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S-r-ideals in commutative rings

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ABSTRACT: The rings considered in this article are commutative with identity. This article is motivated by the results proved by Visweswaran ([10]) on S-primary ideals. In this paper, we introduce the concept of S-r-ideal (resp., S-pr-ideal) of a commutative ring and study its properties. Let R be a commutative ring with $1 \neq 0$ and S be a multiplicatively closed subset of R. Let I be an ideal of R disjoint with S. We say that I is an S-r-ideal (resp., S-pr-ideal) of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in I$ with $Ann(a) = (0)$ implies that $sb \in I$ (resp., $sb \in \sqrt{I}$). We investigate many properties and characterizations of S-r-ideals (resp., S-pr-ideals). We discuss the form of S-r-ideals (resp., S-pr-ideals) in a finite direct product of rings. Furthermore, we study S-r-ideals (resp., S-pr-ideals) in Nagata's idealization ring. Our results allow us to construct original examples of S-r-ideals (resp., S-pr-ideals).

Key Words: S-r-ideal, S-pr-ideal, strongly S-pr-ideal.

Contents

1. Introduction

Throughout this paper, all considered rings are assumed to be commutative with identity $1 \neq 0$ and all ring homomorphisms are assumed to be unital. If A is a subring of B, we suppose that they have the same identity element. As usual, if R is a commutative ring, then $zd(R)$ denotes the set of zero divisors of R and $\text{Reg}(R) = R \setminus zd(R)$ is the set of its regular elements. Recall that a subset S of a ring R is called *multiplicative* if $1 \in S$, $0 \notin S$ and S is closed under multiplication. Note that any multiplicative subset of R satisfies the inclusion relations $\{1\} \subseteq S \subsetneq R$. Recall also that an ideal I of R is said to be *prime* if $I \neq R$ and whenever a and b are elements of R such that $ab \in I$, then $a \in I$ or $b \in I$. Note that I is a prime ideal of R if and only if $R \setminus I$ is a multiplicative subset of R. In [1], D. D. Anderson and E. Smith have defined a proper ideal I of R to be *weakly prime* if $0 \neq ab \in I$ implies $a \in I$ or $b \in I$. Some properties of weakly prime ideals have been settled. On the other hand, A. Hamed and A. Malek have introduced and investigated the concept of S-prime ideals which constitute a generalization of prime ideals (see [4]). More precisely, let R be a commutative ring, S a multiplicative subset of R and I an ideal of R disjoint from S. Then, I is called an *S-prime ideal* of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in I$, then $sa \in I$ or $sb \in I$. Note that if S consists of units of R, then the notions of S-prime and prime ideals coincide. Recall that an ideal I of R is said to be *primary* if for all $a, b \in R$, $ab \in I$ implies $a \in I$ or $b \in \sqrt{I}$. In [10] the author introduced and investigated the concept of S-primary ideals which constitute a generalization of primary ideals (the same notion can be found in [7]). More precisely, let R be a commutative ring, S a multiplicative subset of R and I an ideal of R disjoint from S. Then, I is called an *S-primary ideal* of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in I$, then $sa \in I$ or $sb \in \sqrt{I}$. In commutative algebra, r-ideal and its generalizations have an important role. There have been lots of studies on this issue (See [8, 9]). Recall that a proper ideal I is called an *r-ideal* (resp., *pr-ideal*) if $ab \in I$ with $Ann(a) = 0$ then $b \in I$ (resp., $b \in \sqrt{I}$). In [9], Uregen also studies a special class of pr-ideals fixing the power of element $b \in R$ in the above definition. A proper ideal I is called *uniformly pr-ideal* if there exists $N \in \mathbb{N}$ and whenever $ab \in I$ with $Ann(a) = (0)$ we have $b^N \in I$ in that case N is called order of I and denoted by $ord(I) = N$. Also I is called a strongly pr-ideal if I

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is a *pr*-ideal and $\sqrt{I}^N \subseteq I$ for some $N \in \mathbb{N}$, where $\sqrt{I}$ is the radical of I. In this case N is called the exponent of I and denoted by $exp(I) = N$.

The main goal of the present paper is to complete this circle of ideas by introducing and studying the concept of S-r-ideals and S-pr-ideals of a commutative ring in a way that generalizes essentially all the results concerning r-ideals and pr-ideals. Let R be a commutative ring, S a multiplicative subset of R and I a proper ideal of R disjoint from S. Then we say that I is an *S-r-ideal* (resp., an *S-pr-ideal*) of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in I$ with $Ann(a) = 0$, we have $sb \in I$ (resp., $sb \in \sqrt{I}$). In Section 2, we study the basic properties of S-r ideals. Proposition 2.1 states that if $(I : s)$ is an r-ideal of R for some $s \in S$, then I is an S-r-ideal of R. In the case where the multiplicative set S consists only of nonzero divisors (that is, $S \subseteq \mathrm{Reg}(R)$), an ideal P is S-r-ideal of R if and only if $S^{-1}P$ is an r-ideal of $S^{-1}R$ and $(S^{-1}P) \cap R = (P : s)$ for some $s \in S$. Theorem 2.3 is a counterpart of the celebrated Prime Avoidance Lemma for S-r-ideals. We show that if $I, P_1, ..., P_n$ are ideals of R such that $I \subseteq \bigcup_{i=1}^n P_i$, P_1 is an S-r-ideal, and the others have regular elements, then there exists $s \in S$ such that $sI \subseteq P_1$. We give in Theorem 2.15, a characterization of S-r-ideals of R in terms of the $(S(+)M)$-r-ideal of $R(+)M$. Section 3 is devoted to study the notion of uniformly S-pr-ideals and strongly S-pr-ideals. An ideal I of R disjoint from S is called a *uniformly S-pr-ideal* if there exist $N \in \mathbb{N}$ and $s \in S$ such that whenever $ab \in I$ with $Ann(a) = (0)$, then $(sb)^N \in I$. An ideal I is called a *strongly S-pr-ideal* if I is an S-pr-ideal and there exist $s' \in S$ and $N \in \mathbb{N}$ such that $s'\sqrt{I}^N \subseteq I$. We show that if Q is an S-pr-ideal of R, then $P = \sqrt{Q}$ is an S-r-ideal of R. We also show that the class of S-pr-ideals contains the class of uniformly S-pr-ideals and the class of uniformly S-pr-ideals contains strongly S-pr-ideals (see Proposition 3.1). Finally, we give respectively in Corollaries 3.1 and 3.2, a characterization of uniformly S-pr-ideals and strongly S-pr-ideals in a finite direct product of rings.

2. Basic results

Let R be a commutative ring, S a multiplicative subset of R and I an ideal of R disjoint from S. In this section, we study basic results of an S-r-ideal (resp., S-pr-ideal) of R.

Definition 2.1 Let R be a commutative ring, S a multiplicative subset of R and I an ideal of R disjoint from S. We say that I is an *S-r-ideal* (resp., *S-pr-ideal*) of R if there exists an $s \in S$ such that $ab \in I$ with $Ann(a) = (0)$ implies that $sb \in I$ (resp., $sb \in \sqrt{I}$) for each $a, b \in R$.

Remark 2.1 *(1)* Every r-ideal (resp., pr-ideal) of R disjoint from S is an S-r-ideal(resp., S-pr-ideal) of R.

(2) If S consists of units of R, then the notions of r-ideal (resp., pr-ideal) and S-r-ideals (resp., S-pr-ideals) coincide.

Our next example shows that the reverse of (1) in the previous Remark is not true in general.

Example 2.1 *Let T be a reduced ring such that $\mathbb{Z} \subseteq T$ and I be a prime ideal of T with $I \cap \mathbb{Z} = (0)$. Put $J := X^2I[X]$ and $S := \{X^n \mid n \in \mathbb{N}\}$. Then J is an ideal of $R = \mathbb{Z} + XT[X]$ with $J \cap S = \emptyset$. Now, let $P, Q \in R$ such that $PQ \in J$ and $Ann(P) = (0)$. By [8, Example 2.17], J is a pr-ideal of R which is not an r-ideal. Then $Q \in \sqrt{J} = XI[X]$, since J is a pr-ideal of R; so $Q \in XI[X]$, and hence $XQ \in J$. This implies that J is an S-r-ideal of R.*

In the following, we give an example of an S-prime ideal which is neither an S-r-ideal nor an S-pr-ideal.

Example 2.2 *Let $R = \mathbb{Z}[X]$, $I = 4XR$, and $S = \{2^n \mid n \in \mathbb{N}\}$. By [4, Example 1(3)], I is an S-prime ideal of R which is not prime. Note that for all $s = 2^n \in S$, we have $(2X)(2) \in I$ and $ann(2X) = 0$ but $s(2) \notin \sqrt{I}$. This shows that I is neither an S-r-ideal of R nor an S-pr-ideal of R.*

In the following, we give an example of an S-r-ideal which is not S-prime.

Example 2.3 *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $I = \{\overline{0}\} \times \{\overline{0}\} \times \mathbb{Z}_2$ and $S = \{(\overline{1}, \overline{1}, \overline{1}), (\overline{1}, \overline{1}, \overline{0})\}$. We show that I is an r-ideal of R. Let $a, b \in R$ such that $ab \in I$ with $Ann(a) = (0)$. Since R is a Boolean ring, $a^2 = a$, which implies that $1 - a \in Ann(a) = (0)$; so $a = 1$, and thus $b \in I$. This shows that I is an S-r-ideal of R, since I is disjoint from S. Now, we show that I is not an S-prime ideal of R. Let $\alpha = (\overline{1}, \overline{0}, \overline{1})$ and $\beta = (\overline{0}, \overline{1}, \overline{1})$. Then $\alpha, \beta \in R$ and $\alpha\beta = (\overline{0}, \overline{0}, \overline{1}) \in I$. Note that for each $s \in S$, $s\alpha \notin I$ and $s\beta \notin I$. Thus I is not S-prime.*

The next proposition characterizes the S-r-ideals (resp., S-pr-ideals) of a commutative ring R in the case where $S \subseteq Reg(R)$. First, recall that if I is an ideal of R and $s \in R$, then $(I : s) := \{x \in R \mid sx \in I\}$ is an ideal of R containing I.

Proposition 2.1 *Let R be a commutative ring, $S \subseteq \mathrm{Reg}(R)$ a multiplicative subset of R and I an ideal of R disjoint from S. Then the following assertions are equivalent:*

(1) I is an S-r-ideal of R.

(2) $(I : s)$ is an r-ideal of R for some $s \in S$.

(3) $S^{-1}I$ is an r-ideal of $S^{-1}R$ and $(S^{-1}I) \cap R = (I : s)$ for some $s \in S$.

Proof:

(1)⇒(2). Since I is an S-r-ideal of R, there exists $s \in S$ such that if $ab \in I$ with $Ann(a) = (0)$, then $sb \in I$. We show that $(I : s)$ is an r-ideal of R. Let $x, y \in R$ such that $xy \in (I : s)$ with $Ann(x) = (0)$. Then $sxy \in I$. It follows that $s^2y \in I$, then $y \in (I : s)$, since $Ann(s^2) = 0$. This implies that $(I : s)$ is an r-ideal of R.

(2) ⇒ (1). Assume that $(I : s)$ is an r-ideal of R for some $s \in S$. Let $a, b \in R$ such that $ab \in I$ with $Ann(a) = (0)$. Then $ab \in (I : s)$, which implies that $b \in (I : s)$; so $sb \in I$. Therefore, I is an S-r-ideal of R.

(1) ⇒ (3). First, we prove that $S^{-1}I$ is an r-ideal of $S^{-1}R$. Let $x, y \in S^{-1}R$ such that $xy \in S^{-1}I$ with $Ann(x) = (0)$. Write $x = \dfrac{a}{s_1}$ and $y = \dfrac{b}{s_2}$ for some elements $a, b \in R$ and $s_1, s_2 \in S$. Thus, $xy = \frac{p}{s_3}$ where $p \in I$ and $s_3 \in S$. It follows that $abs_3 = s_1s_2p \in I$. As I is an S-r-ideal of R and $Ann(a) = (0)$, $ss_3b \in I$ for some $s \in S$. This implies that $y = \dfrac{b}{s_2} = \dfrac{ss_3b}{ss_3s_2} \in S^{-1}I$. Now we show that $(S^{-1}I) \cap R = (I : s)$. Let $\alpha \in (I : s)$. Then $\alpha \in R$ and $s\alpha \in I$. As $S \subseteq \mathrm{Reg}(R)$, the mapping $\varphi : R \longrightarrow S^{-1}R$ defined by $\varphi(r) = \frac{r}{1}$ is an injective ring homomorphism. Thus $\alpha = \frac{\alpha}{1} = \frac{s\alpha}{s} \in S^{-1}P$. This implies that $\alpha \in (S^{-1}I) \cap R$. Conversely, let $\alpha \in (S^{-1}I) \cap R$. Then $\alpha \in R$ and $\alpha = \frac{p}{t}$ for some elements $p \in I$ and $t \in S$. Since $\alpha t = p \in I$ with $Ann(t) = (0)$, it follows that $s\alpha \in I$, and hence $\alpha \in (I : s)$.

(3) ⇒ (1). Assume (3), and let $a, b \in R$ such that $ab \in I$ with $Ann(a) = (0)$. Since $\frac{a}{1}\frac{b}{1} \in S^{-1}I$ and $Ann(\dfrac{a}{1}) = (0)$, it follows that $\frac{b}{1} \in S^{-1}I$, which implies that $b = \dfrac{b}{1} \in S^{-1}I \cap R = (I : s)$. Thus $sb \in I$ and hence I is an S-r-ideal of R. □

Corollary 2.1 *Let R be a commutative ring, $S \subseteq \mathrm{Reg}(R)$ a multiplicative subset of R and I an ideal of R disjoint from S. Set $\overline{S} := \{s + I \mid s \in S\}$. If $Z(R/I) \cap \overline{S} = \emptyset$, then I is an S-r-ideal if and only if I is an r-ideal.*

Proof: The "if" part is immediate, since any r-ideal of R disjoint from S is an S-r- ideal (see Remark 2.1). For the "only if" part, we first claim that $(I : s) = I$ for all $s \in S$ and we conclude by Proposition 2.1. The inclusion relation $I \subseteq (I : s)$ is straightforward. Conversely, let $s \in S$ and let $x \in (I : s)$. Then $sx \in I$. Hence, $(s + I)(x + I) = I$. Notice that $s + I \neq I$ (because $I \cap S = \emptyset$) and $s + I \notin Z(R/I)$, since $Z(R/I) \cap \overline{S} = \emptyset$. Therefore, we get $x + I = I$. Thus $x \in I$. This shows that $(I : s) \subseteq I$, and hence $I = (I : s)$ for any $s \in S$. □

Example 2.4 *Let R be a commutative ring, S a multiplicative subset of R such that $S \cap \mathrm{Reg}(R) \neq \emptyset$ and I an r-ideal of R. Then for each $s \in S \cap Reg(R)$, sI is an S-r-ideal of R. If moreover $I \neq (0)$ and $\bigcap_{n=1}^{\infty} Rs^n = (0)$, then sI is not an r-ideal of R.*

Proof: Let $s \in S \cap \mathrm{Reg}(R)$. It is convenient to denote sI by J. As $J \subseteq I$, we have $J \cap S = \emptyset$. Since $Ann(s) = (0)$, we get that $(sI : s) = I$. Thus $(J : s) = I$ is an r-ideal of R, and hence $J = sI$ is an S-r-ideal of R. Assume that $I \neq (0)$ and $\bigcap_{n=1}^{\infty} Rs^n = (0)$. We verify that $P = sI$ is not an r-ideal of R. Observe that $I \neq sI$. For if $I = sI$, then $I = s^n I$ for each $n \geqslant 1$. From $\cap_{n=1}^{\infty} Rs^n = (0)$, it follows that $I = (0)$ and this is a contradiction to the assumption that $I \neq (0)$. Hence $I \neq sI$. Let $a \in I \backslash sI$. Note that $sa \in sI = P$. As $a \notin P$ and $Ann(s) = (0)$, we obtain that sI is not an r-ideal of R. □

Proposition 2.2 *Let R be a commutative ring, $S \subseteq \mathrm{Reg}(R)$ a multiplicative subset of R and I an ideal of R disjoint from S. Then the following assertions are equivalent:*

(1) I is an S-pr-ideal of R.

(2) $(I : s)$ is a pr-ideal of R for some $s \in S$.

Proof: (1) $\Rightarrow$ (2). Assume that I is an S-pr-ideal of R. There exists $s \in S$ such that if $ab \in I$ with $Ann(a) = (0)$, we have $bs \in \sqrt{I}$. We claim that $(I : s)$ is a pr-ideal of R. Since $s \notin I$, we get that $(I : s) \neq R$. Let $x, y \in R$ such that $xy \in (I : s)$ with $Ann(x) = (0)$. Then $sxy \in I$. It follows that $sy \in \sqrt{I}$, then $s^m y^m = (sy)^m \in I$ for some integer $m \geq 1$. Since s is a regular element, $Ann(s^m) = (0)$. It follows that $y^m \in (I : s)$ which is equivalent to $y \in \sqrt{(I : s)}$. Hence $(I : s)$ is a pr-ideal of R.

(2) $\Rightarrow$ (1). Assume (2), and let $a, b \in R$ such that $ab \in I$ with $Ann(a) = (0)$. Note that $ab \in (I : s)$ and $Ann(a) = (0)$. Since $(I : s)$ is a pr-ideal of R, $b \in \sqrt{(I : s)}$, which implies that $sb^k \in I$ for some integer $k \geq 1$. Thus $(sb)^k = s^k b^k \in I$. This implies that $sb \in \sqrt{I}$. Therefore, I is an S-pr-ideal of R. This completes the proof. □

Let $R \subseteq T$ be an extension of commutative rings. Let a be an element of R. We denote by $Ann_T(a) = \{\alpha \in T \mid \alpha a = 0\}$. Then $Ann_T(a)$ is an ideal of T.

Proposition 2.3 *Let R be a commutative ring and S a multiplicative subset of R. Then the following hold:*

1. *Let J be an ideal of R such that $J \cap S \neq \emptyset$. If I is an S-r-ideal of R, then so is JI.*
2. *Let $R \subseteq T$ be an extension of commutative rings. If J is an S-r-ideal of T, then $I = J \cap R$ is an S-r-ideal of R.*
3. *Let $f : R \longrightarrow T$ be an isomorphism of commutative rings. Then $f(I)$ is an $f(S)$-r-ideal of T, whenever I is an S-r-ideal of R, and $f^{-1}(J)$ is an S-r-ideal of T, whenever J is an $f(S)$-r-ideal of T.*
4. *Every S-r-ideal is included in $zd(R)$.*
5. *Let I be an S-prime ideal of R. If $S \subseteq \mathrm{Reg}(R)$, then I is an S-r-ideal of R if and only if $(I : t) \subseteq zd(R)$ for some $t \in S$.*

Proof: (1). Let $a, b \in R$ such that $ab \in JI$ with $Ann(a) = (0)$. As I is an S-r-ideal of R, there exists $s \in S$ such that $sb \in I$. Pick $t \in J \cap S$ (such t exists since $J \cap S \neq \emptyset$). Then $(ts)b = t(sb) \in JI$. Therefore, JI is an S-r-ideal of R.

(2). **Case 1**. $I = (0)$, in this case I is an r-ideal of R and $S \cap I = \emptyset$, implies that I is an S-r-ideal of R.

Case 2. $I \neq (0)$. Let $a, b \in R$ such that $ab \in I$ with $Ann_R(a) = (0)$.

We show that $sb \in I$ for some $s \in S$. Clearly $ab \in J$. It is shown in [8, Proposition 3.17], that $Ann_T(a) = (0)$. As J is an S-r-ideal of T, there exists $s \in S$ such that $sb \in J$, then clearly $sb \in J \cap R$. It follows that $I = J \cap R$ is an S-r-ideal of R.

(3). It is not difficult to check that $f^{-1}(J) \cap S = \emptyset$. Now, let $a, b \in R$ such that $ab \in f^{-1}(J)$ with $Ann(a) = (0)$. Since f is bijective, we get $Ann(f(a)) = (0)$. Then there exists an $s \in S$ such that $f(s)f(b) \in J$, since J is an $f(S)$-r-ideal of T. This implies that $sb \in f^{-1}(J)$. Hence $f^{-1}(J)$ is an S-r-ideal of R. Now, let $f(a)f(b) \in f(I)$ with $Ann(f(a)) = (0)$. Then $Ann(a) = (0)$, there exists $x \in I$ such that $f(ab) = f(x)$, then $ab = x \in I$, thus $sb \in I$. Since I is an S-r-ideal, $f(s)f(b) \in f(I)$, hence $f(I)$ is $f(S)$-r-ideal. This completes the proof.

(4). Suppose that $I \nsubseteq zd(R)$. Then there exists $a \in I$ with $Ann(a) = (0)$. Since I is an S-r-ideal, there exists $s \in S$ such that $s = s \cdot 1 \in I$, a contradiction, since $S \cap I = \emptyset$. Hence $I \subseteq zd(R)$.

(5). Suppose that I is an S-prime ideal with respective to t. If I is an S-r-ideal, then $(I : t)$ is an r-ideal, by Proposition 2.1. Then by (4), $(I : t) \subseteq zd(R)$. Conversely, assume that $(I : t) \subseteq zd(R)$. Let $ab \in I$ with $Ann(a) = (0)$. Then we have $ta \in I$ or $tb \in I$. It is easy to see that $ta \notin I$. This implies that $tb \in I$, and so I is an S-r-ideal. □

Proposition 2.4 *Let R be a commutative ring and S a multiplicative subset of R. Let I be an r-ideal of R disjoint from S, and P be an ideal of R such that $I \subseteq P$. Then P is an S-r-ideal of R if and only if P/I is an $\overline{S}$-r-ideal of R/I.*

Proof: Note that $P \cap S = \emptyset$ if and only if $(P/I) \cap \overline{S} = \emptyset$.

For the "if" part, let $a + I, b + I \in R/I$ such that $(a + I)(b + I) \in P/I$ with $Ann(a + I) = (\overline{0})$. Then $(ab + I) \in P/I$, and so $ab \in P$. As I is an S-r-ideal of R and $Ann(a) = (0)$, there exists $s \in S$ such that $sb \in P$. Therefore, $(s + I)(b + I) \in P/I$.

For the "only if" part, since P/I is an $\overline{S}$-r-ideal of R/I, there exists $t \in S$ such that for all $a+I, b+I \in R/I$ if $(a + I)(b + I) \in P/I$ with $Ann(a + I) = \overline{0}$, then $(t + I)(b + I) \in P/I$. We need to show that P is an S-r-ideal of R. For, let $x, y \in R$ such that $xy \in P$ with $Ann(x) = (0)$. Then $(x + I)(y + I) \in P/I$. Since I is an r-ideal, it follows that $Ann(x+I) = \overline{0}$, so $(ty + I) = (t+I)(y+I) \in P/I$. This implies that $ty \in P$, completing the proof. □

Theorem 2.1 *Let R be a commutative ring, S a multiplicative subset of R and I an ideal of R disjoint from S. Then I is an S-r-ideal if and only if there exists $s \in S$ such that for every two ideals J, K of R, with K regular and $JK \subseteq I$, we have $sJ \subseteq I$.*

Proof: For the "only if" part, let $a, b \in R$ such that $ab \in I$ with $Ann(a) = 0$. Then aR is regular. As $(aR)(bR) \subseteq I$ by assumption, $s(bR) \subseteq I$. Hence $sb \in I$.
For the "if" part, as I is an S-r-ideal, there exists $s \in S$ such that for all $a, b \in R$, if $ab \in I$ with $Ann(a) = 0$, then $sa \in I$. We suppose that for all $t \in S$, there exist J_t, K_t two ideals of R with K_t regular and $J_tK_t \subseteq I$ but $tJ_t \nsubseteq I$. Since $s \in S$, there exist two ideals of R J_s, K_s with K_s regular and $J_sK_s \subseteq I$ but $sJ_s \nsubseteq I$. Therefore, there exist $a_s \in K_s \cap Reg(R)$ and $b_s \in J_s$ such that $sb_s \notin I$ with $a_sb_s \in J_sK_s \subseteq I$, which is absurd because I is an S-r-ideal. This completes the proof. □

Recall from [4] that a multiplicative set S of a commutative ring R is called a strongly multiplicative set if for each family $(s_\alpha)_{\alpha\in\Lambda}$ of element of S we have

$$(\bigcap_{\alpha\in\Lambda} s_\alpha R) \cap S \neq \emptyset.$$

Theorem 2.2 *Let R be a commutative ring and $S \subseteq R$ a strongly multiplicative set. Let $(P_\alpha)_{\alpha\in\Lambda}$ be a chain of S-r-ideals of R. Then $P = \bigcap_{\alpha\in\Lambda} P_\alpha$ is an S-r-ideal of R.*

Proof: For each $\alpha \in \Lambda$, there exists $s_\alpha \in S$ such that for all $a, b \in R$, $ab \in P_\alpha$ with $Ann(a) = 0$, implies $s_\alpha b \in P_\alpha$.
Since S is a strongly multiplicative set, $(\bigcap_{\alpha\in\Lambda} s_\alpha R) \cap S \neq \emptyset$. Let $t \in (\bigcap_{\alpha\in\Lambda} s_\alpha R) \cap S$. We will show that for all $a, b \in R$ such that $ab \in P$ with $Ann(a) = 0$ implies $tb \in P$. Suppose that $tb \notin P$. Then $tb \notin P_\beta$ for some $\beta \in \Lambda$. Let $\alpha \in \Lambda$. We have $P_\alpha \subseteq P_\beta$ or $P_\beta \subseteq P_\alpha$.

First case. $P_\alpha \subseteq P_\beta$. Since $tb \notin P_\beta$, it follows that $tb \notin P_\alpha$, so $s_\alpha b \notin P_\alpha$ absurd since $ab \in P_\alpha$ and $Ann(a) = 0$.

Second case. $P_\beta \subseteq P_\alpha$. As $ab \in P_\beta$, $Ann(a) = 0$ and $tb \notin P_\beta$, it follows that $s_\beta b \notin P_\beta$ which is absurd. In each case we have $tb \in P$. □

Proposition 2.5 *Let R be a commutative ring and S a multiplicative subset of R. Suppose that $P_1, ..., P_n$ are prime ideals of R, which are not comparable. If $\bigcap_{i=1}^n P_i$ is an S-r-ideal of R, then for each $1 \leq i \leq n$, P_i is an S-r-ideal of R.*

Proof: Let $rx \in P_i$ with $Ann(r) = 0$ and take $y \in (\bigcap_{j\neq i} P_j) \setminus P_i$. Hence, $rxy \in \bigcap_{i=1}^n P_i$. Since $\bigcap_{i=1}^n P_i$ is an S-r-ideal, we infer that $sxy \in \bigcap_{i=1}^n P_i$ for some $s \in S$, and therefore $sxy \in P_i$. This implies that $sx \in P_i$, then P_i is an S-r-ideal. □

The following result is a counterpart of the celebrated Prime Avoidance Lemma for S-r-ideals.

Theorem 2.3 *Let R be a commutative ring, S a multiplicative subset of R and I an ideal of R. Let $P_1, ..., P_n$ be n ideals of R such that $I \subseteq \bigcup_{i=1}^n P_i$. If P_1 is an S-r-ideal and the others are regular, there exists $s \in S$ such that $sI \subseteq P_1$.*

Proof: As P_1 is an S-r-ideal, there exists $s \in S$ such that $(P_1 : s)$ is an r-ideal. We have $I \subseteq \bigcup_{i=1}^n P_i \subseteq \bigcup_{i=1}^n (P_i : s)$ and for $i \neq 1$ $(P_i : s)$ contains a regular element. By the r-ideal Avoidance Lemma [8, Theorem 3.8], $I \subseteq (P_1 : s)$. This implies that $sI \subseteq P_1$. □

Proposition 2.6 *Let R be a commutative ring and S be a multiplicative subset of R. Assume that I is an S-r-ideal of R and let T be a multiplicative subset of R with $S_T \cap I_T = \emptyset$. Then the following statements hold:*

(1) I_T is an S_T-r-ideal of R_T.

(2) If morover, $T \subseteq \mathrm{Reg}(R)$, then I_T is an S-r-ideal of R_T.

Proof: (1) Let $x, y \in R_T$ such that $xy \in I_T$ with $Ann(x) = 0$. Write $x = \dfrac{a}{t_1}$ and $y = \dfrac{b}{t_2}$ for some elements $a, b \in R$ and $t_1, t_2 \in T$. So $\dfrac{ab}{t_1 t_2} = \dfrac{p}{t}$, where $p \in I$ and $t \in T$. There exists $t' \in T$ with $t'tab = t't_1t_2p \in I$. By hypothesis, I is an S-r-ideal of R, $s_1tt'b \in I$ for some $s_1 \in S$, since $Ann(a) = 0$. Then $s_1tt' = p_1 \in P$ for some $s_1 \in S$. Let $\alpha = \frac{s_1}{1}$. Then we have $\alpha.y = \frac{s_1}{1}.\frac{btt'}{tt't_2} = \frac{s_1tt'b}{tt't_2} = \frac{p_1}{tt't_2} \in I_T$, hence I_T is an S_T-r-ideal of R_T.
(2). As $T \subseteq \mathrm{Reg}(R)$, the mapping $\varphi : R \longrightarrow R_T$ defined by $\varphi(r) = \frac{r}{1}$ is an injective ring homomorphism. Thus we have $S \cap I_T = \emptyset$ since $S_T \cap I_T = \emptyset$. □

Let R be a commutative ring with identity and M a unitary $R-$ module. The idealization of M in R (or trivial extension of R by M) is a commutative ring $R(+)M = \{(r, m) \mid r \in R \text{ and } m \in M\}$ under the usual addition and the multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ for all $(r_1, m_1), (r_2, m_2) \in R(+)M$ [2]. If I is an ideal of R and N is a submodule of M, then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$ (see [2]). In that case, $I(+)N$ is called a homogeneous ideal of $R(+)M$. Note that $zd(R(+)M) = \{(a,m) \mid a \in zd(R) \bigcup Z(M)\}$, where $Z(M) = \{a \in R \mid am = 0 \text{ } for \text{ } some \text{ } 0 \neq m \in M\}$ (see [2]). It is easy to show that if S is a multiplicative subset of R, then $S(+)M$ is a multiplicative subset of $R(+)M$.

Theorem 2.4 *Let R be a commutative ring, M a unitary R-module, I an ideal of R and S a multiplicative subset of R with $S \cap I = \emptyset$. Then*

1. *If I is an S-r-ideal (resp., S-pr-ideal) of R, then $I(+)M$ is an $(S(+)M)$-r-ideal (resp., $(S(+)M)$-pr-ideal) of $R(+)M$.*
2. *If $Z(M) = zd(R)$ and $I(+)M$ is an $(S(+)M)$-r-ideal (resp., $(S(+)M)$-pr-ideal) of $R(+)M$, then I is an S-r-ideal (resp., S-pr-ideal) of R.*

Proof: Notice that $(S(+)M) \cap (I(+)M) = \emptyset$ if and only if $S \cap I = \emptyset$.

(1). Since I is an S-r-ideal (resp., S-pr-ideal) of R, there exists an $s \in S$ such that for all $x, y \in R$, if $xy \in I$ with $Ann(x) = (0)$, then $sy \in I$ (resp., $sy \in \sqrt{I}$). Let $(a,m),(b,n) \in R(+)M$ such that $(a,m)(b,n) = (ab, an+bm) \in I(+)M$ with $Ann((a,m)) = ((0,0))$. Since I is an S-r-ideal (resp., S-pr-ideal) and $Ann(a) = (0)$, we get $sb \in I$ (resp., $sb \in \sqrt{I}$). This implies that $(s,0)(b,n) = (sb,sn) \in I(+)M$ (resp., $(s,0)(b,n) = (sb,sn) \in \sqrt{I}(+)M = \sqrt{I(+)M}$). Hence $I(+)M$ is an $(S(+)M)$-r-ideal (resp., $(S(+)M)$-pr-ideal) of $R(+)M$.

(2). Suppose that $I(+)M$ is an $(S(+)M)$-r-ideal (resp., $(S(+)M)$-pr-ideal). Let $a, b \in R$ such that $ab \in I$ with $Ann(a) = (0)$. Then $(a,0)(b,0) = (ab,0) \in I(+)M$. Since $Z(M) = zd(R)$, $Ann((a,0)) = ((0,0))$. This implies that $(s,m)(b,0) \in I(+)M$ (resp., $(s,m)(b,0) \in \sqrt{I}(+)M$) for some $(s,m) \in S(+)M$,. Therefore, $sb \in I$ (resp., $sb \in \sqrt{I}$), and hence I is an S-r-ideal (resp., S-pr-ideal) of R. □

In the particular case when S consists of units of R we recover the following well-known result.

Corollary 2.2 *Let R be a commutative ring, M a unitary R-module such that $Z(M) = zd(R)$ and I an ideal of R. Then $I(+)M$ is an r-ideal of $R(+)M$ if and only if I is an r-ideal of R.*

In what follows, we establish a relationship between S-pr-ideals and S-r-ideals of R.

Proposition 2.7 *Let R be a commutative ring, S a multiplicative subset of R and Q an ideal of R. Then Q is an S-pr-ideal of R if and only if $I = \sqrt{Q}$ is an S-r-ideal R. In this case Q is called an I-S-r-*ideal *of R.*

Proof: One can easily check that $\sqrt{Q} \cap S = \emptyset$ if and only if $Q \cap S = \emptyset$. Assume that Q is an S-pr-ideal of R, and let $a, b \in R$ such that $ab \in I$ with $Ann(a) = (0)$. There exists a positive integer $n \geq 1$ such that $a^n b^n = (ab)^n \in Q$. Clearly $Ann(a^n) = (0)$. By hypothesis, there exists an integer $m \geq 1$ such that $sb^{nm} \in Q$; so $sb \in I = \sqrt{Q}$. Hence I is an S-r-ideal of R. Conversely, assume that $I = \sqrt{Q}$ is an S-r-ideal R. Since I is an S-r-ideal of R, there exists $s \in S$ such that for all $a, b \in R$, if $ab \in I$ with $Ann(a) = (0)$, then $sb \in I$. Now, let $x, y \in R$ such that $xy \in Q$ and $Ann(x) = (0)$. As $xy \in \sqrt{Q}$, $sy \in \sqrt{Q}$. This shows that Q is an S-pr-ideal of R, and hence the proof is completed. □

Proposition 2.8 *Let R be a commutative ring and $S \subseteq Reg(R)$ a multiplicative subset of R. An S-primary ideal Q is an S-pr-ideal if and only if $(Q : t) \subseteq zd(R)$ for some $t \in S$.*

Proof: Let Q be an S-primary ideal of R. Then there exists $t \in S$ such that for all $a, b \in R$ with $ab \in Q$, we have either $ta \in Q$ or $tb \in \sqrt{Q}$. Assume that Q is an S-pr-ideal. Then by Proposition 2.7, $\sqrt{Q}$ is an S-r-ideal with respective to t; so $(\sqrt{Q} : t)$ is an r-ideal (by Proposition 2.1). Now, by [8, Remark 2.3(d)], $(\sqrt{Q} : t) \subseteq zd(R)$, and hence $(Q : t) \subseteq zd(R)$. Conversely, assume that $(Q : t) \subseteq zd(R)$. Let $ab \in Q$ with $Ann(a) = (0)$. Then $ta \in Q$ or $tb \in \sqrt{Q}$. If $ta \in Q$, then $a \in (Q : t) \subseteq zd(R)$, absurd since $Ann(a) = (0)$. Hence $tb \in \sqrt{Q}$; so Q is an S-pr-ideal. □

In the following example we justify that the condition "$S \subseteq \text{Reg}(R)$" is essential in the previous proposition.

Example 2.5 *Let $R = \mathbb{Z}/12\mathbb{Z}$, $I = 2\mathbb{Z}/12\mathbb{Z}$ and $S = \{\overline{1}, \overline{3}, \overline{9}\}$. Then S is a multiplicative subset of R with $S \cap I = \emptyset$ and $S \nsubseteq Reg(R)$. It is easy to show that I is an S-r-ideal (resp., S-pr-ideal) of R with respective to $s = \overline{3}$. Note that for each $t \in S$, $(I : t) = I \nsubseteq zd(R) = \{\overline{0}\}$. This shows that the condition "$S \subseteq \text{Reg}(R)$" is essential in the previous Proposition.*

Theorem 2.5 *Let R_1 and R_2 be commutative rings and let S_1 and S_2 be multiplicative subsets of R_1 and R_2, respectively. Set $R = R_1 \times R_2$ and $S = S_1 \times S_2$. Then the following assertions are equivalent:*

(1) $I = I_1 \times I_2$ is an S-pr-ideal of R.

(2) $I_1 = R_1$ and P_2 is an S_2-pr-ideal of R_2 or $I_2 = R_2$ and I_1 is an S_1-pr-ideal of R_1 or I_1 is an S_1-pr-ideal of R_1 and I_2 is an S_2-pr-ideal of R_2 .

Proof: (1) $\Rightarrow$ (2). Suppose that I is an S-*pr*-ideal and $I_2 = R_2$. Let $ab \in I_1$ with $Ann(a) = (0)$. Then note that $Ann((a, 1)) = ((0, 0))$ and also $(a, 1)(b, 0) = (ab, 0) \in I$. Since I is an S-*pr*-ideal, there exists $(s_1, s_2) \in S$ such that $(s_1, s_2)(b, 0) = (s_1 b, 0) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$ and so $s_1 b \in \sqrt{I_1}$. Hence, I_1 is an S_1-*pr*-ideal. Similarly, one can easily show that I_2 is an S_2- *pr*-ideal of R_2 when $I_1 = R_1$. Assume that I_1, I_2 are proper ideals. Similarly we can show that each I_i $i = 1, 2)$ is an S_i-*pr*-ideal of R_i.
(2) $\Rightarrow$ (1). Let I_i be an S_i-*pr*-ideal, for every $i = 1, 2$.
Let $(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2) \in I_1 \times I_2$ with $Ann((a_1, a_2)) = ((0, 0))$. This implies that $Ann(a_1) = Ann(a_2) = (0)$. Note that $a_i b_i \in I_i$ with $Ann(a_i) = (0)$. Since I_i is an S_i-*pr*-ideal, we conclude that there exists $(s_1, s_2) \in S$ such that $s_i b_i \in \sqrt{I_i}$. Then $(s_1, s_2)(b_1, b_2) = (s_1 b_1, s_2 b_2) \in \sqrt{I_1 \times I_2}$. Thus I is an S-*pr*-ideal. In other cases, one can similarly prove that I is an S-*pr*-ideal. □

Theorem 2.6 *Let $R_1, R_2, ..., R_n$ be commutative rings and S_i be a multiplicative subset of R_i for $i = 1, ..., n$, respectively. Set $R = R_1 \times R_2 \times \cdots \times R_n$, and $S = S_1 \times S_2 \times \cdots \times S_n$. Let $I = I_1 \times I_2 \times \cdots \times I_n$, where I_i's are ideals of R_i's, respectively. Then the following assertions are equivalent:*

(1) I is an S-pr-ideal of R.

(2) There exist $k_1, k_2, ..., k_t \in \{1, 2, \ldots, n\}$ such that $I_k = R_k$ for each $k \in \{k_1, \ldots, k_t\}$ and I_k is an S_k-pr-ideal for each $k \in \{1, 2, \ldots, n\} \setminus \{k_1, \ldots, k_t\}$.

Proof: We use the mathematical induction on n. If $n = 1$, the claim is true. If $n = 2$, the claim follows from the previous theorem.

Assume that the claim is true for all $k < n$. Let $I = I_1 \times I_2 \times \cdots \times I_n$. Now put $L = I_1 \times I_2 \times \cdots \times I_{n-1}$ and $S' = S_1 \times S_2 \times \cdots \times S_{n-1}$. Then by the previous theorem, $I = L \times I_n$ is an $(S' \times S_n)$-*pr*-ideal if and only if $L = R_1 \times R_2 \times \cdots \times R_{n-1}$ and I_n is an S_n-*pr*-ideal or $I_n = R_n$ and L is an S'-*pr*-ideal or L, I_n are S'-*pr*-ideal and S_n-*pr*-ideal respectively. By induction hypothesis the claim is true. □

3. Uniformly and strongly S-*pr*-ideals

Definition 3.1 Let R be a commutative ring, S a multiplicative subset of R and I an ideal of R disjoint from S. I is called a uniformly S-*pr*-ideal if there exist $s \in S$ and $N \in \mathbb{N}$ such that for all $a, b \in R$ if $ab \in I$ with $Ann(a) = (0)$, then $(sb)^N \in I$.

It is easily obtained by the definition that every uniforly S-*pr*-ideal is an S-*pr*-ideal.

Definition 3.2 Let R be a commutative ring and S a multiplicative subset of R and I an ideal of R disjoint from S. An S-*pr*-ideal I of R is said to be *strongly* S-*pr*-ideal if there exist an $s' \in S$ and $n \in \mathbb{N}$ such that $s'(\sqrt{I})^n \subseteq I$.

Proposition 3.1 *Every strongly S-pr-ideal is a uniformly S-pr-ideal.*

Proof: Let I be a strongly S-pr-ideal. Thus I is an S-pr-ideal. Since I is a strongly S-pr-ideal, there exists $N \in \mathbb{N}$ such that $s'\sqrt{I}^N \subseteq I$ for some $s' \in S$. Let $ab \in I$ with $Ann(a) = (0)$ for some $a, b \in R$. Since I is an S-pr-ideal, $sb \in \sqrt{I}$. As $s'\sqrt{I}^N \subseteq I$, we have $s'(sb)^N \in s'\sqrt{I}^N \subseteq I$, then $(s'sb)^N \in I$. Hence I is a uniformly S-pr-ideal. □

Theorem 3.1 *Let R_1 and R_2 be commutative rings and let S_1 and S_2 be multiplicative subsets of R_1 and R_2, respectively. Put $R = R_1 \times R_2$, and $S = S_1 \times S_2$. Let $I = I_1 \times I_2$, where I_i's are ideals of R_i's, respectively. Then the following assertions are equivalent:*

(1) I is a uniformly S-pr-ideal of R.

(2) $I_1 = R_1$ and P_2 is a uniformly S_2-pr-ideal of R_2 or $I_2 = R_2$ and I_1 is a uniformly S_1-pr-ideal of R_1 or I_1 is a uniformly S_1-pr-ideal of R_1 and I_2 is a uniformly S_2-pr-ideal of R_2 .

Proof: (1) ⇒ (2). Suppose that I is a uniformly S-pr-ideal and $I_2 = R_2$. Let $ab \in I_1$ with $Ann(a) = (0)$. Then note that $Ann((a, 1)) = ((0, 0))$ and also $(a, 1)(b, 0) = (ab, 0) \in I$. Since I is a uniformly S-pr-ideal, we conclude that $(s_1, s_2)^N(b, 0)^N = ((s_1b)^N, 0) \in I_1 \times I_2$, then $(s_1b)^N \in I_1$. Hence I_1 is a uniformly S_1-pr-ideal. Similarly, one can easily show that I_2 is a uniformly S_2- pr-ideal of R_2 when $I_1 = R_1$. Assume that I_1, I_2 are proper ideals. Similarly we can show that each I_i, $(i = 1, 2)$ is a uniformly S_i-pr-ideal of R_i.

(2) ⇒ (1). Let I_i be a uniformly S_i-pr-ideal with $ord(I_i) = N_i$ for every $i = 1, 2$. Put $N = max\{N_1, N_2\}$. Let $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2) \in I_1 \times I_2$ with $Ann((a_1, a_2)) = ((0, 0))$. This implies that $Ann(a_1) = Ann(a_2) = (0)$. Note that $a_ib_i \in I_i$ with $Ann(a_i) = (0)$. Since I_i is a uniformly S_i-pr-ideal with $ord(I_i) = N_i$, we conclude that there exists $(s_1, s_2) \in S$ such that $(s_ib_i)^{N_i} \in I_i$. Then $(s_ib_i)^N \in I_i$ and so $((s_1, s_2)(b_1, b_2))^N = ((s_1b_1)^N, (s_2b_2)^N) \in I_1 \times I_2$. Thus I is a uniformly S-pr-ideal. Also note that $ord(I_1 \times I_2) \leqslant N$. In other cases, one can similarly prove that I is a uniformly S-pr-ideal. □

Corollary 3.1 *Let $R_1, R_2, ..., R_n$ be commutative rings. For each $1 \leq i \leq n$, let S_i be a multiplicative subsets of R_i. Put $R = R_1 \times R_2 \times \cdots \times R_n$ and $S = S_1 \times S_2 \times \cdots \times S_n$. Let $I = I_1 \times I_2 \times \cdots \times I_n$, where I_i's are ideals of R_i's, respectively. Then the following assertions are equivalent:*

1. *I is a uniformly S-pr-ideal of R.*
2. *There exist $k_1, k_2, ..., k_t \in \{1, 2, \ldots, n\}$ such that $I_k = R_k$ for each $k \in \{k_1, \ldots, k_t\}$ and I_k is a uniformly S_k-pr-ideal for each $k \in \{1, 2, \ldots, n\} \setminus \{k_1, \ldots, k_t\}$.*

Proof: We use the mathematical induction on n. Assume that the claim is true for all $k < n$. Suppose that $I = I_1 \times I_2 \times \cdots \times I_n$. Now put $L = I_1 \times I_2 \times \cdots \times I_{n-1}$ and $S' = S_1 \times S_2 \times \cdots \times S_{n-1}$. Then by the previous case, $I = L \times I_n$ is a uniformly $(S' \times S_n)$-pr-ideal if and only if $L = R_1 \times R_2 \times \cdots \times R_{n-1}$ and I_n is a uniformly S_n-pr-ideal or $I_n = R_n$ and L is a uniformly S'-pr-ideal or L, I_n are uniformly S'-pr-ideal and uniformly S_n-pr-ideal, respectively. By induction hypothesis the claim is true. □

Theorem 3.2 *Let R_1 and R_2 be commutative rings and let S_1 and S_2 be multiplicative subsets of R_1 and R_2 respectively. Put $R = R_1 \times R_2$, and $S = S_1 \times S_2$. Let $I = I_1 \times I_2$, where I_i's are ideals of R_i's, respectively. Then the following assertions are equivalent:*

1. *I is a strongly S-pr-ideal of R.*
2. *$I_1 = R_1$ and I_2 is a strongly S_2-pr-ideal of R_2 or $I_2 = R_2$ and I_1 is a strongly S_1-pr-ideal of R_1 or I_1 is a strongly S_1-pr-ideal of R_1 and I_2 is a strongly S_2-pr-ideal of R_2 .*

Proof:

(1) $\Rightarrow$ (2). Suppose that I is a strongly S-pr-ideal and $I_2 = R_2$. Hence I_1 is an S_1-pr-ideal by Theorem 2.6. Moreover, there exist $s' = (s'_1, s'_2) \in S$ and $n \in \mathbb{N}$ such that $s'(\sqrt{I})^n \subseteq I$, then $(s'_1, s'_2)(\sqrt{I_1})^n \times (R_2)^n = s'_1(\sqrt{I_1})^n \times s'_2(R_2)^n \subseteq I_1 \times R_2$. And so $s'_1(\sqrt{I_1})^n \subseteq I_1$, hence I_1 is a strongly S_1-pr-ideal of R_1. Similarly, we can easily show that each $I_2$is a strongly S_2-pr-ideal of $R_2$when $I_1 = R_1$. Assume that I_1, I_2 are proper ideals. Similarly we can show that each I_i $(i = 1, 2)$ is a strongly S_i-pr-ideal of R_i.

(2) $\Rightarrow$ (1). Let I_i be a strongly S_i-pr-ideal with $ord(I_i) = N_i$ for every $i = 1, 2$. Put $N = max\{N_1, N_2\}$. Let $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2) \in I_1 \times I_2$ with $Ann((a_1, a_2)) = ((0, 0))$. This implies that $Ann(a_1) = Ann(a_2) = (0)$. Note that $a_ib_i \in I_i$ with $Ann(a_i) = (0)$. Since I_i is a strongly S_i-pr-ideal with $ord(I_i) = N_i$, we conclude that there exists $(s_1, s_2) \in S$ such that $(s_ib_i)^{N_i} \in I_i$. Then $(s_ib_i)^N \in I_i$, and so $((s_1, s_2)(b_1, b_2))^N = ((s_1b_1)^N, (s_2b_2)^N) \in I_1 \times I_2$. Thus I is a strongly S-pr-ideal. Also note that $ord(I_1 \times I_2) \leqslant N$. In other cases, one can similarly prove that I is a strongly S-pr-ideal. $\square$

We proceed exactly as the proof of Corollary 3.1, we can prove the following result.

Corollary 3.2 *Let $R_1, R_2, ..., R_n$ be commutative rings. For each $i = 1, ..., n$, let S_i be a multiplicative subset of R_i. Put $R = R_1 \times R_2 \times \cdots \times R_n$, and $S = S_1 \times S_2 \times \cdots \times S_n$. Let $I = I_1 \times I_2 \times \cdots \times I_n$, where I_i's are ideals of R_i's, respectively. Then the following assertions are equivalent:*

(1) I is a strongly S-pr-ideal of R.

(2) There exist $k_1, k_2, ..., k_t \in \{1, 2, \ \dots\ , \ n\}$ such that $I_k = R_k$ for each $k \in \{k_1, \ \dots\ , \ k_t\}$ and I_k is a strongly S_k-pr-ideal for each $k \in \{1, \ 2, \ \dots\ , \ n\} \setminus \{k_1, \ \dots\ , \ k_t\}$.

Now, we characterize the uniformly S-pr-ideal (resp., strongly S-pr-ideal) notion in terms of the uniformly $(S(+)M)$-pr-ideal (resp., strongly $(S(+)M)$-pr-ideal) of $R(+)M$.

Proposition 3.2 *Let R be a ring and M be an R-module. Then*

(1) If I is a uniformly S-pr-ideal, then $I(+)M$ is a uniformly $(S(+)M)$-pr-ideal of $R(+)M$.

(2) If $Z(M) = zd(R)$ and $I(+)M$ is a uniformly $(S(+)M)$-pr-ideal of $R(+)M$, then I is a uniformly S-pr-ideal of R.

Proof: (1) Let $(a, m)(b, n) = (ab, an+bm) \in I(+)M$ with $Ann(a, m) = ((0, 0))$. Then we have $Ann(a) = 0$ and $ab \in I$. Since I is a uniformly S-pr-ideal, there exist $N \in \mathbb{N}$ and $s \in S$ such that $(sb)^N \in I$, which implies that $((s, 0)(b, n))^N = (s^N, 0)(b^N, Nb^{N-1}n) = ((sb)^N, Ns^Nb^{N-1}n) \in I(+)M$. Hence $I(+)M$ is a uniformly S-pr-ideal of $R(+)M$.
(2) Let $ab \in I$ with $Ann(a) = (0)$. Then $(a, 0)(b, 0) = (ab, 0) \in I(+)M$. Since $Z(M) = zd(R)$, note that $Ann((a, 0)) = ((0, 0))$. As $I(+)M$ is uniformly $(S(+)M)$-pr-ideal, we conclude that $((s, m)(b, 0))^N = ((sb)^N, Ns^{N-1}b^Nm) \in I(+)M$ for some $(s, m) \in S(+)M$. This implies that $(sb)^N \in I$, and so I is a uniformly S-pr-ideal. $\square$

Corollary 3.3 *Let R be a ring and M be an R-module such that $Z(M) = zd(R)$. Then $I(+)M$ is a uniformly $(S(+)M)$-pr-ideal of $R(+)M$ if and only if I is a uniformly S-pr-ideal of R.*

Proposition 3.3 *Let R be a ring and M be an R-module. If I is a strongly S-pr-ideal, then $I(+)M$ is a strongly $(S(+)M)$-pr-ideal of $R(+)M$.*

Proof: Let I be a strongly S-pr-ideal. Then I is uniformly S-pr-ideal by Proposition 3.1. By Proposition3.8 (1), $I(+)M$ is a uniformly $(S(+)M)$-pr-ideal, and so $I(+)M$ is an S-pr-ideal. Since I is a strongly S-pr-ideal, we have $s'(\sqrt{I})^N \subseteq I$ for some $N \in \mathbb{N}$ and some $s' \in S$. By [1,Theorem 3.2], $\sqrt{I(+)M} = \sqrt{I}(+)M$, and so $(s', 0)\sqrt{I(+)M}^N = (s', 0)(\sqrt{I}(+)M)^N \subseteq (s'\sqrt{I}^N)(+)M \subseteq I(+)M$.
$\square$

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References

1. Anderson, D.D, Smith, E. , *Weakly prime ideals.* Houston.J. Math. 29 (4),831-840,(2003).
2. Anderson, D.D., Winders, M. , *Idealization of a module.* J. Commut. Algebra. 1 (1), 3-56,(2009).
3. Atani, S.E, Farzalipour, F. , *On weakly primary ideals.* GMJ. 12 (3): 423-429, (2005).
4. Hamed, A., Malek, A. , *S-prime ideals of a commutative ring.* Beitr. Algebra Geom. 61 (3), 533-542,(2020).
5. Gilmer, R. , *Multiplicative Ideal Theory.* Dekker, New York,(1972).
6. Kaplansky, I. ,*Commutative Rings.* rev. ed., Univ. Chicago Press, Chicago,(1974).
7. Massaoud, E. , *S-primary ideals of a commutative ring.* Commun. Algebra. 50(3),988-997,(2022).
8. Mohamadian, R. , *r-ideals in commutative rings.* Turk. J. Math. 39,733-749,(2015).
9. Uregen, R. N., *On uniformly pr-ideals in commutative rings.* Turk. J. Math. 43,1878-1886, (2019).
10. Visweswaran, S. , *Some results on S-primary ideals of a commutative ring.* Beitr Algebra Geom. 63(2), 247-266, (2021).

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