$

Later, for existence of the solution, one has to proved that the mapping $\mathcal{Z}:\mathcal{M}_{\mathcal{L}}\to\mathcal{M}_{\mathcal{L}}$ is completely continuous. Thus, for employing the Arzelà- Ascoli theorem on time scale, the following steps are necessary to prove:

Step 1: The mapping $\mathcal{Z}:\mathcal{M}_{\mathcal{L}}\to\mathcal{M}_{\mathcal{L}}$ is continuous.
Consider a sequence (h_n) such that $h_n\to h$, then for $t\in\mathcal{J}_T$, we obtain

$$\begin{aligned}
&\|\mathcal{Z}h_n-\mathcal{Z}h\|_\xi\\
&=\sup_{t\in\mathcal{J}_T}|\mathcal{Z}(h_n(t))-\mathcal{Z}(h(t))|\\
&\le\big|\mathcal{H}\big(t,x_0+\mathbb{I}^\psi(h_n(t)),h_n(t)\big)-\mathcal{H}\big(t,x_0+\mathbb{I}^\psi(h(t)),h(t)\big)\big|\\
&\le F_1\big|\mathbb{I}^\psi h_n(t)-\mathbb{I}^\psi h(t)\big|+F_2|h_n(t)-h(t)|\\
&\le\frac{F_1T^\psi}{\Gamma(\psi+1)}|h_n(t)-h(t)|+F_2|h_n(t)-h(t)|\\
&\le\frac{F_1T^\psi}{\Gamma(\psi+1)}\|h_n-h\|_\xi+F_2\|h_n-h\|_\xi.
\end{aligned}$$

Thus we get

$$\|\mathcal{Z}h_n-\mathcal{Z}h\|_\xi\le\frac{F_1T^\psi}{\Gamma(\psi+1)}\|h_n-h\|_\xi+F_2\|h_n-h\|_\xi. \tag{3.10}$$

Since $\|h_n-h\|\to0$ as $n\to\infty$, we get $\|\mathcal{Z}(h_n)-\mathcal{Z}(h)\|_\xi\to0$, as $n\to\infty$. Thus the mapping $\mathcal{Z}:\mathcal{M}_{\mathcal{L}}\to\mathcal{M}_{\mathcal{L}}$ is continuous.
Step 2: The mapping $\mathcal{Z}:\mathcal{M}_{\mathcal{L}}\to\mathcal{M}_{\mathcal{L}}$ is uniformly bounded. For $h\in\mathcal{M}_{\mathcal{L}}$, we have

$$\begin{aligned}
\|\mathcal{Z}h\|_\xi&=\sup_{t\in\mathcal{J}_T}|\mathcal{Z}(h(t))|\\
&\le|\mathcal{H}(t,x_0+\mathbb{I}^\psi h(t),h(t)|\\
&\le|E_1(t)|+E_2|x_0+\mathbb{I}^\psi h(t)|+E_3|h(t)|\\
&\le|E_1(t)|+E_2|x_0|+E_2|\mathbb{I}^\psi h(t)|+E_3|h(t)|\\
&<\|E_1\|_\xi+E_2|x_0|+E_2\Big|\frac{1}{\Gamma(\psi)}\int_0^t(t-\rho(s))^{\psi-1}h(s)\nabla s\Big|+E_3||h||_\xi\\
&\le\|E_1\|_\xi+E_2\|x_0\|_\xi+\frac{E_2T^\psi}{\Gamma(\psi+1)}\|h\|_\xi+E_3\|h\|_\xi\\
&\le\|E_1\|_\xi+E_2\|x_0\|_\xi+\Big(\frac{E_2T^\psi}{\Gamma(\psi+1)}+E_3\Big)\|h\|_\xi\\
&\le\mathcal{L},
\end{aligned}$$

that is

$$\|\mathcal{Z}h\|_\xi\le\mathcal{L}. \tag{3.11}$$

Thus from (3.8) we can say that the mapping $\mathcal{Z}$ is uniformly bounded.
Step 3: The mapping $\mathcal{Z}:\mathcal{M}_{\mathcal{L}}\to\mathcal{M}_{\mathcal{L}}$ is equicontinuous.
Let $h\in\mathcal{M}_{\mathcal{L}}$ and $t_1,t_2\in\mathcal{J}_T$ such that $t_1<t_2$, then we have

$$\begin{aligned}
\|(\mathcal{Z}h)(t_2)-(\mathcal{Z}h)(t_1)\|_\xi&=\sup_{t\in\mathcal{J}_T}|\mathcal{Z}(h(t_2))-\mathcal{Z}(h(t_1))|\\
&\le\big|\mathcal{H}\big(t_2,x_0+\mathbb{I}^\psi h(t_2)),h(t_2)\big)-\mathcal{H}\big(t_1,x_0+\mathbb{I}^\psi h(t_1)),h(t_1)\big)\big|\\
&\le\frac{F_1T^\psi}{\Gamma(\psi+1)}|h(t_2)-h(t_1)|+F_2|h(t_2)-h(t_1)|,
\end{aligned}$$

that is

$$\|(\mathcal{Z}h)(t_2)-(\mathcal{Z}h)(t_1)\|_\xi\le\frac{F_1T^\psi}{\Gamma(\psi+1)}|h(t_2)-h(t_1)|+F_2|h(t_2)-h(t_1)|.$$

when $t_2 \to t_1$, then $|h(t_2) - h(t_1)| \to 0$. So $\|(\mathcal{Z}h)(t_2) - (\mathcal{Z}h)(t_1)\|_\xi \to 0$. Thus the mapping $\mathcal{Z}$ is equicontinuous.

So as a consequences of Step 1 – Step 3 and using the Definition 2.10, we conclude that the mapping $\mathcal{Z}$ is continuous completely. Hence, by virtue of Schauder's fixed point theorem, the IVFDE (1.1) has a solution. □

Theorem 3.3 *Consider that the assumption (L1) hold, and if*

$$\tfrac{F_1 T^\psi}{\Gamma(\psi+1)} + F_2 < 1,$$

then the IVFDE (1.1) *consist a unique solution.*

Proof: For $h, r \in \mathcal{M}_\mathcal{L}$ and using the Propositions 3.2 and 3.3 we get

$$\begin{aligned}
&\|\mathcal{Z}(h) - \mathcal{Z}(r)\|_\xi \\
&= \sup_{t\in\mathcal{J}_T} |\mathcal{Z}(h(t)) - \mathcal{Z}(r(t))| \\
&\leq F_1|\mathcal{H}(t, x_0 + \mathbb{I}^\psi h(t), h(t)) - \mathcal{H}(t, x_0 + \mathbb{I}^\psi r(t), r(t))| \\
&\leq F_1\left|\mathbb{I}^\psi h(t) - \mathbb{I}^\psi r(t)\right| + F_2|h(t) - r(t)| \\
&\leq \frac{F_1 T^\psi}{\Gamma(\psi+1)}|h(t) - r(t)| + F_2|h(t) - r(t)| \\
&\leq \frac{F_1 T^\psi}{\Gamma(\psi+1)}\|h - r\|_\xi + F_2\|h - r\|_\xi \\
&\leq \Big(\frac{F_1 T^\psi}{\Gamma(\psi+1)} + F_2\Big)\|h - r\|_\xi,
\end{aligned}$$

that is

$$\|\mathcal{Z}(h) - \mathcal{Z}(r)\|_\xi \leq \Big(\frac{F_1 T^\psi}{\Gamma(\psi+1)} + F_2\Big)\|h - r\|_\xi.$$

Since $\frac{F_1 T^\psi}{\Gamma(\psi+1)} + F_2 < 1$, which is a contraction mapping, this hints the existence of the unique solution of the IVFDE (1.1). □

4. Example

Example 4.1 Consider an initial value problem on $\mathbb{T} = [0, 1] \cup [2, 3]$

$$\begin{cases} {}^C D^{(\psi)} x(t) = \frac{e^{-2t}}{e^{3t}+7}[1 + x(t) + {}^C D^{0.5} x(t)] \\ x(0) = 1 \end{cases} . \tag{4.1}$$

For $t \in [0, 3] \cap \mathbb{T}_\mathcal{K}$ and $\psi = \frac{1}{2}$, Now,

$$\mathcal{H}(t, x(t), {}^C D^{0.5} x(t)) = \frac{e^{-2t}}{e^{3t} + 7}[1 + x(t) + {}^C D^{0.5} x(t)],$$

which satisfies (L1). Again for ${}^C D^{0.5} x(t) = h(t),\ h \in C(\mathcal{J}_T, \mathbb{R})$ we get

$$\begin{aligned}
\mathcal{H}(t, x(t), h(t)) =& \frac{e^{-2t}}{e^{3t} + 7}[1 + x(t) + h(t)] \\
\leq& \frac{1}{8} + \frac{1}{8}|x(t)| + \frac{1}{8}|h(t)|,
\end{aligned}$$

or

$$\mathcal{H}(t, x(t), h(t)) \le \frac{1}{8} + \frac{1}{8}|x(t)| + \frac{1}{8}|h(t)|. \tag{4.2}$$

Which satisfies (L2) with $E_1 = \frac{1}{8},\ E_2 = \frac{1}{8},\ E_3 = \frac{1}{8}$. Again for ${}^{C}D^{0.5}x_i(t) = h_i(t),\ i = 1, 2$.

$$\begin{aligned}|\mathcal{H}(t, x_1, h_1) - \mathcal{H}(t, x_2, h_2)| &= \frac{e^{-2t}}{e^{3t} + 7}\big(|x_1 - x_2| + |h_1 - h_2|\big)\\ &\le \frac{1}{8}|x_1 - x_2| + \frac{1}{8}|h_1 - h_2|,\end{aligned}$$

that is

$$|\mathcal{Z}(t, u_2, v_2) - \mathcal{Z}(t, u_1, v_1)| \le \frac{1}{8}|u_2 - u_1| + \frac{1}{8}|v_2 - v_1|.$$

which satisfies (L3) with $F_1 = \frac{1}{8},\ F_2 = \frac{1}{8}$, Now putting these values in Theorem 3.3 we get

$$\begin{aligned}\frac{F_1 T^{\psi}}{\Gamma(\psi + 1)} + F_2 &= \frac{\frac{1}{8}T^{\frac{1}{2}}}{\Gamma(\frac{1}{2} + 1)} + \frac{1}{8}\\ &\le \frac{3^{\frac{1}{2}}}{8 \times 0.887} + \frac{1}{8}\\ &< 1,\end{aligned}$$

satisfies the Banach contraction principle. Hence the equation (4.1) has a solution which is unique.

5. Conclusion

In this manuscript, we have discussed the implicit type nonlinear fractional dynamic equation with an initial condition, which is the new concept in time scale context. Since we have taken the initial condition in terms of real numbers, so the Caputo ∇- fractional derivative is more useful than the Riemann- Liouville ∇- fractional derivative in practical field. If we take the initial conditions in terms of integer order, then the results of existence and uniqueness solution of the dynamic equation (1.1) will be same in both the cases.

As an extension of this research, the characteristics of the solutions such as dependency, stability analysis, positiveness are our future scope of study.

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Bikash Gogoi,
Department of Mathematics, Sibsagar University,
Sivasagar, 785665, Assam, India.
E-mail address: bikashg@sibsagaruniversity.ac.in

and

Utpal Kumar Saha,
Department of Basic and Applied Science,
National Institute of Technology Arunachal Pradesh,
India.
E-mail address: uksahanitap@gmail.com

and

Bipan Hazarika,
Department of Mathematics,
Gauhati University, Guwahati 781014, Assam,
India.
E-mail address: bh_gu@gauhati.ac.in