Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–22.
ISSN-0037-8712
doi:10.5269/bspm.67883

A new approximation method for PDE constraint optimal control problem solutions

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ABSTRACT: In this article, we present a method to solve the optimal control with elliptic partial differential equation constraint, based on the new method of spectral element. In this new method, we use Muntz polynomials as interpolation polynomials. Using this method, we get the discrete form of the problem, which is itself a large scale constrained optimization problem. We use the split Bregman method to solve this optimization problem. In the end, we will check the accuracy and efficiency of the method with some numerical examples.

Key Words: Muntz polynomials, spectral element method, Split Bregman method, optimal control with elliptic partial differential equation constraint.

Contents

Submitted April 14, 2023. Published July 12, 2025
2010 *Mathematics Subject Classification:* 65M60, 78M22, 78M50, 49J20 .

1. Introduction

Modeling many problems in various disciplines of science and engineering leads to partial differential equations. In most cases, these problems do not have an exact solution or the exact solution is complicated. The development of numerical methods to approximate the solution to these problems has been very important and has always been of interest to mathematicians and engineers. In the following, we briefly introduce some numerical methods for solving partial differential equations.

1.1. Spectral methods

The spectral method is a conventional method for solving partial differential equations, which was first introduced by Navier for elastic sheet problems in 1825. The main property of spectral methods is the high accuracy and rapid asymptotic rate of convergence, which is called exponential convergence. In spectral methods, the solution is approximated on one general domain. Spectral methods are a member of the larger class of methods known as the weighted residual methods. In spectral methods, the unknown function is approximated by a sequence of basis functions. The selection of basis functions in spectral methods distinguishes it from other approximation methods such as finite difference methods and finite elements. Spectral methods can be divided into two general categories:

1.1.1. Collocation methods. In these methods, a set of points is selected as colocation or interpolation points, and the unknown function is approximated in these points. These methods was first introduced by Slater [1], Barta and Frazer [2,3] solved differential equations with this method. Lansuz used Chebyshev points for colocation points for the first time [4].

1.1.2. Non-Interpolating method. Non-Interpolating methods include Tau and Galerkin methods. In these methods, there are no nodes or collocation points, and the series coefficients of the unknown function interpolator are obtained by using the internal multiplication of the unknown function and the bases. In these methods, orthogonal bases will speed up the calculations. The origin of the Galerkin method is generally associated with a paper published in 1915 by Boris Galerkin, a Russian mechanical engineer, on the elastic equilibrium of rods and thin plates [5]. The use of the Galerkin methods increased rapidly during the 1950s. The Galerkin method has been used to solve many problems in structural mechanics, dynamics, heat flow, hydrodynamic stability, magnetohydrodynamics, heat and mass transfer, acoustics, microwave theory, neutron transport, etc. Problems governed by ordinary differential equations, partial differential equations, integral equations, and integro-differential equations have been investigated via Galerkin formulations. Steady, unsteady, and eigenvalue problems have proved to be equally amenable to the Galerkin treatment. Essentially, any problem for which governing equations can be written down is a candidate for the Galerkin method [6,7,8,9,10,11,12,13,14] The bases of the Galerkin method must be smooth functions with derivatives of any order. These bases must also apply to the boundary conditions of the problem. The tau method is similar to the Galerkin method, except that the tau method does not need the bases to apply in boundary conditions. The tau method is often used for problems with non-periodic boundary conditions.

1.2. Finite element methods

It is very difficult to find an exact date for the invention of the finite element method. However, it can be said that this method has been used for the first time in solving complex elasticity and structural analysis problems in civil and aeronautical engineering. The first works in this field were presented by Hrennikoff [15] and Courant [16] in the early 1940s. In the later 1950s, the Chinese mathematician Feng Kang independently re-invented the finite element method and called it the 'finite difference method based on variation principle'. The essential characteristic of the method introduced by these pioneers is: the mesh discretization of a continuous domain into a set of discrete sub-domains, usually called elements. Courant's approach divides the domain into finite triangular subregions to solve second-order elliptic partial differential equations that arise from the problem of torsion of a cylinder. Courant's contribution was evolutionary, drawing on a large body of earlier results for partial differential equations developed by Rayleigh, Ritz, and Galerkin. A rigorous mathematical basis to the finite element method was provided in 1973 with the publication by Strang and Fix [17] The method has since been generalized for the numerical modeling of physical systems in a wide variety of engineering disciplines, e.g., electromagnetism, heat transfer, and fluid dynamics [18,19]. The most important property of the finite element method is its flexibility in solving problems on unstructured domains.

1.3. Spectral element methods

In 1984, Patra by combining the good flexibility of the low-order finite element methods on unstructured domains and high accuracy and exponential convergence of the spectral methods, introduced a new method so-called 'spectral element method' [20]. In the introduced method, Patra used Lagrange polynomials through the zeros of the Gauss-Lobatto-Legendre integration points as basis functions. Common polynomials used in spectral element methods as basis functions are Chebyshev and Legendre polynomials. The main advantage of using Legendre polynomials over Chebyshev polynomials is the diagonal mass matrix obtained in the Legendre spectral element methods [21]. The use of this method in solving various problems of science and engineering has increased widely in recent years. Zhu et al. used the Chebyshev spectral element method coupled with the implicit Newmark time integral method is investigated to simulate acoustic field [22]. Yibiao Li and Xin Kai Li, developed a new Chebyshev spectral element method in their paper, in which exact quadratures are used to overcome a shortfall of the Gauss–Chebyshev quadrature in variational spectral element projections. They tested the method with the Stokes and the Cauchy–Riemann problems, and showed that the numerical accuracy is much better than that from the Gauss–Lobatto–Legendre spectral element method [23]. Quaglino et al. use the Legendre spectral element methods for fast and accurate training of Neural Ordinary Differential Equations [24]. Lotfi and Alipanah solve the sine-Gordon equation in one dimension with Legendre spectral element method [25]. In [26], authors propose a new numerical scheme based on Legendre spectral element method for the solution of two and three-dimensional time fractional nonlinear damped Klein-Gordon equation. In [27], a reduced-order modal spectral element method has been investigated to solve a two-dimensional viscoelastic equation. Dehghan et al. used Legendre spectral element method to solve the Sobolev equations that have several applications in physics and mechanical engineering, and performed error analysis for the introduced method [28]. The above papers show the application of the spectral element method in solving various problems. In this paper, we introduce a new basis for the spectral element method and evaluate the efficiency of the new method in solving fluid mechanics problems.

1.4. PDE Constraint Optimal Control

In recent decades, optimal control problems with partial differential equation constraints have been studied extensively. These issues are very complex, and the numerical solution to such problems is of great importance. In this paper, we will discuss the numerical solution of the following convex optimal control problem:

$$\begin{aligned} &\min_{u\in K} \quad g(y)+\rho(u) \\ &s.t \quad -\Delta y = f+u \quad \text{in}\Omega, \quad y|_{\partial\Omega}=0. \end{aligned} \tag{1.1}$$

where g, ρ, and f are known functions, g and ρ are convex functions, Ω is a rectangular domain, and K is a closed convex set. Eq. (1.1) appears, for example, in temperate control problems [29].

Finite element approximation of optimal control problems plays a very important role in numerical methods for these problems. There have been extensive studies on this aspect, see, for example, [30,31,32,33,34,35,36,37]. According to the accuracy of spectral methods and using the idea of the finite element method, Chen et al., in [38], used the spectral element method to solve Eq. (1.1). They provide the boundaries for the a priori and a posteriori error of the method. The Legendre spectral element method has been widely used to solve various problems, for example, [39,40,41,42,43,44,45,46,47]. Here, we use the Muntz spectral element method (MSEM) for the discretization of the Eq. (1.1).
The discrete form is obtained as follows:

$$\begin{aligned} &\min g(Y)+\rho(U) \\ &s.t \quad Y = BU+F \end{aligned} \tag{1.2}$$

where $A, B \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times 1}$ are known matrices. In the MSEM, the accuracy of the solution increases as the degree of polynomials or the number of nodal points gets bigger. So, in order to obtain sufficient accuracy, it is necessary to select as large a number of nodal points as possible. This makes the issue of the discrete form to be relatively large-scale.

To solve Eq. (1.2), there are many optimization methods [38,48,49]. In this paper, we use the split Bregman method (SBM) presented by Goldstein and Osher in [50] for solving Eq. (1.2). Solving the optimization problem (1.2) with traditional methods is difficult and requires a lot of CPU time and memory, and results in less accuracy. The Bregman method is an iterative method that can be used to solve the constrained optimization problem [51]. The characteristic of the Bregman iterative method is its speed in solving problems with high dimensions. It should be noted that the problems derived from MSEM discretization are large, so we use the Bregman method to solve the optimization problem. The objective function of the optimal control problem with elliptic partial differential equation constraint is the sum of two convex functions, so we use the SBM [50], which is useful for this kind of problem. The SBM is faster than conventional methods, needs less memory, and is more accurate. Actually, the SBM converts the constrained optimization problem to an unconstrained one. Solving the unconstrained problem is much easier than the constrained one.

1.5. The main aim of this paper

In this paper, we use the MSEM to discretize the optimal control problems with elliptic constraints. Then we propose the iterative SBM for the resulted in large-scale constrained optimization problem. Several examples are included to show the efficiency and accuracy of the proposed technique. finally, the numerical results and computational cost are compared with the SQP algorithm.

This paper is organized as follows: In section 2, we describe the MSEM approximation for the convex optimal control problem (1.1). In section 3, we explain the Bregman iterative method and its application for constrained optimization. In section 4, we solve the optimization part of problem (1.1) using the iterative SBM and the convergence theorem is presented. Numerical results and discussion are given in section 5.

2. Muntz Spectral Element Method (MSEM)

2.1. Muntz Polynomials

In 1885, Weierstrass introduced one of the most important theorems in approximation theory:

Theorem 2.1 (Weierstrass Approximation Theorem) *Let f be a continuous real-valued function defined on the real interval $[0,1]$, and let $\varepsilon > 0$. Then there exists a polynomial p on $[0,1]$ such that*

$$\|f-p\|_{sup} := \sup\{|f(x)-p(x)| : x \in [0,1]\} < \varepsilon.$$

In other words, Span$\{x^n : n \in \mathbb{Z}_{\geq 0}\}$ is dense in $C([0,1])$ with the supremum norm.

The natural question of how many exponents are needed to have a density in $C([0,1])$ had been asked by one of the greatest approximation theorists: Bernstein. In 1912, Bernstein conjectured the following: given a family of distinct real numbers $\{0 = \lambda_0 < \lambda_1 < \dots\}$, then Span$\{x^{\lambda_j} : j \in \mathbb{Z}_{\geq 0}\}$ is dense in $C([0,1])$ if and only if

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty.$$

Bernstein showed that the necessary and sufficient conditions for density respectively are

$$\sum_{j=1}^{\infty} \frac{1+\log\lambda_j}{\lambda_j} = \infty,$$

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{j\log\lambda_j} = 0.$$

Bernstein's conjecture was proved by Herman Muntz in 1914 [52] and Otto Szász in 1916 [53].

Theorem 2.2 (Muntz-Szász) *Let* $\{0 = \lambda_0 < \lambda_1 < \dots\}$ *be a sequence of real numbers satisfying* $\lim_{j\to\infty} \lambda_j = \infty$*. Then* $Span\{x^{\lambda_j} : j \in \mathbb{Z}_{\geq 0}\}$ *is dense in* $C([0,1])$ *if and only if*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty.$$

Let $\{0 = \lambda_0 < \lambda_1 < \dots\}$, the classical Muntz-Szász theorem states that the Muntz polynomials are linear combinations of the Muntz system $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$, denoted by $M_n(\lambda)$:

$$M_n(\lambda) = \mathrm{Span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}.$$

Muntz orthogonal system was first introduced by Armenian mathematicians [54,55]. In this paper, we consider the Muntz Legendre polynomials, which are orthogonal. These polynomials are orthogonal on the interval $[0,1]$ with respect to the weight function $w(x) = 1$, and are used in the numerical solving of various problems [56,57,58].

2.2. Orthogonal Muntz Legendre Polynomials

Suppose we have $\lambda_k > -\frac{1}{2}$ in a sequence $\{\lambda_0, \lambda_1, \lambda_2, \dots\}$, and we also put $\Lambda_n = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$. If a simple contour Γ contains and surrounds all the zeros of the denominator in the following function:

$$W_n(s) = \prod_{k=0}^{n-1} \frac{s + \overline{\lambda}_k + 1}{s - \lambda_k} \cdot \frac{1}{s - \lambda_n}, \quad n \in \mathbb{N} \cup \{0\},$$

then we define the n-th Muntz Legendre polynomial as follows [54,55]:

$$P_n(x) = P_n(x; \Lambda_n) = \frac{1}{2\pi i} \oint_{\Gamma} W_n(s) x^s \, ds.$$

In the case of $n = 0$, we consider $P_0(x) = 1$. If $\lambda_k \neq \lambda_l$ for $k \neq l$, these polynomials can be defined as follows [59]:

$$P_n(x) = \sum_{k=0}^{n} C_{n,k} x^{\lambda_k}, \quad C_{n,k} = \frac{\prod_{l=0}^{n-1} (\lambda_k + \overline{\lambda}_l + 1)}{\prod_{\substack{l=0 \\ l \neq k}}^{n} (\lambda_k - \lambda_l)}. \tag{2.1}$$

The Muntz Legendre polynomials satisfy the following relation [60]:

$$x P_n'(x) - x P_{n-1}'(x) = \lambda_n P_n(x) + (1 + \overline{\lambda}_{n-1}) P_{n-1}(x).$$

These polynomials satisfy the following orthogonality condition [60]:

$$(P_n, P_m) = \int_0^1 P_n(x) P_m(x) \, dx = \frac{\delta_{n,m}}{\lambda_n + \overline{\lambda}_n + 1}.$$

The first five Muntz Legendre polynomials for $\lambda_k = \frac{k}{5}$ are:

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= 6x^{\frac{1}{5}} - 5, \\
P_2(x) &= 28x^{\frac{2}{5}} - 42x^{\frac{1}{5}} + 15, \\
P_3(x) &= 168x^{\frac{1}{5}} - 252x^{\frac{2}{5}} + 120x^{\frac{3}{5}} - 35.
\end{aligned}$$

Figures 1 and 2 show graphs of Muntz Legendre polynomials P_n with different values of n and λ_k.

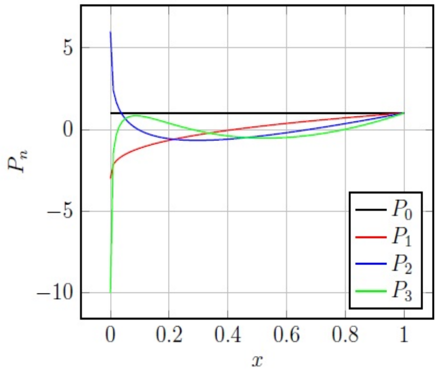

Figure 1: Graph of the Muntz Legendre polynomials with $n = 0, 1, 2, 3$ and $\lambda_k = \frac{k}{3}$.

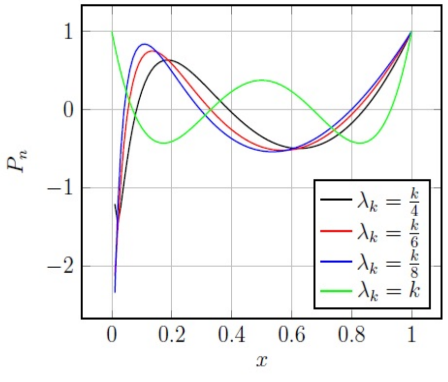

Figure 2: Graph of the Muntz Legendre polynomials with P_4 and $\lambda_k = \frac{k}{4}, \frac{k}{4}, \frac{k}{4}, \frac{k}{4}$.

2.3. Muntz Spectral Element Methods (MSEM)

To solve problems with the spectral element method, first, we divide the domain into N_e non-overlapping subdomains (elements) Ω_e, which satisfy the following conditions [25]:

$$\overline{\Omega} = \bigcup_{e=1}^{N_e} \overline{\Omega}_e, \quad \bigcap_{e=1}^{N_e} \Omega_e = \emptyset.$$

We define the approximation space U_h as follows:

$$U_h = \{u \in U : u|_{\Omega_e} \in \mathbb{P}_N(\Omega_e)\},$$

where $\mathbb{P}_N$ is a polynomial space of dimension less than or equal to N.

In spectral element methods, convergence speed is achieved by increasing the degree of polynomials or increasing the number of elements N_e. Basis functions are typically considered to be Lagrangian interpolation polynomials of high degree defined at Gauss-Lobatto integration points on each element. Here, we are going to introduce a new basis for the spectral element method, using a special combination of Muntz polynomials.

After dividing the domain into N_e non-overlapping elements, on each element, define the approximate solution of order N, u_e, as follows:

$$u_e = \sum_{j=0}^{N-2} u_{e,j}\varphi_j, \quad 1 \le e \le N_e,$$

where φ_j is defined as follows:

$$\varphi_j = Q_j - Q_{j+2}, \quad j = 0, 1, \ldots, N-2. \tag{2.2}$$

Here, Q_j is a combination of Muntz-Legendre polynomials (2.1) with $\lambda_k = k$ for $k = 0, \ldots, j$, mapped to the interval $[-1, 1]$, and defined as follows:

$$Q_j(x) = P_j\left(\frac{x+1}{2}\right) = \sum_{k=0}^{j} C_{j,k}\left(\frac{x+1}{2}\right)^k = \sum_{k=0}^{j}\left(\left(\frac{1}{2}\right)^k C_{j,k}x^k + C_{j,k}\left(\frac{1}{2}\right)^k\right),$$

$$C_{j,k} = \frac{(-1)^{j+k}}{k!(j-k)!}\prod_{l=0}^{j-1}(k+l+1).$$

Figure 3 shows the graph of φ_j for $j = 0, 1, 2, 3, 4$. Let x_e and x_{e-1} be the endpoints of the e-th element. Then, for transformation between $[-1, 1]$ and the e-th element, we use the mapping functions:

$$t(x) = \frac{(t_e - t_{e-1})x}{2} + \frac{t_e + t_{e-1}}{2}, \quad -1 \le x \le 1,$$

$$x(t) = \frac{2t - (t_e + t_{e-1})}{t_e - t_{e-1}}, \quad t_{e-1} \le t \le t_e.$$

2.3.1. The Mass Matrix. If $h_e = t_e - t_{e-1}$, then the elemental mass matrix M_{ij}^e on Ω_e is constructed as:

$$M_{ij}^e = \int_{t_{e-1}}^{t_e} \varphi_i(t)\varphi_j(t)\,dt = \frac{h_e}{2}\int_{-1}^{1} \varphi_i(x)\varphi_j(x)\,dx.$$

Expanding this, we get:

$$M_{ij}^e = \frac{h_e}{2}\int_{-1}^{1} (Q_i(x) - Q_{i+2}(x))(Q_j(x) - Q_{j+2}(x))\,dx.$$

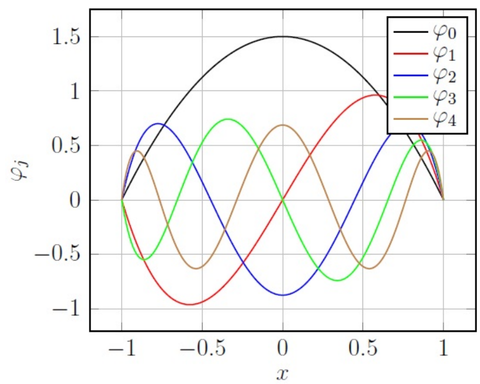

Figure 3: Graph of φ_j for $j = 0, 1, 2, 3, 4$.

This can be further simplified as:

$$\begin{aligned} M_{ij}^e =& \frac{h_e}{2}\left(\int_{-1}^{1} Q_i(x)Q_j(x)\,dx + \int_{-1}^{1} Q_{i+2}(x)Q_{j+2}(x)\,dx\right) \\ &- \frac{h_e}{2}\left(\int_{-1}^{1} Q_i(x)Q_{j+2}(x)\,dx + \int_{-1}^{1} Q_{i+2}(x)Q_j(x)\,dx\right). \end{aligned}$$

Using orthogonality properties, we obtain:

$$M_{ij}^e = \frac{h_e}{2}\left(\frac{2\delta_{ij}}{2i+1} + \frac{2\delta_{ij}}{2i+5}\right) - \frac{h_e}{2}\left(\frac{2\delta_{i(j+2)}}{2i+1} + \frac{2\delta_{(i+2)j}}{2j+5}\right).$$

The mass matrix is a symmetric tridiagonal matrix with:

$$M_{ij}^e = \begin{cases} h_e\left(\frac{1}{2i+1} + \frac{1}{2i+5}\right), & i = j, \\ h_e\left(-\frac{1}{2i+1}\right), & i = j+2, \\ h_e\left(-\frac{1}{2i+5}\right), & i = j-2, \\ 0, & \text{Otherwise.} \end{cases}$$

The structure of the mass matrix for $N = 10$ is as follows:

$$M^e = \begin{bmatrix} \bullet & * & & & & & & & & \\ * & \bullet & * & & & & & & & \\ & * & \bullet & * & & & & & & \\ & & * & \bullet & * & & & & & \\ & & & * & \bullet & * & & & & \\ & & & & * & \bullet & * & & & \\ & & & & & * & \bullet & * & & \\ & & & & & & * & \bullet & * & \\ & & & & & & & * & \bullet & \\ & & & & & & & & * & \bullet \end{bmatrix}.$$

2.3.2. The Stiffness Matrix. Let $h_e = t_e - t_{e-1}$, then the stiffness matrix on each element is calculated as follows:

$$S_{ij}^e = \int_{t_{e-1}}^{t_e} \frac{d\varphi_i(t)}{dt}\frac{d\varphi_j(t)}{dt}\,dt = \frac{2}{h_e}\int_{-1}^{1} \frac{d\varphi_i(x)}{dx}\frac{d\varphi_j(x)}{dx}\,dx.$$

This results in a diagonal matrix:

$$S^e_{ij} = \begin{cases} h_e^{-1}(8i+12), & i=j, \\ 0, & i \neq j. \end{cases}$$

The structure of the stiffness matrix for $N = 10$ is as follows:

$$S^e = \begin{bmatrix} \bullet & & & & & & & & & \\ & \bullet & & & & & & & & \\ & & \bullet & & & & & & & \\ & & & \bullet & & & & & & \\ & & & & \bullet & & & & & \\ & & & & & \bullet & & & & \\ & & & & & & \bullet & & & \\ & & & & & & & \bullet & & \\ & & & & & & & & \bullet & \\ & & & & & & & & & \bullet \end{bmatrix}.$$

2.3.3. Condition Number Comparison. Table 1 shows the condition number of mass and stiffness matrices in two methods: Legendre Spectral Element Method (LSEM) and Muntz Spectral Element Method (MSEM) for $N = 5$. According to the data in Table 1, the condition number of the mass matrix in the two methods is almost the same, but the condition number of the stiffness matrix of LSEM is much larger than the corresponding matrix in MSEM..

Table 1: Comparison of the condition number of mass and stiffness matrices in the two methods (MSEM and LSEM with $N = 5$)

Method	**Condition Number (Mass)**	**Condition Number (Stiffness)**
MSEM	2.52	1.66
LSEM	8.3229	9.64×10^{16}

3. The Elliptic Optimal Control Problem and MSEM Approximation

In this section, we obtain the discrete form of Eq. (1.1) using the MSEM. Suppose that the state space is $Y = H^1_0(\Omega)$ and the control space is $U = L^2(\Omega)$. In this case, using the inner product $(\cdot,\cdot)_U$, the weak form of Eq. (1.1) will be as follows:

Find (y, u) such that

$$\begin{aligned} &\min_{u \in K} \quad g(y) + \rho(u) \\ &s.t \quad a(y,w) = (f+u,w), \quad w \in Y, \end{aligned} \tag{3.1}$$

where

$$a(v,w) = \int_\Omega (\nabla v)\cdot(\nabla w), \quad v,w \in Y,$$

and

$$(f,w) = \int_\Omega fw, \quad f,w \in U.$$

The control problem (3.1) has a unique solution (y, u) [61], and (y, u) is the solution of (3.1) if and only if there exists a costate variable $p \in Y$ such that (y, u, p) satisfies the following conditions:

$$a(y,w) = (f+u,w), \quad w \in Y,$$

$$a(p,q) = (g'(y),q), \quad q \in Y,$$

$$(\rho'(u)+p, v-u)_U \geq 0, \quad v \in K = L^2(\Omega_U),$$

where g' and ρ' are the derivatives of g and ρ, respectively.

Now, we obtain the MSEM approximation of Eq. (3.1). In MSEM, similar to the finite element method, the domain Ω is divided into N_e non-overlapping subdomains Ω_e (in the 2-D case, after converting the subdomain, we consider the Ω_e subdomains to be rectangular $[-1,1]\times[-1,1]$ and use triangle cell shapes) as

$$\overline{\Omega} = \bigcup_{i=1}^{N_e} \overline{\Omega}_i, \quad \bigcap_{i=1}^{N_e} \Omega_i = \emptyset.$$

Define the spaces of state and control approximations as follows:

$$U_h^e = \{u \in U : u|_{\Omega_e} \in \mathbb{P}_N,\, e = 1,\dots,N_e\},$$

$$Y_h^e = \{y \in Y : y|_{\Omega_e} \in \mathbb{P}_N,\, e = 1,\dots,N_e\},$$

where $\mathbb{P}_N$ is the space of polynomials of degree less than or equal to N. Basis functions are considered as $\varphi_j = Q_j - Q_{j+2}$ in (2.2).

Now, for each element, the approximate solution of order N is defined as:

$$u_h^e = \sum_{j=0}^{N-2} u_j^e \varphi_j, \quad 1 \le e \le N_e. \quad (3.3) \tag{3.2}$$

Then, the MSEM approximation of Eq. (3.1) is obtained as follows:

For each element Ω_e, find $(u_h^e, y_h^e) \in U_h^e \times Y_h^e$ such that

$$\min_{u_h^e \in U_h^e} \left(g(y_h^e) + \rho(u_h^e) \right),$$

subject to

$$a(y_h^e, w_h^e) = (f + u_h^e, w_h^e), \quad w_h^e \in Y_h^e. \tag{3.3}$$

The control problem (3.3) has a unique solution (y_h^e, u_h^e), and (y_h^e, u_h^e) is the solution of (3.3) if and only if there exists a costate variable $p_h^e \in Y_h^e$ such that (y_h^e, p_h^e, u_h^e) satisfies the following conditions [61]:

$$a(y_h^e, w_h^e) = (f + u_h^e, w_h^e), \quad w_h \in Y_h^e \subset Y = H_0^1(\Omega),$$

$$a(p_h^e, q_h^e) = (g'(y_h^e), q_h^e), \quad q_h \in Y_h^e \subset Y = H_0^1(\Omega),$$

$$(\rho'(u_h^e) + p_h^e, v_h^e - u_h^e)_U \ge 0, \quad v_h^e \in K_h^e \subset U = L^2(\Omega_U).$$

Using Eq. (3.2), the matrix form of Eq. (3.3) on each element is obtained as:

$$\min\, g(Y_h^e) + \rho(U_h^e),$$

subject to

$$S_e Y_h^e = M_e U_h^e + F_e, \quad e = 1,\dots,N_e, \tag{3.4}$$

where $F_e(i) = \int_{\Omega_e} f\varphi_i\, dx$. The vectors U_h^e and Y_h^e contain the approximate solution of order N on the element Ω_e, M_e is the local diagonal mass matrix, and S_e is the local stiffness matrix on the element Ω_e.

In order to obtain a discrete form on the general domain, we must assemble the local matrices M_e and S_e and obtain the general matrices M and S. Thus, Eq.(3.2) becomes:

$$\min\, g(Y_h) + \rho(U_h),$$

subject to

$$SY_h = MU_h + F,$$

where U_h and Y_h are the vectors of the approximate solutions on the general domain Ω.

4. Solving the Optimization Part of Elliptic Optimal Control Problem using the Iterative SBM

4.1. Bregman Iterative Method

The Bregman method is an iterative optimization method introduced by Bregman in 1967. He used this method for finding the extreme of convex functions [?]. Osher et al. used this method to solve some problems in image processing [51]. The Bregman iterative method is widely used for solving the problem of basis pursuit:

$$\min_u \|u\|_1, \quad \text{subject to} \quad Au = f,$$

as discussed in [67,68,69]. The application of this method in solving medical images has been investigated in [70]. This method can be used to solve constrained optimization problems [51]. The Bregman iterative method is defined based on Bregman distance.

Definition 4.1 (Subgradient) *Suppose $g : \chi \to \mathbb{R}$ is a convex function and $u \in \chi$. An element $x \in \chi^*$ (where χ^* is the dual of χ) is called a subgradient of g at u if for all $v \in \chi$,*

$$g(v) - g(u) - \langle x, u - v\rangle \geq 0,$$

where $\langle \cdot, \cdot \rangle$ is an inner product.

The set of all subgradients of g at u is called the subdifferential of g at u and is denoted by $\partial g(u)$.

Definition 4.2 (Bregman Distance [66]) *Suppose $g : \chi \to \mathbb{R}$ is a convex function, $u, v \in \chi$, and $x \in \partial g(v)$. Then the Bregman distance between points u and v is defined by*

$$D_x^g(u, v) = g(u) - g(v) - \langle x, u - v\rangle.$$

The Bregman distance satisfies the following conditions [66]:

1. If $u, v \in \chi$ and $x \in \partial g(v)$, then $D_x^g(u, v) \geq 0$.
2. $D_x^g(v, v) = 0$.
3. If w lies between u and v, then $D_x^g(u, v) \geq D_x^g(w, v)$.

Now, suppose there are two convex functions f and g on $\mathbb{R}^n$ such that

$$\min_{u\in\mathbb{R}^n} f(u) = 0.$$

Consider the unconstrained optimization problem:

$$\min_{u\in\mathbb{R}^n} g(u) + \alpha f(u). \tag{4.1}$$

Eq. (4.1) can be solved using the Bregman iterative method [66]:

$$u^{k+1} = \min_u D_x^g(u, u^k) + \alpha f(u) = \min_u g(u) - \langle x^k, u - u^k\rangle + \alpha f(u), \tag{4.2}$$

where x^0 is an initial value. In [51], the authors analyze the convergence of the Bregman iterative scheme.

Theorem 4.1 (Convergence of Bregman Iterative Method [51]) *If f and g are convex functions, f is differentiable, and Eq. (4.2) has a solution, then*

$$f(u^{k+1}) \leq f(u^k),$$

$$f(u^k) \leq f(u^*) + \frac{g(u^*)}{k},$$

where u^* is a minimizer of f.

Now, consider the constrained optimization problem:

$$\min_u g(u), \quad \text{subject to} \quad Au = b, \tag{4.3}$$

where A is a linear operator and b is a vector. Problem (4.3) can be transformed into an unconstrained problem:

$$\min_u g(u) + \frac{\alpha}{2}\|Au - b\|_2^2, \tag{4.4}$$

which is a special case of problem (4.1). Obviously, Eqs. (4.3) and (4.4) have the same solution. The Bregman iteration (4.2) for problem (4.4) is:

$$u^{k+1} = \min_u D_x^g(u, u^k) + \frac{\alpha}{2}\|Au - b\|_2^2 = \min_u g(u) - \langle x^k, u - u^k\rangle + \frac{\alpha}{2}\|Au - b\|_2^2,$$

$$x^{k+1} = x^k - \alpha A^T(Au^{k+1} - b). \tag{4.5}$$

If A is linear, the complicated formula (4.5) can be simplified as [51]:

$$u^{k+1} = \min_u g(u) + \frac{\alpha}{2}\|Au - b^k\|_2^2,$$

$$b^{k+1} = b^k + b - Au^k. \tag{4.6}$$

Equations (4.5) and (4.6) and Theorem 4.1 give the following result:

$$\lim_{k\to\infty} Au^k = b.$$

The following theorem shows that a solution to (4.6) is a solution to the original constrained problem (4.3).

Theorem 4.2 (Solution to Constrained Problem [50]) *Let $g : \mathbb{R}^n \to \mathbb{R}$ be convex and $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Consider the algorithm (4.6). Suppose that some iterate, u^*, satisfies $Au^* = b$. Then u^* is a solution to the original constrained problem (4.3).*

4.2. Split Bregman Method

Goldstein and Osher used the Split Bregman Method (SBM) for solving the problem [50]:

$$\min_u |\phi(u)| + g(u),$$

where $|\cdot|$ denotes the ℓ_1-norm, and both $|\phi(u)|$ and $g(u)$ are convex functions. They considered the following problem:

$$\min_{u,d} |d| + g(u), \quad \text{subject to} \quad d = \phi(u).$$

To solve this problem, first, we convert it into an unconstrained problem:

$$\min_{u,d} |d| + g(u) + \frac{\alpha}{2}\|d - \phi(u)\|_2^2.$$

If we let $\Theta(u,d) = |d| + g(u)$, and define $\Psi(u,d) = d - \phi(u)$, then the Bregman iteration will be as follows:

$$(u^{k+1}, d^{k+1}) = \min_{u,d} D_x^\Theta(u, u^k, d, d^k) + \frac{\alpha}{2}\|d - \phi(u)\|_2^2 = \min_{u,d} \Theta(u,d) - \langle x_u^k, u - u^k\rangle - \langle x_d^k, d - d^k\rangle + \frac{\alpha}{2}\|d - \phi(u)\|_2^2,$$

$$x_u^{k+1} = x_u^k - \alpha(\nabla\phi)^T(\phi(u^{k+1}) - d^{k+1}),$$
$$x_d^{k+1} = x_d^k - \alpha(d^{k+1} - \phi(u^{k+1})).$$

By applying the simplification form presented in (4.5), we have the SBM:

$$\begin{cases} (u^{k+1}, d^{k+1}) = \min_{u,d} |d| + g(u) + \frac{\alpha}{2}\|d - \phi(u) - b^k\|_2^2, \\ b^{k+1} = b^k + (\phi(u^{k+1}) - d^{k+1}). \end{cases} \tag{4.7}$$

In order to implement the algorithm (4.7), we must be able to solve the problem:

$$(u^{k+1}, d^{k+1}) = \min_{u,d} |d| + g(u) + \frac{\alpha}{2}\|d - \phi(u) - b^k\|_2^2.$$

As a result, we need to perform the following three steps:

1. Step 1: $u^{k+1} = \min_u g(u) + \frac{\alpha}{2}\|d^k - \phi(u) - b^k\|_2^2$,
2. Step 2: $d^{k+1} = \min_d |d| + \frac{\alpha}{2}\|d - \phi(u^{k+1}) - b^k\|_2^2$,
3. Step 3: $b^{k+1} = b^k + (\phi(u^{k+1}) - d^{k+1})$.

The speed of the SBM is largely dependent on how fast we can solve each of these subproblems [50].

4.3. Solving the Optimization Part of Elliptic Optimal Control Problem using the Iterative SBM

We consider the convex optimal control problem:

$$\begin{aligned} &\min \quad \tfrac{1}{2}\int_\Omega (y - y_0)^2\,dx + \tfrac{1}{2}\int_{\Omega_U} u^2\,dx \\ &s.t \quad -\Delta y = f + u \quad \text{in} \quad \Omega, \quad y|_{\partial\Omega} = 0. \end{aligned} \tag{4.8}$$

We assume that $\Omega_U = \Omega = [-1, 1]$. We also use the same meshes for the approximation of the state and the control variables. Assume that the state variable y and control variable u are approximated in the MSEM space Y_h and U_h, respectively, with φ_i as basis functions. Thus, the problem (4.8) is discretized as the following optimization problem:

$$\begin{aligned} &\min \quad \tfrac{1}{2}(Y_h - Y_0)^T Q (Y_h - Y_0) + \tfrac{1}{2}U_h^T Q U_h, \\ &s.t \quad AY_h = BU_h + F, \end{aligned} \tag{4.9}$$

where

$$Q_{ij} = M_{ij} = B_{ij} = \int_\Omega \varphi_i \varphi_j\,dx,$$

$$A_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\,dx,$$

$$F_i = \int_\Omega f\varphi_j\,dx.$$

With these assumptions, the iterative Bregman method to solve the problem (4.9) is obtained as:

$$\begin{cases} (Y_h^{k+1}, U_h^{k+1}) = \min G(Y_h) - G(Y_h^k) - \langle x_Y^k, Y_h - Y_h^k\rangle + M(U_h) - \langle x_U^k, U_h - U_h^k\rangle + \frac{\alpha}{2}\|BU_h + F - AY_h\|_2^2, \\ x_Y^{k+1} = x_Y^k + \alpha B^T(BU_h^{k+1} + F - AY_h^{k+1}), \\ x_U^{k+1} = x_U^k + \alpha A^T(BU_h^{k+1} + F - AY_h^{k+1}). \end{cases}$$

Here, $G(Y) = \frac{1}{2}(Y_h - Y_0)^T Q(Y_h - Y_0)$ and $M(U) = \frac{1}{2}U_h^T Q U_h$.

The Split Bregman Method (SBM) is:

$$\begin{cases} Y_h^{k+1} = \min G(Y_h) + \langle \lambda^k, -AY_h\rangle + \frac{\alpha}{2}\|BU_h^k + F - AY_h\|_2^2, \\ U_h^{k+1} = \min M(U_h) + \langle \lambda^k, BU_h\rangle + \frac{\alpha}{2}\|BU_h + F - AY_h^{k+1}\|_2^2, \\ \lambda^{k+1} = \lambda^k + \alpha(BU_h^{k+1} + F - AY_h^{k+1}). \end{cases}$$

The speed of the SBM is largely dependent on how fast we can solve subproblems. To solve these subproblems, one can use direct methods and some methods that are suitable for solving linear systems.

4.4. Convergence Theorem

The Lagrangian of the primal problem (4.9) is defined as:

$$L(Y_h, U_h, \lambda) = G(Y_h) + M(U_h) + \langle \lambda, BU_h + F - AY_h \rangle.$$

Now, define the convex function $q : \mathbb{R}^d \to (-\infty, \infty)$ as follows:

$$q(\lambda) = \inf\ L(Y_h, U_h, \lambda).$$

The dual problem to (4.9) is:

$$\max\ q(\lambda). \tag{4.10}$$

Finding optimal solutions of (4.9) and (4.10) is equivalent to finding a saddle point of L. More precisely, (Y_h^*, U_h^*) is an optimal primal solution and λ^* is an optimal dual solution if and only if:

$$L(Y_h^*, U_h^*, \lambda) \leq L(Y_h^*, U_h^*, \lambda^*) \leq L(Y_h, U_h, \lambda^*), \quad \forall Y_h, U_h, \lambda.$$

Theorem 4.3 (Convergence of SBM [71]) *Let λ^0 and Y^0 be arbitrary, and let $\alpha > 0$. For given sequences $\{\mu_k\}$ and $\{\nu_k\}$ such that:*

$$\mu_k \geq 0, \quad \nu_k \geq 0, \quad \sum_{k=0}^{\infty} \mu_k < \infty, \quad \sum_{k=0}^{\infty} \nu_k < \infty,$$

suppose:

$$\|Y_h^{k+1} - \min\ G(Y_h) + \langle \lambda^k, -AY_h \rangle + \frac{\alpha}{2} \|BU_h^k + F - AY_h\|_2^2\| \leq \mu_k,$$

and:

$$\|U_h^{k+1} - \min\ M(U_h) + \langle \lambda^k, BU_h \rangle + \frac{\alpha}{2} \|BU_h + F - AY_h^{k+1}\|_2^2\| \leq \nu_k.$$

If there exists a saddle point of $L(Y_h, U_h, \lambda)$, then:

$$\lambda^k \to \lambda^*, \quad U_h^k \to U_h^*, \quad Y_h^k \to Y_h^*,$$

where $(Y_h^*, U_h^*, \lambda^*)$ is such a saddle point. On the other hand, if no such saddle point exists, then at least one of the sequences $\{\mu_k\}$ or $\{\nu_k\}$ must be unbounded.

5. Numerical Examples

In this section, we carry out some numerical examples to demonstrate our theoretical results for the one-dimensional case. Our examples were solved using MATLAB. In SBM, we set $\alpha = 1$. The stop condition for SBM is considered as:

$$\frac{\|BUh + F - AYh\|}{\|F\|} \leq 10^{-10}.$$

We also use the SQP method for solving the problems and compare it with the results obtained by the SBM.

5.1. Example 6.1

In this example, we let:

$$y_0 = 0, \quad f = -\left(1 + \pi^4\right) p.$$

The exact solutions are [38]:

$$p = \sin(\pi x), \quad y = y_0 + \pi^2 p, \quad u = \max\{0, \bar{p}\} - p.$$

Table 2 shows the L_2-error for Y and U with different values of N and $N_e = 100$. Figures 4 and 5 show the graphs of the exact solution and the approximate solution of y and u with $N = 4$ and $N_e = 20$, respectively. Figures 6 and 7 show the graphs of the absolute error $\|Y - Y_h\|$ and $\|U - U_h\|$, with $N = 4$ and $N_e = 20$, respectively.

It should be noted that the CPU time with 400 nodal points for the SQP method is 142.76 seconds in comparison to SBM, which needs 17.02 seconds to achieve the approximate solution.

Table 2: Numerical results for Example 6.1 with $N_e = 100$

N	With SBM			With SQP		
	$\|Y - Y_h\|_2$	$\|U - U_h\|_2$	Time (s)	$\|Y - Y_h\|_2$	$\|U - U_h\|_2$	Time (s)
1	2.2497e-02	4.6069e-03	4.707041	2.2497e-02	4.6069e-03	4.939759
2	2.0375e-06	2.3864e-07	7.357864	4.7086e-06	3.4908e-04	14.673594
3	1.8452e-08	4.6903e-08	11.922072	7.6608e-06	5.8120e-04	51.720935
4	1.6958e-08	3.4372e-08	17.024132	9.0156e-05	1.6078e-03	142.755224
5	1.2845e-08	2.4030e-08	57.960903	8.4974e-05	2.0295e-03	287.938128

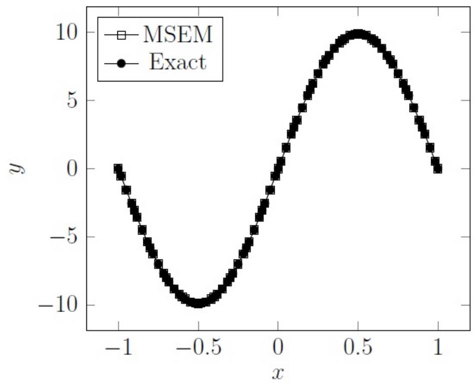

Figure 4: The exact solution and its MSEM solution with SBM of y, for Example 6.1 with $N = 4$ and $N_e = 20$.

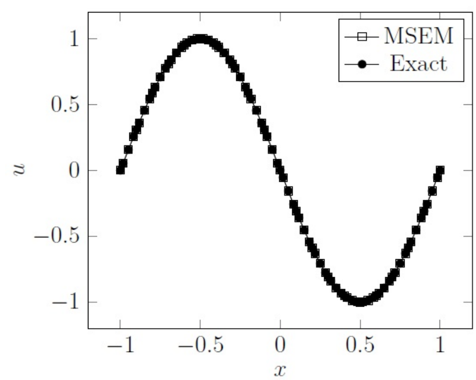

Figure 5: The exact solution and its MSEM solution with SBM of u, for Example 6.1 with $N = 4$ and $N_e = 20$.

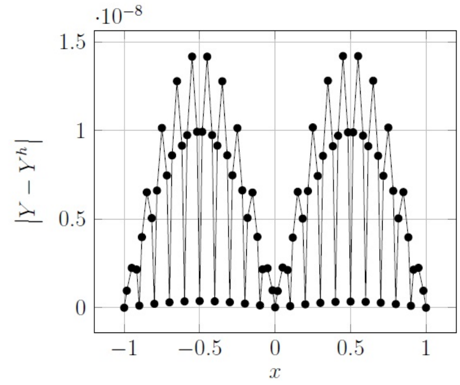

Figure 6: Absolute error $\|Y - Y_h\|$, for Example 6.1 with $N = 4$ and $N_e = 20$.

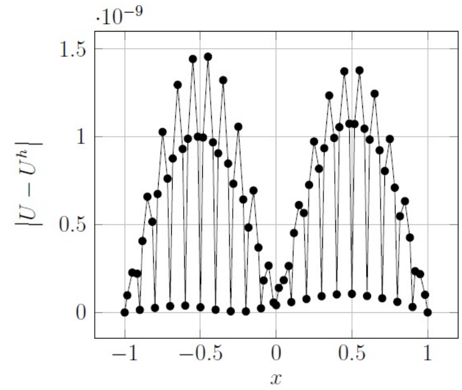

Figure 7: Absolute error $\|U - U_h\|$, for Example 6.1 with $N = 4$ and $N_e = 20$.

5.2. Example 6.2

In this example, we set:

$$y_0 = |x|, \quad f = -\left(1+\pi^4\right)\sin(\pi x) + \frac{|x|^3}{6} - \frac{1}{6}.$$

The exact solutions are [38]:

$$p = \sin(\pi x) + \frac{|x|^3}{6} - \frac{1}{6}, \quad y = \pi^2 \sin(\pi x), \quad u = -p.$$

Table 3 shows the L_2 norm errors for Y and U with $N = 1, 2, 3, 4$. Figures 8, 9, 10, and 11 show the plots of errors for Y and U. In Figures 12 and 13, we show the L_2-error of Y and U as a function of N (the degree of the polynomials), for a fixed value of $N_e = 20$, respectively.

Table 3: Numerical results for Example 6.2 with $N_e = 100$

N	With SBM			With SQP		
	$\Vert Y - Y_h\Vert_2$	$\Vert U - U_h\Vert_2$	Time (s)	$\Vert Y - Y_h\Vert_2$	$\Vert U - U_h\Vert_2$	Time (s)
1	2.2498e-02	4.6188e-03	13.956775	2.2497e-02	4.6250e-03	17.370773
2	2.0394e-06	2.3940e-07	23.948239	2.6321e-05	3.1900e-04	53.208305
3	3.8962e-08	5.5200e-08	49.151134	8.5320e-05	8.2462e-04	116.320883
4	3.2305e-08	4.4158e-08	50.746974	3.3278e-04	3.0183e-03	214.158470

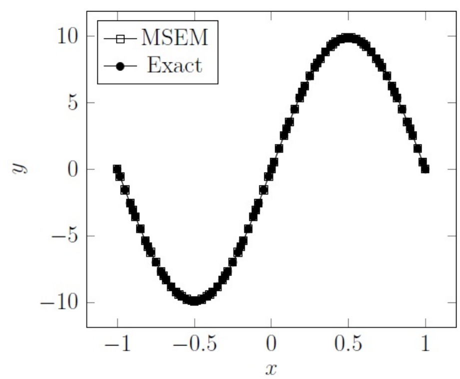

Figure 8: The exact solution and its MSEM solution with SBM of y, for Example 6.2 with $N = 4$ and $N_e = 20$.

6. Conclusion

Spectral polynomials are useful tools for solving ordinary and partial differential equations. Also, the incorporation of the finite element method with spectral polynomials, i.e., the use of spectral polynomials as a new shape function in the finite element method, is very efficient for obtaining a numerical algorithm with high accuracy. In this paper, we constructed a Muntz Spectral Element Method (MSEM) for the solution of the one-dimensional elliptic optimal control problem. We used the MSEM for discretizing the spatial space and then solved the discrete form, which is an optimization problem, using the Split Bregman Method (SBM). The numerical results show the speed and accuracy of SBM for solving this kind of problem.

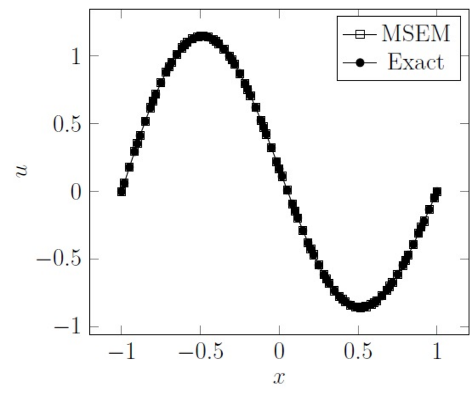

Figure 9: The exact solution and its MSEM solution with SBM of u, for Example 6.2 with $N = 4$ and $N_e = 20$.

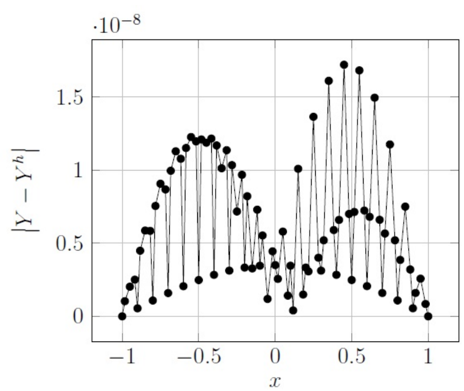

Figure 10: Absolute error $\|Y - Y_h\|$, for Example 6.2 with $N = 4$ and $N_e = 20$.

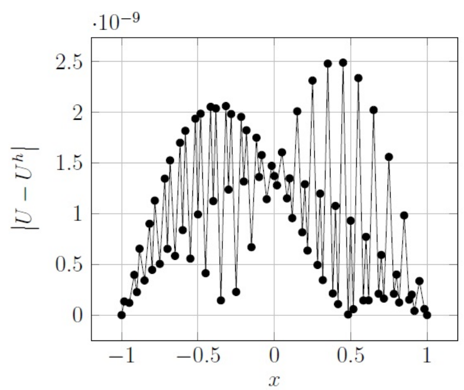

Figure 11: Absolute error $\|U - U_h\|$, for Example 6.2 with $N = 4$ and $N_e = 20$.

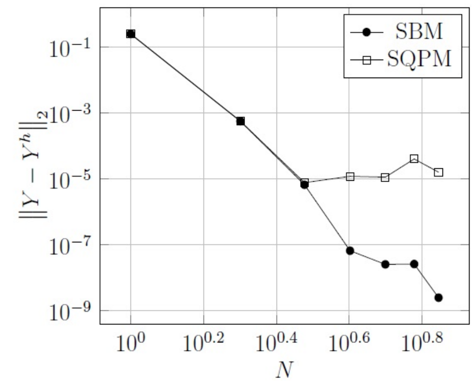

Figure 12: The L_2-error of y as a function of N with $N_e = 20$.

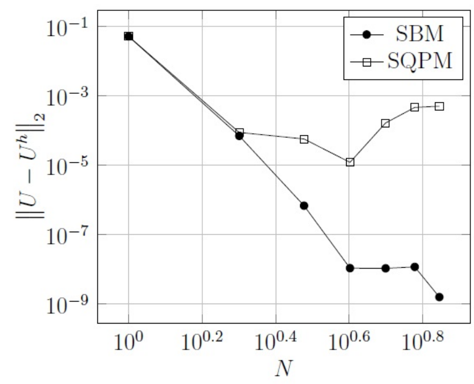

Figure 13: The L_2-error of u as a function of N with $N_e = 20$.

Availability of Data and Material

The results and numerical data obtained in this paper have been fully tested. These results are obtained using MATLAB R2017a (win64) software and Windows 8 operating system on an Intel(R) Core(TM) i7 CPU, 1.73 GHz processor with 4 GB RAM. The authors declare that all data and material in the paper are available and veritable.

Competing interests

The author declares that he has no competing interests.

Funding

Not applicable.

Acknowledgements

The author is thankful to the honorable editors and reviewers.

References

1. J. C. Slater, *Electronic Energy Bands in Metals*, Harvard University, Phys. Rev. 45, (1934), 794–801.
2. J. Barta, *Über die Randwertaufgabe der gleichmäßig belasteten Kreisplatte*, Z. Angew. Math. Mech, 17(1): 1937, 184–185.
3. R. A. Frazer, S. W. Jones, and W. N. P. Skan, *Approximations to Functions and to the Solutions of Differential Equations*, Reports and memoranda, HSMO, 1937.
4. C. Lanczos, *Trigonometric Interpolation of Empirical and Analytical Functions*, 1938.
5. B. Galerkin, *Vestnik inzhenerov*, Technology, 19:897–908, 1915.
6. L. Collatz, *Gewöhnliche Differentialgleichungen*, Teubner, Stuttgart, 6th ed., 1981.
7. B. A. Finlayson, *The Method of Weighted Residuals and Variational Principles, with Application in Fluid Mechanics, Heat and Mass Transfer*, Math. Sci. Eng., Elsevier Science, 1972.
8. B. Y. Guo, *Spectral Methods and Their Applications*, World Scientific Publishing Company, Incorporated, 1998.
9. W. J. Duncan, *Galerkin's Method in Mechanics and Differential Equations*, Reports and memoranda, H.M. Stationery Office, 1938.
10. W. J. Duncan, *Note on Galerkin's Method for the Treatment of Problems Concerning Elastic Bodies*, Phill. Mag., 25(11):628–633, November 2008.
11. W. J. Duncan, *Application of the Galerkin Method to the Torsion Flexure of Cylinders and Prisms*, Phill. Mag., 25(11):636–649, November 2008.
12. W. J. Duncan and D. D. Lindsay, *Methods for Calculating the Frequencies of Overtones*, Reports and memoranda, H.M. Stationery Office, 1939.
13. C. A. J. Fletcher, *Computational Galerkin Methods*, Springer Series in Computational Physics, Springer-Verlag, 1984.
14. D. Gottlieb, M. Y. Hussaini, and S. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*, Mathematics in Science and Engineering, Elsevier Science, (1986).
15. A. Hrennikoff, *Solution of Problems of Elasticity by the Framework Method*, Journal of Applied Mechanics, 8(4): (1941), 169–175.
16. R. Courant, *Variational Methods for the Solution of Problems of Equilibrium and Vibrations*, Bulletin of the American Mathematical Society, 49: (1943), 1–23. doi:10.1090/s0002-9904-1943-07818-4.
17. G. Strang and G. Fix, *An Analysis of the Finite Element Method*, Prentice Hall, (1973). ISBN 978-0-13-032946-2.
18. O. C. Zienkiewicz, R. L. Taylor, and J. Z. Zhu, *The Finite Element Method: Its Basis and Fundamentals*, Butterworth-Heinemann, (2013). ISBN 978-0-08-095135-5.
19. K. J. Bathe, *Finite Element Procedures*, Cambridge, MA: Klaus-Jürgen Bathe, (2006). ISBN 978-0979004902.
20. A. T. Patera, *A Spectral Element Method for Fluid Dynamics: Laminar Flow in a Channel Expansion*, Journal of Computational Physics, 54, (1984), 468–488.
21. E. Priolo and G. A. Seriani, *A Numerical Investigation of Chebyshev Spectral Element Method for Acoustic Wave Propagation*, Proceedings of the 13th IMACS Conference, 54, (1991), 154–172.

22. C. Zhu, G. Qin, and J. Zhang, *Implicit Chebyshev Spectral Element Method for Acoustic Wave Equations*, Finite Elements in Analysis and Design, Volume 47, Issue 2, (2011), 184–194. https://doi.org/10.1016/j.finel.2010.09.004.

23. Y. Li and X. K. Li, *The Chebyshev Spectral Element Approximation with Exact Quadratures*, Journal of Computational and Applied Mathematics, Volume 296, (2016), 320–333. https://doi.org/10.1016/j.cam.2015.09.021.

24. A. Quaglino, M. Gallieri, J. Masci, and J. Koutn, *Accelerating Neural ODEs with Spectral Elements*, ArXiv, abs/1906.07038, (2019).

25. M. Lotfi and A. Alipanah, *Legendre Spectral Element Method for Solving Sine-Gordon Equation*, Advances in Differential Equations, 2019, 113, (2019). https://doi.org/10.1186/s13662-019-2059-7.

26. M. Saffarian and A. Mohebbi, *Numerical Solution of Two and Three Dimensional Time Fractional Damped Nonlinear Klein–Gordon Equation Using ADI Spectral Element Method*, Applied Mathematics and Computation, Volume 405, (2021), 126–182. https://doi.org/10.1016/j.amc.2021.126182.

27. M. Abbaszadeh and M. Dehghan, *The Proper Orthogonal Decomposition Modal Spectral Element Method for Two-Dimensional Viscoelastic Equation*, Thin-Walled Structures, Volume 161, (2021), 1–14. https://doi.org/10.1016/j.tws.2020.107429.

28. M. Dehghan, N. Shafieeabyaneh, and M. Abbaszadeh, *Application of Spectral Element Method for Solving Sobolev Equations with Error Estimation*, Applied Numerical Mathematics, Volume 158, (2020), 439–462. https://doi.org/10.1016/j.apnum.2020.08.010.

29. G. Duvaut and J. L. Lions, *The Inequalities in Mechanics and Physics*, Springer, Berlin, (1973).

30. F. S. Falk, *Approximation of a Class of Optimal Control Problems with Order of Convergence Estimates*, Journal of Mathematical Analysis and Applications, (1973), 28–48.

31. D. A. French and J. T. King, *Approximation of an Elliptic Control Problem by the Finite Element Method*, Numerical Functional Analysis and Optimization, (1991), 299–314.

32. T. Geveci, *On the Approximation of the Solution of an Optimal Control Problem Governed by an Elliptic Equation*, RAIRO Analyse Numérique, (1979), 313–328.

33. W. Alt, *On the Approximation of Infinite Optimization Problems with an Application to Optimal Control Problems*, Applied Mathematics and Optimization, (1984), 15–27.

34. K. Malanowski, *Convergence of Approximations vs. Regularity of Solutions for Convex, Control Constrained, Optimal Control Systems*, Applied Mathematics and Optimization, (1982), 69–95.

35. D. Tiba, *Lectures on the Optimal Control of Elliptic Equations*, University of Jyvaskyla Press, Jyvaskyla, (1995).

36. J. Haslinger and P. Neittaanmaki, *Finite Element Approximation for Optimal Shape Design*, Wiley, Chichester, (1989).

37. M. Lotfi, *Finite Element and Split Bregman Methods for Solving a Family of Optimal Control Problems with Partial Differential Equation Constraint*, International Journal of Difference Equations, Vol. 15, No. 1, (2020), 11–29.

38. Y. Chen, N. Yi, and W. Liu, *A Legendre–Galerkin Spectral Method for Optimal Control Problems Governed by Elliptic Equations*, SIAM Journal on Numerical Analysis, 46(5), (2008), 2254–2275.

39. M. Lotfi and A. Alipanah, *Legendre Spectral Element Method for Solving Volterra-Integro Differential Equations*, Results in Applied Mathematics, Vol. 7, (2020), 1–11.

40. M. Lotfi and A. Alipanah, *Using Legendre Spectral Element Method with Quasi-Linearization Method for Solving Bratu's Problem*, Computational Methods in Differential Equations, Vol. 7, (2019), 580–588.

41. M. Lotfi and A. Alipanah, *Legendre Spectral Element Method for Solving Sine-Gordon Equation*, Advances in Differential Equations, Vol. 113, (2019), 1–20.

42. Q. Ai, H. Y. Li, and Z. Q. Wang, *Diagonalized Legendre Spectral Methods Using Sobolev Orthogonal Polynomials for Elliptic Boundary Value Problems*, Applied Numerical Mathematics, 127, (2018), 196–210.

43. Y. Wang, G. Qin, and Z. Q. Wang, *An Improved Time-Splitting Method for Simulating Natural Convection Heat Transfer in a Square Cavity by Legendre Spectral Element Approximation*, Computers & Fluids, 174, (2018), 122–134.

44. A. Khan, C. S. Upadhyay, and M. Gerritsma, *Spectral Element Method for Parabolic Interface Problems*, Computer Methods in Applied Mechanics and Engineering, 337, (2018), 66–94.

45. Q. Zhuang and L. Chen, *Legendre–Galerkin Spectral-Element Method for the Biharmonic Equations and Its Applications*, Computers and Mathematics with Applications, Volume 74, (2017), 2958–2968.

46. F. Zeng, H. Ma, and T. Zhao, *Alternating Direction Implicit Legendre Spectral Element Method for Schrödinger Equations*, Journal of Shanghai University (Natural Science Edition), 17(6), (2011), 724–727.

47. F. X. Giraldo, *Strong and Weak Lagrange–Galerkin Spectral Element Methods for the Shallow Water Equations*, Computers & Mathematics with Applications, 45, (2003), 97–121.

48. S. B. Hazra, *Large-Scale PDE-Constrained Optimization in Applications*, Springer-Verlag Berlin Heidelberg, (2010).

49. M. Hinze, *Optimal and Instantaneous Control of the Instationary Navier-Stokes Equations*, Habilitation thesis, Technische Universität Berlin, Berlin, Germany, (2000).

50. T. Goldstein and S. Osher, *The Split Bregman Method for L1-Regularized Problems*, SIAM Journal on Imaging Sciences, (2009), 323–343.

51. S. Osher, M. Burger, D. Goldfarb, J. Xu, and W. Yin, *An Iterative Regularization Method for Total Variation-Based Image Restoration*, MMS, (2005), 460–489.

52. Ch. H. Muntz, *Über den Approximationssatz von Weierstrass*, H. A. Schwarz's Festschrift, Berlin, pp., (1914), 303–312.

53. O. Szász, *Über die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen*, Math. Ann. 77: (1916), 482–496. doi:10.1007/BF01456964.

54. G. V. Badalyan, *Generalization of Legendre Polynomials and Some of Their Applications*, Akad. Nauk. Armyan. SSR Izv. Fiz.-Mat. Estest. Tekhn. Nauk, 8(5), (1956), 1–28, (1955) and 9(1), 3–22, (Russian, Armenian summary).

55. A. K. Taslakyan, *Some Properties of Legendre Quasipolynomials with Respect to a Muntz System*, Mathematics, Erevan Univ., Erevan, 2, (1984), 179–189 (Russian, Armenian summary).

56. J. Shen and Y. Wang, *Muntz–Galerkin Methods and Applications to Mixed Dirichlet–Neumann Boundary Value Problems*, SIAM Journal on Scientific Computing, 38(4): (2016), 2357–2381.

57. S. Hosseinpour, A. Nazemi, and E. Tohidi, *Muntz-Legendre Spectral Collocation Method for Solving Delay Fractional Optimal Control Problems*, Journal of Computational and Applied Mathematics, 351, (2019), 344–363.

58. S. Sabermahani and Y. Ordokhani, *A New Operational Matrix of Muntz-Legendre Polynomials and Petrov Galerkin Method for Solving Fractional Volterra-Fredholm Integrodifferential Equations*, Computational Methods in Differential Equations, 8, (2020), 408–423.

59. P. Mokhtary, F. Ghoreishi, and H. M. Srivastava, *The Muntz Legendre Tau Method for Fractional Differential Equations*, Applied Mathematical Modelling, 40(2), (2016), 671–684.

60. G. V. Milovanovic, *Muntz Orthogonal Polynomials and Their Numerical Evaluation*, International Series of Numerical Mathematics, Vol. 131, (1999).

61. J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin, (1971).

62. J. S. Hesthaven, S. Gottlieb, and D. Gottlieb, *Spectral Methods for Time-Dependent Problems*, Cambridge University Press, (2007).

63. A. Ern and J. L. Guermond, *Theory and Practice of Finite Elements*, New York, NY: Springer-Verlag, (2004).

64. L. N. Hand and J. D. Finch, *Analytical Mechanics*, Cambridge University Press, (2008).

65. C. Pozrikidis, *Introduction to Finite and Spectral Element Methods Using Matlab*, Chapman and Hall/CRC, (2005).

66. L. M. Bregman, *The Relaxation Method of Finding the Common Point of Convex Sets and Its Application to the Solution of Problems in Convex Programming*, USSR Computational Mathematics and Mathematical Physics, (1967), 200–217.

67. S. Osher, Y. Mao, B. Dong, and W. Yin, *Fast Linearized Bregman Iterations for Compressed Sensing and Sparse Denoising*, UCLA CAM Report, (2010), 08–37.

68. W. Yin, S. Osher, D. Goldfarb, and J. Darbon, *Bregman Iterative Algorithms for L1-Minimization with Applications to Compressed Sensing*, SIAM Journal on Imaging Sciences, (2008), 142–168.

69. J. F. Cai, S. Osher, and Z. Shen, *Linearized Bregman Iterations for Compressed Sensing*, UCLA CAM Report, (2009), 1515–1536.

70. L. He, T. Chiun Chang, and S. Osher, *MR Image Reconstruction from Sparse Radial Samples by Using Iterative Refinement Procedures*, Proceedings of the 13th Annual Meeting of ISMRM, (2006), 06–35.

71. E. Esser, *Applications of Lagrangian-Based Alternating Direction Methods and Connections to Split Bregman*, UCLA CAM Report, (2009), 09–31.

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