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Ricci bi-conformal vector fields on four-dimensional Lorentzian Damek-Ricci spaces

Shahroud Azami

ABSTRACT: In this paper, we obtain all Ricci bi-conformal vector fields on the four-dimensional Lorentzian Damek-Ricci spaces and we show that four-dimensional Lorentzian Damek-Ricci spaces have not nontrivial Ricci bi-conformal vector fields as gradient vector filed. Also, we determine which of them are Killing vector fields.

Key Words: Ricci bi-conformal vector fields, Damek-Ricci spaces, left-invariant metrics.

Contents

1. Introduction

Let (M,g) be a smooth n-dimensional pseudo-Riemannian manifold. A vector field X on a Riemannian manifold (M,g) is called a Killing vector field if the Lie derivative with relation to X of the metric g vanishes, that is

$$\mathcal{L}_X g = 0,$$

where $\mathcal{L}_X$ is the Lie derivative in the direction of X. Killing vector fields and generalized of them are important in differential geometry and physics. Recently, various generalizations of Killing vector fields have been investigated. A vector field X on a Riemannian manifold (M,g) is called conformal vector field if there is a smooth function ψ on M that named a potential function, such that $\mathcal{L}_X g = 2\psi g$. When $\psi = 0$, X is a Killing vector field. Conformal vector fields are thoroughly studied in [4,10,16]. Another generalizations of Killing vector fields are generalized Kerr-Schild vector fields. The generalized Kerr-Schild vector field is defined by

$$\mathcal{L}_X g = \alpha g + \beta l \otimes l, \qquad \mathcal{L}_X l = \gamma l,$$

where α, β, γ are smooth functions over M. Coll et al. [6] studied the generalized Kerr-Schild vector field. A symmetric tensor h on M is called a square root of g if $h_{ik} h^k_j = g_{ij}$. Garcia-Parrado and Senovilla [11] using square root of g defined bi-conformal vector fields. A vector field X is said to be a bi-conformal vector field if it satisfies the following equations:

$$\mathcal{L}_X g = \alpha g + \beta h, \qquad \mathcal{L}_X h = \alpha h + \beta g,$$

where h is a symmetric square root of g and α, β are smooth functions. The functions α and β are called gauges [6,11] of the symmetry and they play a role analogous to the factor ψ appearing in the definition of the conformal vector fields. After then, De et al. in [8] using the metric tensor field g and the Ricci tensor field S defined Ricci bi-conformal vector fields as follows.

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Definition 1.1 *A vector field X on a Riemannian manifold (M, g) is called Ricci bi-conformal vector field if it satisfies the following equations*

$$(\mathcal{L}_X g)(Y, Z) = \alpha g(Y, Z) + \beta S(Y, Z), \tag{1.1}$$

and

$$(\mathcal{L}_X S)(Y, Z) = \alpha S(Y, Z) + \beta g(Y, Z), \tag{1.2}$$

for any vector fields Y, Z and some smooth functions α and β.

For the first time Damek and Ricci in [7] introduced Damek-Ricci spaces which are semidirect products of Heisenberg groups with the real line. They proved that the conjecture posed by Lichnerowicz fails in the non-compact case by the authors provided the examples of harmonic manifolds that were not symmetric. The geometry of these spaces has been studied by many authors. In [9], Degla and Todjihounde studied the nonexistence of a proper biharmonic curve in a four-dimensional Damek-Ricci space. In [1], they studied the dispersive properties of the linear wave equation on Damek-Ricci spaces and their applications to nonlinear Cauchy problems. Also, see [3,5,12]. Tan and Deng [15] considered the four-dimensional Lorentzian Damek- Ricci spaces and proved that these spaces did not even admit a left-invariant Ricci soliton. Also, they investigated harmonicity of invariant vector fields and curvature properties. Sidhoumi [14] considered the left-invariant Lorentzian metrics admitted by the four-dimensional Damek-Ricci spaces and proved the existence of the vector field for which the soliton equation holds. Also, A. Mostefaoui and N. Sidhoumi [13] classified homogeneous structures on Damek-Ricci spaces equipped with the left nvariant metric.

The paper is organized in the following way. In Section 2, we give some basic concepts on four-dimensional Damek-Ricci spaces and their metrics in global coordinates, we also describe their Levi-Civita connection, Ricci tensor, and Lei derivative of metric tensor and Ricci tensor in direction an arbitrary vector field. In Section 3, Ricci bi-conformal vector fields of four-dimensional Damek-Ricci spaces are characterized via a system of partial differential equations. In particular, we prove that some of Ricci bi-conformal vector fields admit gradient and Killing vector fields resulting in expanding and shrinking Ricci solitons.

2. Geometry of four-dimensional Damek-Ricci spaces

In this section we recall We start with a short description of four-dimensional Damek-Ricci spaces (see [2,7]). First, we recall the so-called generalized Heisenberg group since Damek-Ricci spaces depend on it.

2.1. Generalized Heisenberg group

The generalized Heisenberg algebras are defined as follows. Suppose that $\mathfrak{b}$ and $\mathfrak{z}$ are the finite-dimensional real vector spaces and $\mathfrak{n} = \mathfrak{b} \oplus \mathfrak{z}$. We define the bracket in $\mathfrak{n}$ as follows

$$[U + X, V + Y] = \beta(U, V),$$

where $\beta : \mathfrak{b} \times \mathfrak{b} \to \mathfrak{z}$ is a skew-symmetric bilinear map. This product defines a Lie algebra structure on $\mathfrak{n}$. We equip $\mathfrak{b}$ with a positive inner product and $\mathfrak{z}$ with a positive or Lorentzian inner product denoting the product metric by $\langle , \rangle_{\mathfrak{n}}$. Consider a linear map $J : Z \in \mathfrak{z} \to J_Z \in End(\mathfrak{b})$ which is defined by

$$\langle J_Z U, V \rangle_{\mathfrak{n}} = \langle \beta(U, V), Z \rangle_{\mathfrak{n}} \text{ for all } U, V \in \mathfrak{b} \text{ and } Z \in \mathfrak{z}.$$

Then $\mathfrak{n}$ is a two-step nilpotent Lie algebra with center $\mathfrak{z}$. The Lie algebra $\mathfrak{n}$ is called a generalized Riemannian Heisenberg algebra, and the associated simply connected nilpotent Lie group, endowed with the induced left-invariant Riemannian metric, is called a generalized Riemannian Heisenberg group, if the inner product in $\mathfrak{z}$ is positive and

$$J_Z^2 = \begin{cases} -\langle Z, Z \rangle_{\mathfrak{n}} id_{\mathfrak{b}}, & \text{when } Z \text{ is spacelike,} \\ \langle Z, Z \rangle_{\mathfrak{n}} id_{\mathfrak{b}}, & \text{when } Z \text{ is timelike.} \end{cases}$$

2.2. Damek-Ricci spaces

Now, let $\epsilon = \pm 1$ and $\mathfrak{a}_\epsilon$ be a one-dimensional pseudo-Riemannian real vector space, which is Lorentzian for $\epsilon = 1$ and Riemannian for $\epsilon = -1$ and let $\mathfrak{n}_{-\epsilon} = \mathfrak{b} \oplus \mathfrak{z}$ be a generalized Heisenberg algebra which is Lorentzian if $\epsilon = 1$ and Riemannian if $\epsilon = -1$. Consider a new vector space $\mathfrak{a}_\epsilon \oplus \mathfrak{n}_{-\epsilon}$. Let $r, s \in \mathbb{R}, U, V \in \mathfrak{b}$ and $X, Y \in \mathfrak{z}$. We define the Lorentzian product $\langle ., . \rangle$ and a Lie bracket $[., .]$ on $\mathfrak{a}_\epsilon \oplus \mathfrak{n}_{-\epsilon}$ by

$$\langle rA + U + X, sA + V + Y \rangle = \langle U + X, V + Y \rangle_{\mathfrak{n}_{-\epsilon}} + \epsilon rs,$$
$$[rA + U + X, sA + V + Y] = [U, V]_{\mathfrak{n}_{-\epsilon}} + \frac{1}{2} rV - \frac{1}{2} sU + rY - sX$$

for a non zero vector A in $\mathfrak{a}_\epsilon$. Therefore $\mathfrak{a}_\epsilon \oplus \mathfrak{n}_{-\epsilon}$ becomes a solvable Lie algebra. The corresponding simply connected Lie group, equipped with the induced leftinvariant Lorentzian metric, is called a Lorentzian Damek-Ricci space and will be denoted by $\mathbb{S}_\epsilon$. Let $(\mathbb{S}^4_\epsilon, g_\epsilon)$ denoted the four-dimensional Lorentzian Damek [5] equipped with the left-invariant Lorentzian metric

$$g_\epsilon = e^{-t} dx^2 + e^{-t} dy^2 + \epsilon e^{-2t} (dz + \frac{c}{2} y dx - \frac{c}{2} x dy)^2, \tag{2.1}$$

in global coordinates (x, y, z, t) where $c \in \mathbb{R}$. Note that Damek-Ricci spaces of dimension four are diffeomorphic to $\mathbb{R}^4$ because they are simply connected solvable Lie groups.

We have an orthonormal basis of left invariant vector fields on $(\mathbb{S}^4_\epsilon, g_\epsilon)$ with respect to the metric g_ϵ given by

$$e_1 = e^{\frac{t}{2}} (\frac{\partial}{\partial x} - \frac{cy}{2} \frac{\partial}{\partial z}), \ e_2 = e^{\frac{t}{2}} (\frac{\partial}{\partial y} + \frac{cx}{2} \frac{\partial}{\partial z}), \ e_3 = e^t \frac{\partial}{\partial z}, \ e_4 = \frac{\partial}{\partial t}, \tag{2.2}$$

and

$$g(e_1, e_1) = g(e_2, e_2) = 1, \ g(e_3, e_3) = -g(e_4, e_4) = \epsilon.$$

The Lie brackets of these vector fields are determined by

$$[e_1, e_2] = ce_3, \quad [e_1, e_3] = 0, \quad [e_1, e_4] = -\frac{1}{2} e_1,$$
$$[e_2, e_3] = 0, \quad [e_2, e_4] = -\frac{1}{2} e_2, \quad [e_3, e_4] = -e_3.$$

We denote the Levi-Civita connection of $(\mathbb{S}^4_\epsilon, g_\epsilon)$ by ∇ and its curvature tensor by R in which defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]} Z \tag{2.3}$$

for all vector fields X, Y and Z. The Ricci tensor of $(\mathbb{S}^4_\epsilon, g_\epsilon)$ is defined by

$$S(X, Y) = \sum_{k=1}^{4} g(e_k, e_k) g(R(e_k, X)Y, e_k), \tag{2.4}$$

with respect to orthonormal basis $\{e_1, e_2, e_3, e_4\}$. Using the Koszul formula, the Levi-Civta connection ∇ of the space $\mathbb{S}^4_\epsilon$ is described by

$$\nabla_{e_i} e_j = \begin{pmatrix} -\frac{\epsilon}{2} e_4 & \frac{c}{2} e_3 & -\frac{\epsilon c}{2} e_2 & -\frac{1}{2} e_1 \\ -\frac{c}{2} e_3 & -\frac{\epsilon}{2} e_4 & \frac{c\epsilon}{2} e_1 & -\frac{1}{2} e_2 \\ -\frac{c\epsilon}{2} e_2 & \frac{c\epsilon}{2} e_1 & -e_4 & -e_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.5}$$

and the Ricci tensor of $\mathbb{S}^4_\epsilon$ is determined by

$$S = \begin{pmatrix} \frac{\epsilon}{2} & 0 & 0 & 0 \\ 0 & \frac{\epsilon}{2} & 0 & 0 \\ 0 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \tag{2.6}$$

with respect to the basis $\{e_1, e_2, e_3, e_4\}$.

We consider an arbitrary vector field $X = X^1e_1 + X^2e_2 + X^3e_3 + X^4e_4$ on $(\mathbb{S}^4_\epsilon, g_\epsilon)$, where $X^i = X^i(x, y, z, t),\ i = 1, 2, 3$ are smooth functions. Suppose that $\partial_x = \frac{\partial}{\partial x},\quad \partial_y = \frac{\partial}{\partial y},\quad \partial_z = \frac{\partial}{\partial z}$ and $\partial_t = \frac{\partial}{\partial t}$. The Lie derivative of the metric g in direction vector field X is given by

$$
\begin{aligned}
(\mathcal{L}_X g)(e_1, e_1) &= 2e_1(X^1) - X^4,\\
(\mathcal{L}_X g)(e_1, e_2) &= e_1(X^2) + e_2(X^1),\\
(\mathcal{L}_X g)(e_1, e_3) &= \epsilon(cX^2 + e_1(X^3)) + e_3(X^1),\\
(\mathcal{L}_X g)(e_1, e_4) &= \frac{1}{2}X^1 - \epsilon e_1(X^4) + e_4(X^1),\\
(\mathcal{L}_X g)(e_2, e_2) &= -X^4 + 2e_2(X^2),\\
(\mathcal{L}_X g)(e_2, e_3) &= \epsilon(-cX^1 + e_2(X^3)) + e_3(X^2),\\
(\mathcal{L}_X g)(e_2, e_4) &= \frac{1}{2}X^2 - \epsilon e_2(X^4) + e_4(X^2),\\
(\mathcal{L}_X g)(e_3, e_3) &= 2\epsilon(-X^4 + e_3(X^3)),\\
(\mathcal{L}_X g)(e_3, e_4) &= \epsilon(X^3 + e_4(X^3) - e_3(X^4)),\\
(\mathcal{L}_X g)(e_4, e_4) &= -2\epsilon e_4(X^4),
\end{aligned}
\tag{2.7}
$$

and the Lie derivative of the Ricci tensor along the vector field X is represented by

$$
\begin{aligned}
(\mathcal{L}_X S)(e_1, e_1) &= \epsilon e_1(X^1) - \frac{\epsilon}{2}X^4,\\
(\mathcal{L}_X S)(e_1, e_2) &= \frac{\epsilon}{2}(e_1(X^2) + e_2(X^1)),\\
(\mathcal{L}_X S)(e_1, e_3) &= \frac{5}{2}(cX^2 + e_1(X^3)) + \frac{\epsilon}{2}e_3(X^1),\\
(\mathcal{L}_X S)(e_1, e_4) &= \frac{\epsilon}{4}X^1 - \frac{3}{2}e_1(X^4) + \frac{\epsilon}{2}e_4(X^1),\\
(\mathcal{L}_X S)(e_2, e_2) &= \frac{\epsilon}{2}(-X^4 + 2e_2(X^2)),\\
(\mathcal{L}_X S)(e_2, e_3) &= \frac{5}{2}(-cX^1 + e_2(X^3)) + \frac{\epsilon}{2}e_3(X^2),\\
(\mathcal{L}_X S)(e_2, e_4) &= \frac{\epsilon}{4}X^2 - \frac{3}{2}e_2(X^4) + \frac{\epsilon}{2}e_4(X^2),\\
(\mathcal{L}_X S)(e_3, e_3) &= 5(-X^4 + e_3(X^3)),\\
(\mathcal{L}_X S)(e_3, e_4) &= \frac{5}{2}(X^3 + e_4(X^3) - \frac{3}{2}e_3(X^4)),\\
(\mathcal{L}_X S)(e_4, e_4) &= -3e_4(X^4).
\end{aligned}
\tag{2.8}
$$

3. Ricci bi-conformal vector fields

In this section, we solve the equation (1.1) and (1.2) on four-dimensional Lorentzian Damek-Ricci spaces. By using (2.1), (2.6), and (2.7) in (1.1), we have

$$2e_1(X^1) - X^4 = \alpha + \frac{\epsilon}{2}\beta, \tag{3.1}$$
$$e_1(X^2) + e_2(X^1) = 0, \tag{3.2}$$
$$\epsilon(cX^2 + e_1(X^3)) + e_3(X^1) = 0, \tag{3.3}$$
$$\frac{1}{2}X^1 - \epsilon e_1(X^4) + e_4(X^1) = 0, \tag{3.4}$$
$$-X^4 + 2e_2(X^2) = \alpha + \frac{\epsilon}{2}\beta, \tag{3.5}$$
$$\epsilon(-cX^1 + e_2(X^3)) + e_3(X^2) = 0, \tag{3.6}$$
$$\frac{1}{2}X^2 - \epsilon e_2(X^4) + e_4(X^2) = 0, \tag{3.7}$$
$$2\epsilon(-X^4 + e_3(X^3)) = \epsilon\alpha + \frac{5}{2}\beta, \tag{3.8}$$
$$\epsilon(X^3 + e_4(X^3) - e_3(X^4)) = 0, \tag{3.9}$$
$$-2\epsilon e_4(X^4) = -\epsilon\alpha - \frac{3}{2}\beta, \tag{3.10}$$

and by applying (2.1), (2.6), and (2.8) in (1.2), we get

$$\epsilon e_1(X^1) - \frac{\epsilon}{2}X^4 = \frac{\epsilon}{2}\alpha + \beta, \tag{3.11}$$
$$\frac{\epsilon}{2}(e_1(X^2) + e_2(X^1)) = 0, \tag{3.12}$$
$$\frac{5}{2}(cX^2 + e_1(X^3)) + \frac{\epsilon}{2}e_3(X^1) = 0, \tag{3.13}$$
$$\frac{\epsilon}{4}X^1 - \frac{3}{2}e_1(X^4) + \frac{\epsilon}{2}e_4(X^1) = 0, \tag{3.14}$$
$$\frac{\epsilon}{2}(-X^4 + 2e_2(X^2)) = \frac{\epsilon}{2}\alpha + \beta, \tag{3.15}$$
$$\frac{5}{2}(-cX^1 + e_2(X^3)) + \frac{\epsilon}{2}e_3(X^2) = 0, \tag{3.16}$$
$$\frac{\epsilon}{4}X^2 - \frac{3}{2}e_2(X^4) + \frac{\epsilon}{2}e_4(X^2) = 0, \tag{3.17}$$
$$5(-X^4 + e_3(X^3)) = \frac{5}{2}\alpha + \epsilon\beta, \tag{3.18}$$
$$\frac{5}{2}(X^3 + e_4(X^3) - \frac{3}{2}e_3(X^4)) = 0, \tag{3.19}$$
$$-3e_4(X^4) = -\frac{3}{2}\alpha - \epsilon\beta. \tag{3.20}$$

From (3.10) and (3.20), we conclude that

$$\beta = 0, \qquad e_4(X^4) = \frac{\alpha}{2}. \tag{3.21}$$

Using (3.4) and (3.14), we infer

$$e_1(X^4) = 0, \qquad \frac{1}{2}X^1 + e_4(X^1) = 0. \tag{3.22}$$

Using (3.7) and (3.17), we have

$$e_2(X^4) = 0, \qquad \frac{1}{2}X^2 + e_4(X^2) = 0. \tag{3.23}$$

Equations (3.7) and (3.17), imply that

$$e_3(X^4)=0, \qquad X^3+e_4(X^3)=0. \tag{3.24}$$

By taking integration of (3.21)-(3.24), we deduce that $X^4=F(t)$ and

$$\alpha=2F'(t),\ \ X^1=G(x,y,z)e^{-\frac{t}{2}},\ \ X^2=K(x,y,z)e^{-\frac{t}{2}},\ \ X^3=L(x,y,z)e^{-t} \tag{3.25}$$

for some smooth functions F, G, K and L. From (3.3) and (3.13), we have

$$e_3(X^1)=0, \qquad cX^2+e_1(X^3)=0. \tag{3.26}$$

Applying $X^1=G(x,y,z)e^{-\frac{t}{2}}$ in $e_3(X^1)=0$ we obtain $\partial_z G(x,y,z)=0$ and $G(x,y,z)=G_1(x,y)$ for some smooth function G_1. Inserting (3.21), (3.25), and $X^1=G_1(x,y)e^{-\frac{t}{2}}$ in (3.1), we arrive at

$$\partial_x G_1(x,y)=F'(t)+\frac{F(t)}{2}=a_1 \tag{3.27}$$

for some constant a_1. Hence, by integration we get $G_1(x,y)=a_1x+G_2(y)$ for some smooth function G_2. Using (3.6) and (3.16), we have

$$e_3(X^2)=0, \qquad -cX^1+e_2(X^3)=0. \tag{3.28}$$

Substituting $X^2=K(x,y,z)e^{-\frac{t}{2}}$ in $e_3(X^2)=0$ we find $K(x,y,z)=K_1(x,y)$ for some smooth function K_1. Inserting (3.21), (3.25), and $X^2=K_1(x,y)e^{-\frac{t}{2}}$ in (3.5), we infer

$$\partial_y K_1(x,y)=F'(t)+\frac{F(t)}{2}=a_1. \tag{3.29}$$

Thus, by integration we deduce $K_1(x,y)=a_1y+K_2(x)$ for some smooth function K_2. Inserting $X^1=(a_1x+G_2(y))e^{-\frac{t}{2}}$ and $X^2=(a_1y+K_2(x))e^{-\frac{t}{2}}$ in (3.2), we conclude

$$G_2'(y)=-K_2'(x)=a_2 \tag{3.30}$$

for some constant a_2. Hence $G_2(y)=a_2y+a_3$ and $K_2(x)=-a_2x+a_4$ for some constants a_3 and a_4. Therefore,

$$X^1=(a_1x+a_2y+a_3)e^{-\frac{t}{2}}, \qquad X^2=(a_1y-a_2x+a_4)e^{-\frac{t}{2}}. \tag{3.31}$$

Applying (3.21), (3.25), and $X^3=L(x,y,z)e^{-t}$ in (3.8), we infer

$$\partial_z L(x,y,z)=F'(t)+F(t)=a_5, \tag{3.32}$$

for some constant a_5. Equations (3.27) and (3.32) imply that $F'(t)=0$ and $F(t)=2a_1=a_5$. By taking integration of (3.32) we deduce $L(x,y,z)=2a_1z+L_1(x,y)$ for some smooth function L_1. Substituting (3.31) and $X^3=(2a_1z+L_1(x,y))e^{-t}$ in (3.28) we find $\partial_y L_1(x,y)=c(a_2y+a_3)$ and integration of it implies that $L_1(x,y)=c(a_2\frac{y^2}{2}+a_3y)+L_2(x)$ for some smooth function L_2. Now, inserting (3.31) and $X^3=(2a_1z+c(a_2\frac{y^2}{2}+a_3y)+L_2(x))e^{-t}$ in (3.26) we get $L_2'(x)=-c(-a_2x+a_4)$ and taking integration of it we obtain $L_2(x)=c(a_2\frac{x^2}{2}-a_4x)+a_6$ for some constant a_6. Therefore

$$X^3=\left(2a_1z+c(a_2\frac{y^2}{2}+a_3y)+c(a_2\frac{x^2}{2}-a_4x)+a_6\right)e^{-t}. \tag{3.33}$$

Hence we have the following theorem.

Theorem 3.1 *The vector field X on $(\mathbb{S}^4_\epsilon, g_\epsilon)$ where g_ϵ is given by (2.1), is Ricci bi-conformal vector field if and only if $\alpha=\beta=0$ and*

$$\begin{aligned} X^1&=(a_1x+a_2y+a_3)e^{-\frac{t}{2}},\\ X^2&=(a_1y-a_2x+a_4)e^{-\frac{t}{2}},\\ X^3&=\left(2a_1z+c(a_2\frac{y^2}{2}+a_3y)+c(a_2\frac{x^2}{2}-a_4x)+a_6\right)e^{-t},\\ X^4&=2a_1, \end{aligned} \tag{3.34}$$

for some constants a_1, a_2, a_3, a_4 and a_6.

Now, let $X = \nabla f$ on $(\mathbb{S}^4_\epsilon, g_\epsilon)$ with potential function f. Then,

$$X = e_1(f)e_1 + e_2(f)e_2 + \epsilon e_3(f)e_3 - \epsilon e_4(f)e_4. \tag{3.35}$$

From Theorem 3.1, the Ricci bi-conformal vector field X on $(\mathbb{S}^4_\epsilon, g_\epsilon)$ is gradient vector field as ∇f if and only if

$$\partial_x f - \frac{cy}{2}\partial_z f = (a_1 x + a_2 y + a_3)e^{-t}, \tag{3.36}$$

$$\partial_y f + \frac{cx}{2}\partial_z f = (a_1 y - a_2 x + a_4)e^{-t}, \tag{3.37}$$

$$\partial_z f = \epsilon\left(2a_1 z + c(a_2\frac{y^2}{2} + a_3 y) + c(a_2\frac{x^2}{2} - a_4 x) + a_6\right)e^{-2t}, \tag{3.38}$$

$$\partial_t f = -2\epsilon a_1. \tag{3.39}$$

By taking derivative of (3.38) with respect to t and using (3.39) we get

$$2a_1 z + c(a_2\frac{y^2}{2} + a_3 y) + c(a_2\frac{x^2}{2} - a_4 x) + a_6 = 0.$$

Hence, $a_1 = a_2 = a_3 = a_4 = a_6 = 0$ a nd by direct integration we obtain $f = b$ for some constant b. Therefore, we have shown the following result.

Corollary 3.1 *Any Ricci bi-conformal vector field X $(\mathbb{S}^4_\epsilon, g_\epsilon)$ is gradient vector field with potential function $f = b$ where b is constant.*

Corollary 3.2 *Any Ricci bi-conformal vector field on $(\mathbb{S}^4_\epsilon, g_\epsilon)$ is a Killing vector field.*

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Shahroud Azami,
Department of Mathematics,
Faculty of Science, Imam Khomeini International University,
Qazvin, Iran.
E-mail address: azami@sci.ikiu.ac.ir