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# On double-switching $ARMA$ processes

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ABSTRACT: In this paper, we introduce a double-switching $ARMA$ model, in which the observed process is an $ARMA$ model subject to Markov switching and a periodic sequence of period $s_2$. We give conditions for the existence of periodic stationary solutions of the double-switching $ARMA$ and higher-order moments of such solutions in the general vector specification. We provide an expression in closed-form of the autocovariance function of this process and its higher power and therefore admit $ARMA$ representation.

Key Words: Markov-switching models, time-varying models, $ARMA$ models, periodic stationarity.

## Contents



## 1. Introduction

There are a large number of modifications to the standard $ARMA$ model for modeling nonlinear time-series models, but it behaves locally linearly (see., Brockwell & Davis [9]) utilizing time-dependent $ARMA$ coefficients, the first is the time-varying models, in particular, the models with periodic time-varying parameters (for more information, see., [1] – [2], [4], [15], [16] and [23]) and the second is the Markov-switching models (for more information, see., [3], [5]- [7], [11], [17]- [19], [21]- [22], [24] and [25]). In this paper, we broaden the well-known double-switching $ARMA$ processes with the $PARMA$ coefficients being a Markov chain with finitely many states. For this, a process $(X_t)_{t\in\mathbb{Z}}$ defined on some probability space $(\Omega,\Im,P)$ is said to be a double-switching $ARMA$ or $MS-ARMA$ process with periodic time-varying coefficients denoted by $DS-ARMA_{(s_1,s_2)}(p,q)$ if it is a solution of the following stochastic difference equation

$$X_t = a_{0,t}(\delta_t) + \sum_{i=1}^{p} a_{i,t}(\delta_t) X_{t-i} + \sum_{j=1}^{q} b_{j,t}(\delta_t) e_{t-j} + e_t, \tag{1.1}$$

In Eq. (1.1), $(\delta_t)_{t\in\mathbb{Z}}$ is a homogeneous, stationary, irreducible, aperiodic Markov chain with finite state space $\mathbb{E}=\{1,...,s_1\}$, which is independent of the independent and identically distributed $(i.i.d)$ sequence $(e_t)_{t\in\mathbb{Z}}$ with $E\{e_t\}=0$ and $E\left\{\log^+|e_t|\right\}<\infty$ where for $x>0$, $\log^+ x=\max(\log x,0)$. In addition, we shall suppose that $e_t$ and $\{(X_{l-1},\delta_t), l\leq t\}$ are independent. The functions $a_{0,t}(\delta_t)$, $a_{i,t}(\delta_t)$ and $b_{j,t}(\delta_t)$, $i\in\{1,...,p\}$, $j\in\{1,...,q\}$ depend on a Markov chain $(\delta_t)_{t\in\mathbb{Z}}$ and a periodic time-varying of period $s_2$. This process is globally nonstationary when $s_2>1$, but is stationary within each period. Our model can be viewed as a mixture of dynamics models, which generalizes various classes of models that have been discussed in the literature, in fact:

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**(a)** Standard $ARMA\,(p,q)$ models: these models are acquired by supposing constant the coefficients in Eq. (1.1) (e.g., Wold [27]).

**(b)** Periodic $ARMA\,(p,q)$ $(PARMA_{s_2}\,(p,q))$ models: these models are acquired by supposing that the state space $\mathbb{E}=\{1\}$ (i.e., $s_1=1$) (e.g., Francq et al. [14]).

**(c)** Markov-switching $ARMA\,(p,q)$ $(MS-ARMA_{s_1}\,(p,q))$ models: these models are acquired by supposing that the period $s_2=1$ in Eq. (1.1) (e.g., Francq and Zakoïan [13]).

**(d)** Mixture $PARMA_{(s_1,s_2)}\,(p,q)$ models: if $(\delta_t)_{t\in\mathbb{Z}}$ is $i.i.d.$ across different dates (e.g., Cavicchioli [10]).

**(e)** Hidden-Markov models $(HMM)$: these models are acquired by setting $X_t = a_{0,0}\,(\delta_t)+b_{0,0}\,(\delta_t)\,e_t$, i.e., $a_{i,t}(.)=b_{j,t}(.)=0$ for all $i,j$ in Eq. (1.1) with $b_{0,0}\,(\delta_t)=1$ generally except that $b_{0,0}\,(\delta_t)\neq 0$ (e.g., Francq and Roussignol [12]).

An overview of the paper is organized as follows. Section 2 provides a state-space representation that is used to derive a sufficient condition for the $DS-ARMA_{(s_1,s_2)}$ process to have a unique stationary (in some sense), causal and ergodic solution having higher-order moments. The autocovariance structure is analyzed in section 3, which allows us to derive an $ARMA$ representation. In Section 4, we show that the power process $(X_t^m)_t$.

## 2. State-space representation and periodic stationarity of $DS-ARMA_{(s_1,s_2)}$ processes

### 2.1. Notations and state-space representation

Let the $r=(p+q)-$vectors $\underline{H}' := (1,\underline{O}'_{(r-1)})$, $\underline{a}_{0,t}(\delta_t)=a_{0,t}(\delta_t)\underline{H}$, $\underline{F}'=\left(1,\underline{O}'_{(p-1)},1,\underline{O}'_{(q-1)}\right)$ and $\underline{X}'_t := (X_t,...,X_{t-p+1},e_t,...,e_{t-q+1}\ )$ and the $r\times r-$matrice $\Gamma_t\,(\delta_t)$:

$$\begin{aligned}\Gamma_t\,(\delta_t) &= \left(\begin{array}{cccccc} a_{1,t}(\delta_t) & \ldots & a_{p,t}(\delta_t) & b_{1,t}(\delta_t) & \ldots & b_{q,t}(\delta_t) \\ I_{(p-1)} & & \underline{O}_{(p-1)} & & O_{(p-1,q)} & \\ & & & \underline{O}'_{(q-1)} & & 0 \\ & O_{(q,p)} & & I_{(q-1)} & & \underline{O}_{(q-1)} \end{array}\right) \\ : &= \left(\begin{array}{cc} A_t(\delta_t) & B_t\,(\delta_t) \\ O_{(q,p)} & J \end{array}\right).\end{aligned}$$

Then (1.1) can be written in the following state-space representation $X_t=\underline{H}'\underline{X}_t$ and

$$\underline{X}_t=\Gamma_t\,(\delta_t)\,\underline{X}_{t-1}+\underline{e}_t(\delta_t),\ t\in\mathbb{Z}, \tag{2.1}$$

where $\underline{e}_t(\delta_t) := \underline{a}_{0,t}(\delta_t)+\underline{F}e_t$, and so the extended process $\left(\widetilde{\underline{X}}_t := \left(\underline{X}'_t,\delta_t\right)', t\in\mathbb{Z}\right)$ is a Markov's chain on $\mathbb{R}^r\times\mathbb{E}$. Now, by iterating (2.1) $s_2-$times, we get

$$\underline{X}_t=\Gamma\,(t)\,\underline{X}_{t-s_2}+\underline{e}(t),\ t\in\mathbb{Z}, \tag{2.2}$$

in which $\Gamma\,(t) := \prod\limits_{j=0}^{s_2-1}\Gamma_{t-j}\,(\delta_{t-j})$ and $\underline{e}(t) := \sum\limits_{k=0}^{s_2-1}\left\{\prod\limits_{j=0}^{k-1}\Gamma_{t-j}\,(\delta_{t-j})\right\}\underline{e}_{t-k}\,(\delta_{t-k})$. Hence, the $r-$dimensional equation (2.1) (or $1-$dimensional equation (1.1)) has a causal, strictly periodically stationary solution (hereafter $SPS$), periodically correlated and periodically ergodic (hereafter resp. $PC$, $PE$ ) iff equation (2.2) has causal, strictly (resp. second-order, ergodic) stationary solution. Finally, since (2.1) (resp. (2.2)) is valid for all $t$ by successive substitution we gain a formal solution given by the series $\underline{X}_t^{(1)}=\sum\limits_{k\geq 0}\left\{\prod\limits_{j=0}^{k-1}\Gamma(t-js_2)\right\}\underline{e}(t-ks_2)$ (resp. $\underline{X}_t^{(2)}=\sum\limits_{k\geq 0}\left\{\prod\limits_{j=0}^{k-1}\Gamma_{t-j}\,(\delta_{t-j})\right\}\underline{e}_{t-k}\,(\delta_{t-k})$). Some notations are utilized throughout the paper:

- The $r^2\times s_1r^2-$matrix $\mathbb{I}'_{(r^2)} := \left(I_{(r^2)}\vdots\ldots\vdots I_{(r^2)}\right)$, where $I_{(r^2)}$ is the identity matrix.

- $O_{(k,l)}$ is the null matrix, in some special cases, we put $O_{(k)} := O_{(k,k)}$ and $\underline{O}_{(k)} := O_{(k,1)}$.
- $\rho(A)$ is the spectral radius of square matrix $A$. Let $\|.\|$ denote any operator norm on the set of $k \times n$ and $k \times 1$ matrices
- $\otimes$ is the usual Kronecker product of matrices and $A^{\otimes n} = A \otimes ... \otimes A$, $n-$times.
- For $\lambda \in ]0,1]$, $|A|^{\lambda} := (|a_{ij}|^{\lambda})$, then $|AB|^{\lambda} \leq |A|^{\lambda} |B|^{\lambda}$, $|A\underline{Y}|^{\lambda} \leq |A|^{\lambda} |\underline{Y}|^{\lambda}$ for any appropriate vector $\underline{Y}$ and $|\sum_i A_i| \leq \sum_i |A_i|$, moreover, if $a_{ij} \leq b_{ij}$ for all $i$ and $j$ then the inequality $A \leq B$.
- If $(A_k, k \in K)$ is squared matrices sequence, we note, for any $l$ and $j$, $\prod\limits_{k=l}^{j} A_k = A_l A_{l+1} \dots A_j$ if $l \leq j$ and $I_{(.)}$ otherwise.
- $\mathbb{P}^{(n)} = \left(p_{ij}^{(n)}\right)_{(i,j)\in\mathbb{E}\times\mathbb{E}}$ is the $n-$step transition probabilities matrix, where $p_{ij}^{(n)} = P(\delta_t = j | \delta_{t-n} = i)$ and $\mathbb{P} = \mathbb{P}^{(1)}$. Moreover, $\Pi' = (\pi(1), ..., \pi(s_1))$ is the initial distribution, where $\pi(i) = P(\delta_0 = i)$, $i = 1, ..., s_1$, such that $\Pi' = \Pi'\mathbb{P}$.
- For any set of non random matrices $A := \{A(i), i \in \mathbb{E}\}$, we note

$$\mathbb{P}^{(n)}(A) = \begin{pmatrix} p_{11}^{(n)} A(1) & \dots & p_{s_1 1}^{(n)} A(1) \\ \vdots & \dots & \vdots \\ p_{1 s_1}^{(n)} A(d) & \dots & p_{s_1 s_1}^{(n)} A(s_1) \end{pmatrix}, \ \Pi(A) = \begin{pmatrix} \pi(1)A(1) \\ \vdots \\ \pi(s_1)A(s_1) \end{pmatrix},$$

with $\mathbb{P}^{(1)}(A) = \mathbb{P}(A)$.

## 2.2. The strict periodic stationarity

Since $E\left\{\log^{+} \|\Gamma(t)\|\right\}$ and $E\left\{\log^{+} \|\underline{e}(t)\|\right\}$ are finite and the process $(\delta_t, e_t)_{t\in\mathbb{Z}}$ is stationary and ergodic, thus from Bougerol and Picard [8], the unique, causal, bounded in probability, strictly stationary and ergodic solution of (2.2) is given by the series $\underline{X}_t^{(1)}$ if and only if the top-Lyapunov exponent $\gamma_L(\Gamma)$ satisfies the following condition

$$\gamma_L(\Gamma) := \inf_{t\geq 1} \left\{ \frac{1}{t} E\left\{ \log \left\| \prod_{j=0}^{k-1} \Gamma(t - j s_2) \right\| \right\} \right\} \overset{a.s.}{=} \lim_{t\longrightarrow\infty} \frac{1}{t} \log \left\| \prod_{j=0}^{t-1} \Gamma(t - j s_2) \right\| < 0.$$

So, (2.2) is called has a unique, causal, strictly stationary and ergodic solution given by $\left(\underline{H}' \underline{X}_t^{(1)}\right)_{t\in\mathbb{Z}}$. The following theorem presents us with the main result for the stochastic difference equation (2.2) due to Bougerol and Picard [8].

**Theorem 2.1** *Let $(\underline{X}_t)_{t\in\mathbb{Z}}$ be the stochastic process defined by (2.2). If $\gamma_L(\Gamma) < 0$ then for all $t \in \mathbb{Z}$, the series $\underline{X}_t^{(1)}$ converges absolutely a.s. and constitute the unique, strictly stationary, ergodic and causal solution for (2.2). Conversely, if (2.2) has a strictly stationary solution, then $\gamma_L(\Gamma) < 0$.*

From the previous theorem, we get the following corollary

**Corollary 2.1** *Under the condition of Theorem 2.1, we find*

- *Equation (2.1) has an unique, causal, SPS and PE solution given by the series $\underline{X}_t^{(2)}$.*
- *The series $\underline{X}_t^{(2)}$ converges absolutely a.s. with $\underline{X}_t^{(1)} \overset{a.s.}{=} \underline{X}_t^{(2)}$.*
- *The multivariate process $\left(\underline{X}'_{s_2 t+1}, ..., \underline{X}'_{s_2 t+s_2}\right)'$ is strictly stationary.*

**Corollary 2.2** *Set* $\Lambda = (\Gamma_t(\delta_t))_{t\in\mathbb{Z}}$ *and let* $\gamma_L^{s_2}(\Lambda)$ *be the top-Lyapunov exponent associated with the sequence of* $s_2-$*periodic random matrices defined as*

$$\gamma_L^{s_2}(\Lambda) := \inf_{t\geq 1}\left\{\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{s_2t-1}\Gamma_{s_2t-j}(\delta_{s_2t-j})\right\|\right\}\right\}.$$

*If* $\gamma_L^{s_2}(\Lambda) < 0$ *then the results of Theorem 2.1 holds true.*

**Corollary 2.3** *Set* $A = (A_t(\delta_t))_{t\in\mathbb{Z}}$ *and let* $\gamma_L(A)$ *be the top-Lyapunov exponent associated with the sequence of random matrices* $A$*. Used same arguments as in Francq and Zakoïan [13], then we have* $\gamma_L(A) = \gamma_L(\Gamma)$*, and hence the results of Theorem 2.1 holds true if* $\gamma_L(A) < 0$*.*

**Remark 2.1** The condition governing the strict stationarity is independent of the moving average part.

Though the condition $\gamma_L(\Gamma) < 0$ could be used as a sufficient condition for the strict stationarity it is of little use in practice since this condition involves the limit of products of infinitely many random matrices. Hence, some simple sufficient conditions ensuring the negativity of $\gamma_L(\Gamma)$ can be given.

**Theorem 2.2** *Consider the* $DS-ARMA_{(s_1,s_2)}(p,q)$ *model (1.1) with state space representation (2.2). Then* $\gamma_L(\Gamma) < 0$ *if one of the following conditions holds true.*

- $E\left\{\left\|\prod_{j=0}^{t-1}|\Gamma(t-js_2)|\right\|\right\} < 1$ for some $t \geq 1$.
- $\log\rho(|\Gamma|) < 1$ where $|\Gamma| := E\{|\Gamma(t)|\}$.

**Proof:** Choosing an absolute norm,i.e., $\|.\|$ a norm such that $\|.\| \leq \||.|\|$, because the top-Lyapunov exponent is independent of the norm. According to Kesten and Spitzer [26] we have $\lim_{t\longrightarrow\infty}\frac{1}{t}\log\left\|\prod_{j=0}^{t-1}\Gamma(t-js_2)\right\| \leq \log\rho(|\Gamma|)$ a.s. and we get

$$\gamma_L(\Gamma) \leq \frac{1}{t}\log E\left\{\left\|\prod_{j=0}^{t-1}\Gamma(t-js_2)\right\|\right\} \leq \log\rho(|\Gamma|) \text{ a.s.}\blacksquare$$

□

**Proposition 2.1** *Consider the* $DS-ARMA_{(s_1,s_2)}(p,q)$ *model. Let* $\Gamma_v^{(\lambda)} := \left(\Gamma_v^{(\lambda)}(i), 1\leq i\leq s_1\right)$ *where* $\Gamma_v^{(\lambda)}(i) := |\Gamma_v(i)|^\lambda$ *for all* $v \in \{1,...,s_2\}$*. Then* $\rho\left(\prod_{v=0}^{s_2-1}\mathbb{P}\left(\Gamma_{s_2-v}^{(\lambda)}\right)\right) < 1$ *implies that* $\gamma_L(\Gamma) < 0$ *and hence the statements of the first assertion of Theorem 2.1 holds.*

**Proof:** Choose a norm $\|.\|$ such that $\|N\|^\lambda \leq \left\||N|^\lambda\right\|$ (e.g., $\|N\| = \sum_{i,j}|n_{ij}|$). Therefore, because $\rho\left(\prod_{v=0}^{s_2-1}\mathbb{P}\left(\Gamma_{s_2-v}^{(\lambda)}\right)\right) < 1$, there exists $0 < \kappa < 1$ such that $\limsup_t\left\|\prod_{v=0}^{s_2-1}\mathbb{P}^t\left(\Gamma_{s_2-v}^{(\lambda)}\right)\right\|^{1/t} < \kappa$. By Jensen inequality and submultiplicativity of the operator $|.|^\lambda$ we obtain

$$\begin{aligned}\gamma_L(\Gamma)\lambda &= \inf_{t\geq 1}\left\{\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{t-1}\prod_{i=0}^{s_2-1}\Gamma_{s_2(t-j)-i}\left(\delta_{s_2(t-j)-i}\right)\right\|^\lambda\right\}\right\}\\ &\leq \lim_{t\to\infty}\frac{1}{t}\log\left\|E\left\{\prod_{j=0}^{t-1}\prod_{i=0}^{s_2-1}\left|\Gamma_{s_2(t-j)-i}\left(\delta_{s_2(t-j)-i}\right)\right|^\lambda\right\}\right\|\\ &\leq \limsup_{t\to\infty}\log\left\|\prod_{v=0}^{s_2-1}\mathbb{P}^t\left(\Gamma_{s_2-v}^{(\lambda)}\right)\right\|^{1/t} < 0.\blacksquare\end{aligned}$$

$\square$

**Corollary 2.4** *If* $\gamma_L(\Gamma) < 0$ *then there is a* $0 < \lambda \leq 1$ *such that* $E\left\{\|\underline{X}_t\|^{\lambda}\right\} < \infty$ *and hence* $E\left\{|X_t|^{\lambda}\right\} < \infty$, $\forall t \in \mathbb{Z}$.

**Corollary 2.5** *For the* $PARMA_{s_2}$ *model, the necessary and sufficient condition reduces to* $\rho\left(\prod_{v=0}^{s_2-1} A_{s_2-v}(1)\right) < 1$.

**Corollary 2.6** *For the* $MS - ARMA_{s_1}$ *model, the necessary and sufficient condition reduces to* $\rho(\mathbb{P}(A_1)) < 1$ *with* $A_1 := (A_1(i), i \in \mathbb{E})$.

**Corollary 2.7** *For the* $DS - ARMA_{(s_1,s_2)}(1,1)$ *model, the necessary and sufficient condition reduces to*

$$\rho\left(\mathbb{P}\left(\sum_{v=0}^{s_2-1} \log\left|\underline{a}_{1,s_2-v}\right|\right)\right) < 1 \textit{ with } \underline{a}'_{1,v} := (a_{1,v}(i), i \in \mathbb{E}).$$

### 2.3. The second-order periodic stationarity

In this subsection, we have general results on the existence of second-order moments for $DS - ARMA_{(s_1,s_2)}(p,q)$ models. The problem of finding conditions ensuring second-order stationarity for weak $MS - ARMA_d(p;q)$ models (resp. $PARMA_s(p;q)$) has been addressed by Francq and Zakoïan [13]. Hence, we offer the sufficient conditions for the existence of causal, $SPS$ and $PE$ solution to Equation (1.1). The results on the second-order stationarity in the following theorem.

**Theorem 2.3** *Consider the* $DS - ARMA_{(s_1,s_2)}$ *process* (1.1) *with state space representation* (2.2) *and assume that* $E\left\{e_t^2\right\} = \sigma^2 < \infty$. *Let* $\Gamma_v^{(2)} := \left(\Gamma_v^{\otimes 2}(i), i \in \mathbb{E}\right)$ *for all* $v \in \{1, ..., s_2\}$. *If*

$$\tau_{(2)} := \rho\left(\prod_{v=0}^{s_2-1} \mathbb{P}\left(\Gamma_{s_2-v}^{(2)}\right)\right) < 1, \tag{2.3}$$

*then,*

- Equation (2.2) has a unique, causal, ergodic and strictly stationary solution given by $\underline{X}_t^{(1)}$ which converges absolutely almost surely and in $\mathbb{L}_2$.
- The series $\underline{X}_t^{(2)}$ is the unique $PC$ solution of (2.1) and $\underline{X}_t^{(1)} \overset{a.s.}{=} \underline{X}_t^{(2)}$.
- The multivariate process $\left(\underline{X}'_{s_2 t+1}, ..., \underline{X}'_{s_2 t+s_2}\right)'$ is second-order stationary process.

**Proof:** To verify that the series defined by $\underline{X}_t^{(1)}$ is well-defined in $\mathbb{L}_2$, it is sufficient to show that the sequence $\underline{X}_t^{(1)}(k) = \left\{\prod_{j=0}^{k-1} \Gamma(t - js_2)\right\} \underline{e}(t - ks_2)$ converges to $\underline{O}_{(r)}$ in $\mathbb{L}_2$ at an exponential rate as $k \longrightarrow \infty$. Hence, we get

$$\begin{aligned} E\left\{\left(\underline{X}_t^{(1)}(k)\right)^{\otimes 2}\right\} &= E\left\{\left\{\prod_{j=0}^{k-1} \Gamma^{\otimes 2}(t - js_2)\right\} \underline{e}^{\otimes 2}(t - ks_2)\right\} \\ &= \sigma^2 \mathbb{I}'_{(r^2)} \left\{\prod_{v=0}^{s_2-1} \mathbb{P}\left(\Gamma_{s_2-v}^{(2)}\right)\right\}^k \left\{\sum_{l=0}^{s_2-1} \left\{\prod_{v=0}^{l-1} \mathbb{P}\left(\Gamma_{s_2-v}^{(2)}\right)\right\}\right\} \Pi\left(\mathbb{I}_{(r^2)}\right). \end{aligned}$$

Under the Condition (2.3) and by Jordan's decomposition, we get

$$\begin{aligned}\left\|\underline{X}_t^{(1)}(k)\right\|_{\mathbb{L}_2}^2 &\leq \sigma^2\left\|\mathbb{I}'_{(r^2)}\right\|\left\|\left\{\prod_{v=0}^{s_2-1}\mathbb{P}\left(\Gamma_{s_2-v}^{(2)}\right)\right\}^k\right\|\left\|\left\{\sum_{l=0}^{s_2-1}\left\{\prod_{v=0}^{l-1}\mathbb{P}\left(\Gamma_{s_2-v}^{(2)}\right)\right\}\right\}\right\|\left\|\Pi\left(\mathbb{I}_{(r^2)}\right)\right\| \\ &\leq Const\ \tau_{(2)}^k,\end{aligned}$$

so, $\underline{X}_t^{(1)}(k)$ converges to $\underline{O}_{(r)}$ in $\mathbb{L}_2$ at an exponential rate as $k \longrightarrow \infty$. Hence, the series $\underline{X}_t^{(1)}$ is the unique solution of (2.2) which converges in $\mathbb{L}_2$ and absolutely almost surely. The rest of assertions are similar as in Theorem 2.1.■ □

**Remark 2.2** Using the same arguments used by Francq and Zakoïan [13] it is clearly to see that

$$\tau_{(2)} := \rho\left(\prod_{v=0}^{s_2-1}\mathbb{P}\left(A_{s_2-v}^{(2)}\right)\right) < 1, \tag{2.4}$$

with $A_v^{(2)} := \left(A_v^{\otimes 2}(i), i \in \mathbb{E}\right)$ for all $v \in \{1, ..., s_2\}$. Hence, the second-order stationarity is independent of the moving average part.

In the following table, we summarize the sufficient conditions for the existence of $E\left\{X_t^2\right\}$ in some particular models

| Specification | Condition (2.4) | Particular case $p = 1$ |
|---|---|---|
| Standard | $\rho\left(A_0^{\otimes 2}(1)\right) < 1$ | $a_{1,0}^2(1) < 1$ |
| $MS-ARMA_{s_1}^{(a)}$ | $\rho\left(\mathbb{P}\left(A_0^{(2)}\right)\right) < 1$ | $\rho\left(\mathbb{P}\left(\underline{a}_{1,0}^{(2)}\right)\right) < 1$ |
| $PARMA_{s_2}$ | $\rho\left(\prod_{v=0}^{s_2-1}A_{s_2-v}^{\otimes 2}(1)\right) < 1$ | $\prod_{v=0}^{s_2-1}a_{1,s_2-v}^2(1) < 1$ |
| Independent-switching | $\rho\left(\prod_{v=0}^{s_2-1}E\left\{A_{s_2-v}^{\otimes 2}(\delta_t)\right\}\right) < 1$ | $\prod_{v=0}^{s_2-1}E\left\{a_{1,v}^2(\delta_t)\right\} < 1$ |

$^{(a)}\underline{a}_{1,0}^{(2)} := \left(a_{1,0}^2(i), i \in \mathbb{E}\right)'$

Table1: Conditions ensuring $E\left\{X_t^2\right\} < \infty$ for certain specifications.

**Example 2.1** For the $DS-ARMA_{(s_1,s_2)}(1,1)$ process. The Condition (2.4) reduces to $\rho\left(\prod_{v=0}^{s_2-1}\mathbb{P}\left(\underline{a}_{1,s_2-v}^{(2)}\right)\right) < 1$. Noting here that when $s_1 = s_2 = 2$ with $p_{21} = p_{12} = \alpha$ and $a = a_{1,2}(1)$, $b = a_{1,2}(2)$, $c = a_{1,1}(1)$, $m = a_{1,1}(2)$, then the Condition (2.4) is equivalent to the following condition: $\alpha^2\left(a^2+b^2\right)\left(c^2+m^2\right) + (1-2\alpha)(ac-bm)^2 < (1-(1-2\alpha)abcm)^2$. In the following table, we summarize the sufficient conditions for the existence of $E\left\{X_t^2\right\}$ in some particular models

| Specification | Condition (2.4) |
|---|---|
| Standard | $a^2 < 1$ |
| $MS-ARMA_{s_1}^{(a)}$ | $(1-\alpha)\left(a^2+b^2\right) - (1-2\alpha)a^2b^2 < 1$ |
| $PARMA_{s_2}$ | $a^2b^2 < 1$ |
| Independent-switching | $a^2c^2\pi(1) + b^2m^2(1-\pi(1)) < 1$ |

Table2: Conditions ensuring $E\left\{X_t^2\right\} < \infty$ for certain specifications.

### 2.4. Existence of higher-order moments

This subsection is extended the last subsection. The conditions ensuring the existence of the $m$th order moment are obtained in the following result

**Theorem 2.4** *Consider the* $DS-ARMA_{(s_1,s_2)}(p,q)$ *process* (1.1) *with state space representation* (2.2). *For any positive integer* $m$, *assume that* $E\{e_t^m\}=\nu_m<\infty$ *and let* $\Gamma_v^{(m)}:=(\Gamma_v^{\otimes m}(i), i\in\mathbb{E})$ *for all* $v\in\{1,...,s_2\}$. *If*

$$\tau_{(m)}:=\rho\left(\prod_{v=0}^{s_2-1}\mathbb{P}\left(\Gamma_{s_2-v}^{(m)}\right)\right)<1. \tag{2.5}$$

*Then, the process* $(\underline{X}_t)_{t\in\mathbb{Z}}$ *defined by* (2.2) *has a unique, causal, ergodic, strictly stationary solution given by* $\underline{X}_t^{(1)}$ *and satisfies* $E\left\{\left|\underline{X}_t^{\otimes m}\right|\right\}<\infty$.

**Proof:** It is easily seen that

$$E\left\{\left(\underline{X}_t^{(1)}(k)\right)^{\otimes m}\right\}=\nu_m\mathbb{I}'_{(r^m)}\left\{\prod_{v=0}^{s_2-1}\mathbb{P}\left(\Gamma_{s_2-v}^{(m)}\right)\right\}^k\left\{\sum_{l=0}^{s_2-1}\left\{\prod_{v=0}^{l-1}\mathbb{P}\left(\Gamma_{s_2-v}^{(m)}\right)\right\}\right\}\Pi\left(\mathbb{I}_{(r^m)}\right).$$

Hence,

$$\left\|\underline{X}_t^{(1)}(k)\right\|_{\mathbb{L}_m}^m\leq Const\ \tau_{(m)}^k,$$

so, under the Condition (2.5) and by Jordan's decomposition, we get $\underline{X}_t^{(1)}(k)$ converges to $\underline{O}_{(r)}$ in $\mathbb{L}_m$ at an exponential rate as $k\longrightarrow\infty$. As a result, for any $t$, $\sum\limits_{k=1}^{n}\underline{X}_t^{(1)}(k)$ converges to $\underline{X}_t^{(1)}$ as $n\longrightarrow\infty$ both in $\mathbb{L}_m$ and absolutely almost surely. The rest of assertions are immediate.■ □

In the following table, we summarize the sufficient conditions for the existence of $E\{|X_t^m|\}$ in some particular models

| Specification | Condition (2.5) | Particular case $p=1$ |
|---|---|---|
| Standard | $\rho\left(A_0^{\otimes m}(1)\right)<1$ | $a_{1,0}^m(1)<1$ |
| $MS-ARMA_{s_1}^{(a)}$ | $\rho\left(\mathbb{P}\left(A_0^{(m)}\right)\right)<1$ | $\rho\left(\mathbb{P}\left(\underline{a}_{1,0}^{(m)}\right)\right)<1$ |
| $PARMA_{s_2}$ | $\rho\left(\prod_{v=0}^{s_2-1}A_{s_2-v}^{\otimes m}(1)\right)<1$ | $\prod_{v=0}^{s_2-1}a_{1,s_2-v}^m(1)<1$ |
| Independent-switching | $\rho\left(\prod_{v=0}^{s_2-1}E\left\{A_{s_2-v}^{\otimes m}(\delta_t)\right\}\right)<1$ | $\prod_{v=0}^{s_2-1}E\left\{a_{1,v}^m(\delta_t)\right\}<1$ |
| $^{(a)}\underline{a}_{1,0}^{(m)}:=\left(a_{1,0}^m(i), i\in\mathbb{E}\right)'$ | | |

Table3: Conditions ensuring $E\{|X_t^m|\}<\infty$ for certain specifications.

## 3. Computation of the second-order moment and $ARMA$ representation

Once the second-order stationary condition is warranted, it can be useful to compute the expectation and covariance function of the process $(X_t)_{t\in\mathbb{Z}}$ generated by a $DS-ARMA_{(s_1,s_2)}(p,q)$ model (1.1) with state-space representation (2.2). We set $\underline{\mu}=E\{\underline{X}_t\}=\sum\limits_{i=1}^{s_1}\pi(i)\underline{\mu}(i)$, $\Gamma^{(n)}(k):=\prod\limits_{v=0}^{k-1}\mathbb{P}\left(\Gamma_{s_1-v}^{(n)}\right)$ and $\underline{\eta}^{(n)\prime}:=\left(\underline{\eta}^{(n)\prime}(i):=\left(E\left\{\underline{e}^{\otimes n}(t)\,|\delta_t=i\right\}\right)', i\in\mathbb{E}\right)$. Starting from Equation (2.2) then we have, $\Pi\left(\underline{\mu}\right)=\mathbb{P}\left(\Gamma^{(1)}(s_2)\right)\Pi\left(\underline{\mu}\right)+\Pi\left(\underline{\eta}^{(1)}\right)$, then under condition $\rho\left(\Gamma^{(1)}(s_2)\right)<1$ (i.e., $I_{(s_1r)}-\Gamma^{(1)}(s_2)$ is invertible), we get $\Pi\left(\underline{\mu}\right)=\left(I_{(s_1r)}-\Gamma^{(1)}(s_2)\right)^{-1}\Pi\left(\underline{\eta}^{(1)}\right)$. Finally, we have $\underline{\mu}=\mathbb{I}'_{(r)}\Pi\left(\underline{\mu}\right)$. Let us compute the variance function of $\underline{X}_t$, let $\underline{V}=E\left\{vec\left(\underline{X}_t\underline{X}_{t-h}\right)\right\}=\sum\limits_{i=1}^{s_1}\pi(i)\underline{V}(i)$, then, from Equation (2.2), we get

$$\Pi\left(\underline{V}\right)=\mathbb{P}\left(\Gamma^{(2)}(s_2)\right)\Pi\left(\underline{V}\right)+\Pi\left(\underline{\eta}^{(2)}\right)+\Pi\left(\underline{D}\right),$$

where $\underline{D}' := \left(\underline{D}'(i), i \in \mathbb{E}\right)$ with $\underline{D}(i) = E\left\{\left(\Gamma(t) \otimes \underline{e}(t) + \underline{e}(t) \otimes \Gamma(t)\right) \underline{X}_{t-s_2} | \delta_t = i\right\}$ for all $i \in \mathbb{E}$, then under Condition (2.3), then we have $\Pi(\underline{V}) = \left(I_{(s_1 r^2)} - \mathbb{P}\left(\Gamma^{(2)}(s_2)\right)\right)^{-1} \left(\Pi\left(\underline{\eta}^{(2)}\right) + \Pi(\underline{D})\right)$.

Now, let $\underline{\Sigma}(h) = E\left\{\underline{X}_t \underline{X}'_{t-h}\right\} = \sum\limits_{i=1}^{s_1} \pi(i) \underline{\Sigma}^{(i)}(h)$, then, from Equation (2.2), we have for any $h > s_2$,

$$\Pi(\underline{\Sigma}(h)) = \mathbb{P}\left(\Gamma^{(1)}(s_2)\right) \Pi(\underline{\Sigma}(h - s_2)) + \sum_{k=0}^{s_2-1} \Gamma^{(1)}(k) \mathbb{P}^{(h-k)}\left(\underline{\eta}^{(1)}\right) \Pi\left(\underline{\widetilde{\mu}}\right),$$

where $\underline{\widetilde{\mu}}' := \left(\underline{\mu}(i), i \in \mathbb{E}\right)$. The autocovariance function of $\left(\underline{X}_t\right)_{t\in\mathbb{Z}}$ is $\underline{\Sigma}(h) = \mathbb{I}'_{(r)} \Pi(\underline{\Sigma}(h))$.

### 3.1. *ARMA* representation

In this subsection, we propose to show that a $MS - ARMA$ process with a time-varying coefficient admit a $ARMA$ representation. Francq and Zakoïan [13] have established an $ARMA$ representation for multivariate $MS - ARMA_{s_1}$ and others nonlinear processes of interest. Obviously sufficient to verify that the autocovariance structure of $\left(\underline{X}_t\right)_{t\in\mathbb{Z}}$ is that of an $ARMA$ representation. For arithmetic simplicity, let us suppose that $a_{0,t}(\delta_t) = 0$. Then we have $\Pi(\underline{\Sigma}(h)) = \mathbb{P}\left(\Gamma^{(1)}(s_2)\right) \Pi(\underline{\Sigma}(h - s_2))$, $h > s_2$. Using the Jordan decomposition (see, e.g., Francq and Zakoïan [13]), it can be seen that $\Pi(\underline{\Sigma}(h)) = \sum\limits_{i=1}^{l} \sum\limits_{j=0}^{r_i-1} h^j \lambda_i^h \Sigma_{ij}$, $h > \max\limits_i r_i$, where the $\Sigma_{ij}$'s are $s_1 r \times s_1 r$ matrices, the $\lambda_i$'s denote the eigenvalues of $\mathbb{P}\left(\Gamma^{(1)}(s_2)\right)$ and $\sum\limits_{i=1}^{l} r_i = s_1 r$. The result follows essentially the same arguments as in Francq and Zakoïan [13], the $ARMA$ representation sentation can be used to obtain the linear prediction of the observed process.

## 4. Covariance structure of higher power for $DS - ARMA_{(s_1,s_2)}(p, q)$

For the identification purpose it is necessary to look at higher-power of the process in order to distinguish between different $ARMA$ representation. So, in this section, once $m^{th}-$order stationarity is guaranteed, it can be useful to compute the expectation and the autocovariance function of $(X_t)_{t\in\mathbb{Z}}$ with state-space representation (2.1). For this purpose, we first establish the following lemma

**Lemma 4.1** *Consider the $DS - ARMA_{(s_1,s_2)}(p, q)$ model (1.1) with state space representation (2.2). Let us define the following matrices $B_{j,t}^{(k)}(i, e_t)$, $j = 0, ..., k = 0, ..., m$ and $i = 1, ..., s_1$, with appropriate dimension such that*

$$\forall k \in \mathbb{N} : \underline{X}_t^{\otimes k} = \left(\Gamma_t(\delta_t) \underline{X}_{t-1} + \underline{e}_t(\delta_t)\right)^{\otimes k} = \sum_{j=0}^{k} B_{j,t}^{(k)}(\delta_t, e_t) \underline{X}_{t-1}^{\otimes j}, \tag{6.1}$$

*where by convention $B_{j,t}^{(k)}(.,.) = O$ if $j > k$ or $j < 0$, $\underline{X}_t^{\otimes 0} = B_{0,t}^{(0)}(.,.) = 1$. Then $B_{j,t}^{(k)}(i, e_t)$ are uniquely determined by the following recursion*

$$B_{0,t}^{(1)}(\delta_t, e_t) = \underline{e}_t(\delta_t), B_{1,t}^{(1)}(\delta_t, e_t) = \Gamma_t(\delta_t),$$
$$B_{j,t}^{(k+1)}(\delta_t, e_t) = B_{0,t}^{(1)}(\delta_t, e_t) \otimes B_{j,t}^{(k)}(\delta_t, e_t) + B_{1,t}^{(1)}(\delta_t, e_t) \otimes B_{j-1,t}^{(k)}(\delta_t, e_t) \text{ for } k > 1.$$

Now, set $\Xi_v^{(m)} = \left(E\left\{\underline{X}_{s_2 t+v}^{\otimes m} | \delta_{s_2 t+v} = i\right\}; i \in \mathbb{E}\right)$,
$\Xi_v^{(k,m)}(h) = \left(E\left\{\underline{X}_{s_2 t+v}^{\otimes k} \otimes \underline{X}_{s_2 t+v-h}^{\otimes m} | \delta_{s_2 t+v} = i\right\}; i \in \mathbb{E}\right)$
and $B_{j,v}^{(m)} = \left(B_{j,v}^{(m,i)} := E\left\{B_{j,s_2 t+v}^{(m)}(i, e_{s_2 t+v}) | \delta_{s_2 t+v} = i\right\}; i \in \mathbb{E}\right)$, for all $v = 1, ..., s_2$, then it is no difficult to see that

$$\Pi\left(\Xi_v^{(m)}\right) = \sum_{j=0}^{m} \mathbb{P}\left(B_{j,v}^{(m)}\right) \Pi\left(\Xi_{v-1}^{(j)}\right) \text{ and } \Pi\left(\Xi_v^{(k,m)}(h)\right) = \sum_{j=0}^{k} \mathbb{P}\left(\widetilde{B}_{j,v}^{(k,m)}\right) \Pi\left(\Xi_{v-1}^{(j,m)}(h-1)\right), k > 1,$$

where $\widetilde{B}_{j,v}^{(k,m)}=\left(B_{j,v}^{(k,i)}\otimes I_{(r^m)};i\in\mathbb{E}\right)$. Moreover, we have

$$\begin{aligned}
W_v^{(m)}(h) &:= \begin{pmatrix} \Pi\left(\Xi_v^{(m,m)}(h)\right) \\ \Pi\left(\Xi_v^{(m-1,m)}(h)\right) \\ \vdots \\ \vdots \\ \Pi\left(\Xi_v^{(0,m)}(h)\right) \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{P}\left(\widetilde{B}_{m,v}^{(m,m)}\right) & \mathbb{P}\left(\widetilde{B}_{m-1,v}^{(m,m)}\right) & \dots & \mathbb{P}\left(\widetilde{B}_{0,v}^{(m,m)}\right) \\ O & \mathbb{P}\left(\widetilde{B}_{m-1,v}^{(m-1,m)}\right) & \dots & \mathbb{P}\left(\widetilde{B}_{0,v}^{(m-1,m)}\right) \\ \vdots & \ddots & \dots & \vdots \\ \vdots & \ddots & \mathbb{P}\left(\widetilde{B}_{1,v}^{(1,m)}\right) & \mathbb{P}\left(\widetilde{B}_{0,v}^{(1,m)}\right) \\ O & \dots & O & I_{(r^m)} \end{pmatrix} W_{v-1}^{(m)}(h-1) \\
&= \Lambda_v\left(m\right)W_{v-1}^{(m)}(h-1).
\end{aligned}$$

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