Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.67928

Study of the stationary solution of non-isothermal Bingham flow with nonlinear boundary conditions in a thin domain*

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ABSTRACT: We study in this manuscript the asymptotic behavior of an incompressible Bingham fluid in a three-dimensional thin domain Ω^ϵ with Tresca friction law coupled with a nonlinear stationary, non-isothermal and incompressible model. Firstly, we demonstrate the results for the existence and uniqueness of the weak solution. Then we reformulate the problem in fixed domain, and we also show the estimates for the velocity field, the pressure, and the temperature. Finally we obtain the limit problem with the specific Reynolds equation and prove the uniqueness of the limit.

Key Words: Asymptotic approach, Coupled problem, Bingham fluid, Temperature, Reynolds equation, variational inequality.

Contents

1. Introduction

In 1916, Bingham fluid model considered a non-Newtonian fluid, with a viscous plastic medium, obeys the general laws of continuum mechanics, and has a special nonlinear constitutive law, such that it moves like a rigid body if a certain function of the stresses does not reach the yield limit, and it behaves like a viscous fluid when the yield limit is reached. It is used for modeling several types of liquids, for example, heavy crude oils, colloidal solutions, powder mixtures, and toothpaste.
Bingham flow modeling has been a permanent source of challenging problems for many decades already, the main breakthrough in this direction being the variational inequality formulation due to Duvaut and Lions (Refs. [12,16,14,18]). The existence, uniqueness and regularity of the solution, as well as its flow structure are investigated in [18]. Further in [6] and [14,16,15] the authors investigate the regularity of the solution for the d-dimensional Bingham fluid flow problem with Dirichlet boundary conditions for the cross section and cavity model, respectively. In [9,8,10], the stationary Bingham fluid flow problems numerical solution is studied.
Other many research papers have been written dealing with the asymptotic analysis of a Bingham fluid flow, for exemple, R. Elmir et al. [13,7] studied the asymptotic behaviour of a Bingham fluid in a thin domain with non linear boundary conditions. The asymptotic stability of weak solutions for the incompressible non-Newtonian fluid motion in $\mathbb{R}^2$ has been studied in [5], other similar works can be found in studies, such as [1,2,3]. More recently, Dilmi et al. [11] worked on the asymptotic analysis of a isothermal Bingham fluid in a thin domain with nonlinear friction of Fourier and Tresca. The coupled non-isothermal problem with mixed boundary conditions in a thin domain with friction law has also been

* This research work is supported by the General Direction of Scientific Research and Technological Development (DGRSDT), Algeria.
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Submitted July 13, 2023. Published April 14, 2025
2010 *Mathematics Subject Classification:* 35R35, 76F10, 78M35, 35B40, 35J85.

studied in [20,22].

The goal of this paper is to study the asymptotic behavior of a coupled system involving an incompressible Bingham fluid and the equation of the heat energy, in a three-dimensional bounded domain with Tresca free boundary friction conditions. The novelty here lies in the addition of the two terms $w^\epsilon \nabla w^\epsilon$ and $w^\epsilon \nabla T^\epsilon$.

The outline of this paper is as follows. In section 2, we introduce the problem and give the basic assumptions, we also recall the weak formulation and existence theorem of weak solution, in section 3, taking into account a small parameter, we introduce a scaling and we give some estimates and convergence, in section 4, we establish the limit variational inequality, the Reynolds equation and the uniqueness of solutions of the limit problem.

2. Statement of the problem and variational formulation

Here, let ω be a fixed region in plan $s = (s_1, s_2) \in \mathbb{R}^2$. We suppose that ω has a Lipschitz boundary and is the bottom of the fluid domain. The upper surface Γ_1^ϵ is defined by $s_3 = \epsilon h(s)$, where $(0 < \epsilon < 1)$ is a small parameter that will tend to zero and h a smooth bounded function such that $0 < \underline{h} \leqslant h(s) \leqslant \bar{h}$ for all $(s, 0) \in \omega$ and Γ_L^ϵ the lateral surface. We denote by Ω^ϵ the domain of the following:

$$\Omega^\epsilon = \{(s, s_3) \in \mathbb{R}^3 : (s, 0) \in \omega, 0 < s_3 < \epsilon h(s)\}.$$

The boundary of Ω^ϵ is Γ^ϵ where $\Gamma^\epsilon = \bar{\Gamma}_1^\epsilon \cup \bar{\Gamma}_L^\epsilon \cup \bar{\omega}$ with $\bar{\Gamma}_L^\epsilon$ is the lateral boundary. The fluid is supposed to be viscoplastic, and the relation between Σ^ϵ and $D(w^\epsilon)$ is given by

$$\begin{cases} \Sigma_{ij}^\epsilon = \tilde{\Sigma}_{ij}^\epsilon - p^\epsilon \delta_{ij}, \\ \tilde{\Sigma}^\epsilon = g^\epsilon (T^\epsilon) \dfrac{D(w^\epsilon)}{|D_{II}(w^\epsilon)|} + 2\Lambda^\epsilon (T^\epsilon) D(w^\epsilon) \text{ if } D(w^\epsilon) \neq 0, \\ |\tilde{\Sigma}^\epsilon| \leq g^\epsilon (T^\epsilon) \text{ if } D(w^\epsilon) = 0. \end{cases}$$

Where Σ^ε represents the constitutive law of a Bingham fluid whose the consistency Λ^ϵ, and the yield limit g^ϵ depend on the temperature, p^ϵ is the pressure, δ_{ij} is the Kronecker symbol and $D(u^\epsilon) = \frac{1}{2}\left(\nabla w^\epsilon + (\nabla w^\epsilon)^T\right)$. For any tensor $D = d_{ij}$, the notation $|D|$ represents the matrix norm: $|D_{II}| = \left(\sum_{i,j}^3 \frac{1}{2} d_{ij} d_{ij}\right)^{\frac{1}{2}}$.

- The low of conservation of momentum

$$w^\epsilon \nabla w^\epsilon - \operatorname{div} \Sigma^\epsilon = f^\epsilon \text{ in } \Omega^\epsilon, \tag{2.1}$$

where $f^\epsilon = (f_i^\epsilon)_{1 \leqslant i \leqslant 3}$ denotes the body forces.

- The equation of the heat energy

$$w^\epsilon \nabla T^\epsilon - \frac{\partial}{\partial s_i}\left(K^\epsilon \frac{\partial T^\epsilon}{\partial s_i}\right) = 2\Lambda^\epsilon (T^\epsilon) d_{ij}(w^\epsilon) d_{ij}(w^\epsilon) + \sqrt{2} g^\epsilon (T^\epsilon) |D(w^\epsilon)| - \alpha^\epsilon (T^\epsilon) \text{ in } \Omega^\epsilon, \tag{2.2}$$

where the specific heat is assumed equal to one, $K^\epsilon > 0$ is the thermal conductivity and the term $-\alpha^\epsilon T^\epsilon$ represents the external heat source with $\alpha^\epsilon > 0$.

- The incompressibility equation

$$\operatorname{div}(w^\varepsilon) = 0 \text{ in } \Omega^\epsilon. \tag{2.3}$$

Our boundary conditions is described as

• At the surface Γ_L^ϵ we suppose

$$w^\epsilon = 0. \tag{2.4}$$

• On $\Gamma_1^\epsilon \cup \omega$, there is a no-flux condition across ω so that

$$w^\epsilon \times n = 0, \tag{2.5}$$

the tangential velocity on ω is unknown and satisfies Tresca boundary conditions with friction coefficient k^ϵ:

$$\begin{cases} |\Sigma_\tau^\epsilon| < k^\epsilon \Rightarrow w_\tau^\epsilon = 0, \\ |\Sigma_\tau^\epsilon| = k^\epsilon \Rightarrow \exists \lambda \geq 0 : w_\tau^\epsilon = -\lambda \Sigma_\tau^\epsilon. \end{cases} \tag{2.6}$$

Here $|.|$ is the Euclidean norm in $\mathbb{R}^2$; $n = (n_1, n_2, n_3)$ is the unit outward normal vector on Γ^ϵ. The normal and the tangential components on the boundary ω are given by

$$w_n^\epsilon = w^\epsilon \cdot n = w_i^\epsilon \cdot n_i, \quad w_{\tau_i}^\epsilon = w_i^\epsilon - w_n^\epsilon n_i,$$
$$\Sigma_n^\epsilon = (\Sigma^\epsilon \cdot n)n = \Sigma_{ij}^\epsilon n_i n_j, \quad \Sigma_{\tau_i}^\epsilon = \Sigma_{ij}^\epsilon n_j - \Sigma_n^\epsilon n_i.$$

For the temperature, we assume that

$$T^\epsilon = 0 \ \text{ on } \Gamma_1^\epsilon \cup \Gamma_L^\epsilon, \tag{2.7}$$

$$\frac{\partial T^\epsilon}{\partial n} = 0 \ \text{ on } \omega. \tag{2.8}$$

To get a weak formulation, we consider the functional framework on Ω^ε

$$V^\epsilon(\Omega^\epsilon) = \left\{ \varphi \in H^1(\Omega^\epsilon)^3 : \varphi = 0 \text{ on } \Gamma_L^\epsilon, \varphi \cdot n = 0 \text{ on } \omega \cup \Gamma_1^\epsilon \right\},$$
$$V_{div}^\epsilon(\Omega^\epsilon) = \{ \varphi \in V^\epsilon(\Omega^\epsilon) : \operatorname{div}(\varphi) = 0 \},$$
$$L_0^2(\Omega^\epsilon) = \left\{ q \in L^2(\Omega^\epsilon) : \int_{\Omega^\epsilon} q\, ds ds_3 = 0 \right\},$$

and

$$H^1_{\Gamma_1^\epsilon \cup \Gamma_L^\epsilon}(\Omega^\epsilon) = \left\{ \Phi \in H^1(\Omega^\epsilon) : \Phi = 0 \text{ on } \Gamma_1^\epsilon \cup \Gamma_L^\epsilon \right\}.$$

A formal application of Green's formula, using (2.1)-(2.8) leads to the weak formulation: Find $w^\epsilon \in V_{div}^\epsilon(\Omega^\epsilon)$, $p^\epsilon \in L_0^2(\Omega^\epsilon)$ and $T^\epsilon \in W^1_{\Gamma_1^\epsilon \cup \Gamma_L^\epsilon}(\Omega^\epsilon)$, such that

$$B(w^\epsilon, w^\epsilon, \varphi - w^\epsilon) + a(T^\epsilon, w^\epsilon, \varphi - w^\epsilon) - (p^\epsilon, div\, \varphi) + j(T^\epsilon, \varphi) - j(T^\epsilon, w^\epsilon) \geq (f^\epsilon, \varphi - w^\epsilon), \forall \varphi \in V^\epsilon(\Omega^\epsilon), \tag{2.9}$$

$$-E(w^\epsilon, T^\epsilon, \Phi) + C(T^\epsilon, \Phi) = F(w^\epsilon, T^\epsilon, \Phi), \quad \forall \Phi \in W^1_{\Gamma_1^\epsilon \cup \Gamma_L^\epsilon}(\Omega^\epsilon), \tag{2.10}$$

where

$$a(T^\epsilon, w^\epsilon, \varphi) = 2 \int_{\Omega^\epsilon} \Lambda^\epsilon(T^\epsilon) D(w^\epsilon) D(\varphi)\, ds ds_3,$$

$$B(w^\epsilon, w^\epsilon, \varphi) = \int_{\Omega^\epsilon} w^\epsilon \nabla w^\epsilon \varphi\, ds ds_3,$$

$$(p^\epsilon, div\, \varphi) = \int_{\Omega^\epsilon} p^\epsilon\, div\, \varphi\, ds ds_3,$$

$$j(T^\epsilon, \varphi) = \int_\omega k^\epsilon |\varphi| \, ds + \sqrt{2} \int_{\Omega^\epsilon} g^\epsilon(T^\epsilon) |D(\varphi)|\, ds ds_3,$$

$$(f^\epsilon, \varphi) = \int_{\Omega^\epsilon} f^\epsilon \varphi\, ds ds_3 = \sum_{i=1}^3 \int_{\Omega^\epsilon} f_i^\epsilon \varphi_i\, ds ds_3,$$

$$E\left(w^{\epsilon},T^{\epsilon},\Phi\right)=\int_{\Omega^{\epsilon}}T^{\epsilon}\nabla\Phi w^{\epsilon}dsds_3,$$

$$C\left(T^{\epsilon},\Phi\right)=\int_{\Omega^{\epsilon}}K^{\epsilon}\nabla T^{\epsilon}\nabla\Phi dsds_3,$$

$$F\left(w^{\epsilon},T^{\epsilon},\Phi\right)=2\int_{\Omega^{\epsilon}}\Lambda^{\epsilon}\left(T^{\epsilon}\right)|D(w^{\epsilon})|^2\Phi dsds_3+2\int_{\Omega^{\epsilon}}g^{\epsilon}\left(T^{\epsilon}\right)|D(w^{\epsilon})|\Phi dsds_3 + \int_{\Omega^{\epsilon}}\alpha^{\epsilon}\left(T^{\epsilon}\right)\Phi dsds_3.$$

We suppose that there exist $\Lambda_*,\ \Lambda^*,\ g^*,\ K_*,\ K^*,\alpha_*,\alpha^*$ in $\mathbb{R}$, such that

$$0\leq\Lambda_*\leq\Lambda^{\epsilon}\leq\Lambda^*,\ \ 0\leq g^{\epsilon}\leq g^* \tag{2.11}$$

and

$$0\leq K_*\leq K^{\epsilon}\leq K^*,\ \ 0\leq\alpha^{\epsilon}\leq\alpha^* \tag{2.12}$$

Theorem 2.1 *Suppose that $f^{\epsilon}\in L^2(\Omega^{\epsilon})^3$, and $k^{\epsilon}\in L^{\infty}(\omega)$, such that $k^{\epsilon}\geq 0$. There exists a unique solution $w^{\epsilon}\in V^{\epsilon}_{div}(\Omega^{\epsilon}), p^{\epsilon}\in L^2_0(\Omega^{\epsilon})$ and $T^{\epsilon}\in H^1_{\Gamma_1^{\epsilon}\cup\Gamma_L^{\epsilon}}(\Omega^{\epsilon})$ to problem (2.9) − (2.10).*

Proof: The proof of the theorem is based on the application of Kakutan Glicksberg fixed point theorem, see for more details [17]. □

3. Change of the domain and some estimates

According to the change of variables $\kappa=\frac{s_3}{\epsilon}$, we define the fixed domain Ω which is independent of ϵ

$$\Omega=\left\{(s,\kappa)\in\mathbb{R}^3:(s,0)\in\omega,0<\kappa<h(s)\right\}.$$

We denote by $\Gamma=\bar{\Gamma}_1\cup\bar{\Gamma}_L\cup\bar{\omega}$ its boundary, then we define the following functions in Ω

$$\hat{w}_i^{\epsilon}(s,\kappa)=w_i^{\epsilon}(s,s_3), i=1,2,\ \hat{w}_3^{\epsilon}(s,\kappa)=\epsilon^{-1}w_3^{\epsilon}(s,s_3) \text{ and } \hat{p}^{\epsilon}(s,\kappa)=\epsilon^2p^{\epsilon}(s,s_3). \tag{3.1}$$

Let us assume that

$$\left.\begin{array}{c}\hat{K}(s,\kappa)=K^{\epsilon}(s,s_3),\hat{g}=\epsilon g^{\epsilon},\ \hat{\Lambda}=\Lambda^{\epsilon},\ \hat{T}^{\epsilon}(s,\kappa)=T^{\epsilon}(s,s_3)\\ \hat{f}(s,\kappa)=\epsilon^2f^{\epsilon}(s,s_3),\ \ \hat{\alpha}(s,\kappa)=\epsilon^2\alpha^{\epsilon}(s,s_3),\ \hat{k}=\epsilon k^{\epsilon}\end{array}\right\}. \tag{3.2}$$

Now, we introduce the functional framework on Ω. For this, we write

$$V(\Omega)=\left\{\ \hat{\varphi}\in H^1\left(\Omega\right)^3:\ \hat{\varphi}=0 \text{ on } \Gamma_L,\ \hat{\varphi}\cdot n=0 \text{ on } \omega\cup\Gamma_1\right\},$$
$$V_{div}(\Omega)=\{\ \hat{\varphi}\in V(\Omega):\text{div}(\ \hat{\varphi})=0\},$$
$$H^1_{\Gamma_1\cup\Gamma_L}\left(\Omega\right)=\left\{\ \hat{\Phi}\in H^1(\Omega):\ \hat{\Phi}=0 \text{ on } \Gamma_1\cup\Gamma_L\right\},$$
$$V_{\kappa}=\left\{\hat{v}\in\left(L^2\left(\Omega\right)\right)^2;\frac{\partial\hat{v}_i}{\partial\kappa}\in L^2\left(\Omega\right):\hat{v}=0\ \text{ on } \Gamma_L\right\},$$

and V_{κ} is the Banach space with the norm

$$\|v\|_{V_{\kappa}}=\left(\sum_{i=1}^{2}\left(\|v_i\|^2_{L^2(\Omega)}+\left\|\frac{\partial v_i}{\partial\kappa}\right\|^2_{L^2(\Omega)}\right)\right)^{\frac{1}{2}}.$$

By injecting the new data and unknown factors in (2.9)–(2.10), then, after multiplication by ϵ, we deduce

$$B_0\left(\hat{w}^\epsilon, \hat{w}^\epsilon, \hat{\varphi}-\hat{w}^\epsilon\right)+a_0\left(\hat{T}^\epsilon, \hat{w}^\epsilon, \hat{\varphi}-\hat{w}^\epsilon\right)-\left(p^\epsilon, div\ \hat{\varphi}\right) + j_0\left(\hat{T}^\epsilon, \hat{\varphi}\right)-j_0\left(\hat{T}^\epsilon, \hat{w}^\epsilon\right) \geq \left(\hat{f}^\epsilon, \hat{\varphi}-\hat{w}^\epsilon\right), \ \forall\, \hat{\varphi} \in V(\Omega), \tag{3.3}$$

$$-E_0\left(\hat{w}^\epsilon, \hat{T}^\epsilon, \hat{\Phi}\right)+C_0\left(\hat{T}^\epsilon, \hat{\Phi}\right)=F_0\left(\hat{w}^\epsilon, \hat{T}^\epsilon, \hat{\Phi}\right), \quad \forall\, \hat{\Phi} \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega), \tag{3.4}$$

where,

$$\begin{aligned}
a_0(\hat{T}^\epsilon, \hat{w}^\epsilon, \hat{\varphi}-\hat{w}^\epsilon) &= \sum_{i,j=1}^{2} \int_\Omega \left[\epsilon^2 \hat{\Lambda}\left(\hat{T}^\epsilon\right)\left(\frac{\partial \hat{w}_i^\epsilon}{\partial s_j}+\frac{\partial \hat{w}_j^\epsilon}{\partial s_i}\right)\right] \frac{\partial(\hat{\varphi}_i-\hat{w}_i^\epsilon)}{\partial s_j} dsd\kappa \\
&+ \sum_{i=1}^{2} \int_\Omega \hat{\Lambda}\left(\hat{T}^\epsilon\right)\left(\frac{\partial \hat{w}_i^\epsilon}{\partial \kappa}+\epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial s_i}\right) \frac{\partial(\hat{\varphi}_i-\hat{w}_i^\epsilon)}{\partial \kappa} dsd\kappa \\
&+ \int_\Omega \left(2\hat{\Lambda}\left(\hat{T}^\epsilon\right)\epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa}\right) \frac{\partial(\hat{\varphi}_3-\hat{w}_3^\epsilon)}{\partial \kappa} dsd\kappa + \\
&+ \sum_{j=1}^{2} \int_\Omega \epsilon^2 \hat{\Lambda}\left(\hat{T}^\epsilon\right)\left(\epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial s_j}+\frac{\partial \hat{w}_j^\epsilon}{\partial \kappa}\right) \frac{\partial(\hat{\varphi}_3-\hat{w}_3^\epsilon)}{\partial s_j} dsd\kappa,
\end{aligned}$$

$$\begin{aligned}
B_0\left(\hat{w}^\epsilon, \hat{w}^\epsilon, \hat{\varphi}-\hat{w}^\epsilon\right) &= \sum_{i,j=1}^{2} \int_\Omega \epsilon^2 \hat{w}_i^\epsilon \frac{\partial \hat{w}_j^\epsilon}{\partial s_i}\left(\hat{\varphi}-\hat{w}^\epsilon\right) dsd\kappa \\
&+ \sum_{i=1}^{2} \int_\Omega \epsilon^4 \hat{w}_i^\epsilon \frac{\partial \hat{w}_3^\epsilon}{\partial s_i}\left(\hat{\varphi}-\hat{w}^\epsilon\right) dsd\kappa + \sum_{i=1}^{2} \int_\Omega \epsilon^2 \hat{w}_3^\epsilon \frac{\partial \hat{w}_i^\epsilon}{\partial \kappa}\left(\hat{\varphi}_i-\hat{w}_i^\epsilon\right) dsd\kappa \\
&+ \int_\Omega \epsilon^4 \hat{w}_3^\epsilon \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa}\left(\hat{\varphi}_3-\hat{w}_3^\epsilon\right) dsd\kappa,
\end{aligned}$$

$$\left(\hat{p}^\epsilon, div\left(\hat{\varphi}-\hat{w}^\epsilon\right)\right)=\int_\Omega \hat{p}^\epsilon div\left(\hat{\varphi}-\hat{w}^\epsilon\right) dsd\kappa,$$

$$j_0(\hat{T}^\epsilon, \hat{\varphi})=\sqrt{2}\int_\Omega \hat{g}\left(\hat{T}^\epsilon\right)\left|\tilde{D}\left(\hat{\varphi}\right)\right| dsd\kappa+\int_\omega \hat{k}|\hat{\varphi}| ds,$$

$$(\hat{f}^\epsilon, \hat{\varphi}-\hat{w}^\epsilon)=\sum_{j=1}^{2}\int_\Omega \hat{f}_i(\hat{\varphi}_i-\hat{w}_i^\epsilon) dsd\kappa+\int_\Omega \epsilon \hat{f}_3(\hat{\varphi}_3-\hat{w}_3^\epsilon) dsd\kappa,$$

$$\begin{aligned}
C_0\left(\hat{T}^\epsilon, \hat{\Phi}\right) &= \int_\Omega \epsilon^2 \hat{K}(\hat{T}^\epsilon)\nabla \hat{T}^\epsilon \nabla \hat{\Phi} dsd\kappa \\
&= \sum_{i=1}^{2}\int_\Omega \epsilon^2 \hat{K}(\hat{T}^\epsilon)\frac{\partial \hat{T}^\epsilon}{\partial s_i}\frac{\partial \hat{\Phi}}{\partial s_i} dsd\kappa+\int_\Omega \hat{K}(\hat{T}^\epsilon)\frac{\partial \hat{T}^\epsilon}{\partial \kappa}\frac{\partial \hat{\Phi}}{\partial \kappa} dsd\kappa,
\end{aligned}$$

$$\begin{aligned}
E_0(\hat{w}^\epsilon, \hat{T}^\epsilon, \hat{\Phi}^\epsilon) &= \int_\Omega \epsilon^2 \hat{T}^\epsilon \nabla \hat{\Phi} \hat{w}^\epsilon dsd\kappa \\
&= \sum_{i=1}^{2}\int_\Omega \epsilon^2 \hat{T}^\epsilon \frac{\partial \hat{\Phi}}{\partial s_i}\hat{w}_i{}^\epsilon dsd\kappa+\int_\Omega \epsilon \hat{T}^\epsilon \frac{\partial \hat{\Phi}}{\partial \kappa}(\epsilon \hat{w}_3^\epsilon) dsd\kappa,
\end{aligned}$$

$$F_0\left(\hat{w}^\epsilon,\hat{T}^\epsilon,\hat{\Phi}\right)=2\int_\Omega\hat{\Lambda}\left(\hat{T}^\epsilon\right)\left|\tilde{D}(\hat{w}^\epsilon)\right|^2\hat{\Phi}dsd\kappa+\sqrt{2}\int_\Omega\hat{g}\left(\hat{T}^\epsilon\right)\left|\tilde{D}(\hat{w}^\epsilon)\right|\hat{\Phi}dsd\kappa$$
$$-\int_\Omega\hat{\alpha}\left(\hat{T}^\epsilon\right)\hat{\Phi}dsd\kappa,$$

where,

$$\left|\widetilde{D}\left(\hat{w}^\epsilon\right)\right|=\left[\frac{1}{4}\sum_{i,j=1}^{2}\epsilon^2\left(\frac{\partial\hat{w}_i^\epsilon}{\partial s_j}+\frac{\partial\hat{w}_j^\epsilon}{\partial s_i}\right)^2+\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\hat{w}_i^\epsilon}{\partial\kappa}+\epsilon^2\frac{\partial\hat{w}_3^\epsilon}{\partial s_i}\right)^2+\epsilon^2\left(\frac{\partial\hat{w}_3^\epsilon}{\partial\kappa}\right)^2\right]^{\frac{1}{2}}.$$

3.1. A priori estimates on the velocity and the pressure

Theorem 3.1 *Let the assumptions of theorem* (2.1) *and* (2.11)-(2.12) *hold, then, there exists a constant* $C>0$ *independent of* ϵ, *such that*

$$\epsilon^2\sum_{i,j=1}^{2}\left\|\frac{\partial\hat{w}_i^\epsilon}{\partial s_j}\right\|_{L^2(\Omega)}^2+\sum_{i=1}^{2}\left\|\frac{\partial\hat{w}_i^\epsilon}{\partial\kappa}\right\|_{L^2(\Omega)}^2+\epsilon^2\left\|\frac{\partial\hat{w}_3^\epsilon}{\partial\kappa}\right\|_{L^2(\Omega)}^2+\epsilon^4\sum_{i=1}^{2}\left\|\frac{\partial\hat{w}_3^\epsilon}{\partial s_i}\right\|_{L^2(\Omega)}^2\leq C. \tag{3.5}$$

$$\|\hat{w}_i{}^\epsilon\|_{L^2(\Omega)}^2\leq C\quad for\ i=1,2 \tag{3.6}$$

$$\|\epsilon\hat{w}_3{}^\epsilon\|_{L^2(\Omega)}^2\leq C, \tag{3.7}$$

$$\left\|\frac{\partial\hat{p}^\epsilon}{\partial s_i}\right\|_{H^{-1}(\Omega)}\leq C\quad for\ i=1,2 \tag{3.8}$$

$$\left\|\frac{\partial\hat{p}^\epsilon}{\partial\kappa}\right\|_{H^{-1}(\Omega)}\leq\epsilon C. \tag{3.9}$$

Proof: Choosing $\varphi=0$ in inequality (2.9), we find

$$\begin{aligned}B\left(w^\epsilon,w^\epsilon,w^\epsilon\right)+a(T^\epsilon,w^\epsilon,w^\epsilon)+\int_\omega k^\epsilon\left|w^\epsilon\right|\ ds\\+\sqrt{2}\int_{\Omega^\epsilon}g^\epsilon\left(T^\epsilon\right)\left|D\left(w^\epsilon\right)\right|dsds_3\leq\left(f^\epsilon,w^\epsilon\right),\end{aligned} \tag{3.10}$$

as $B\left(w^\epsilon,w^\epsilon,w^\epsilon\right)=0$, we obtain

$$a(T^\epsilon,w^\epsilon,w^\epsilon)+\int_\omega k^\epsilon\left|w^\epsilon\right|\ ds+\sqrt{2}\int_{\Omega^\epsilon}g^\epsilon\left(T^\epsilon\right)\left|D\left(w^\epsilon\right)\right|dsds_3\leq\left(f^\epsilon,w^\epsilon\right). \tag{3.11}$$

By Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}a(T^\epsilon,w^\epsilon,w^\epsilon)+\int_\omega k^\epsilon\left|w^\epsilon\right|\ ds+\sqrt{2}\int_{\Omega^\epsilon}g^\epsilon\left(T^\epsilon\right)\left|D\left(w^\epsilon\right)\right|dsds_3\\\leq\Lambda^*C_k\left\|\nabla w^\epsilon\right\|_{L^2(\Omega^\epsilon)}^2+\frac{(\epsilon\bar{h})^2}{4\left(\Lambda^*C_k\right)}\left\|f^\epsilon\right\|_{L^2(\Omega^\epsilon)}^2.\end{aligned} \tag{3.12}$$

Multiplying (17) by ε, and as $\varepsilon^2\left\|f^\varepsilon\right\|_{L^2(\Omega^\varepsilon)}^2=\varepsilon^{-1}\|\hat{f}\|_{L^2(\Omega)}^2$, we have

$$\begin{aligned}\epsilon a(T^\epsilon,w^\epsilon,w^\epsilon)+\epsilon\int_\omega k^\epsilon\left|w^\epsilon\right|\ ds+\sqrt{2}\epsilon\int_{\Omega^\epsilon}g^\epsilon\left(T^\epsilon\right)\left|D\left(w^\epsilon\right)\right|dsds_3\\\leq\Lambda^*C_k\epsilon\left\|\nabla w^\epsilon\right\|_{L^2(\Omega^\epsilon)}^2+\frac{(\bar{h})^2}{4\left(\Lambda^*C_k\right)}\left\|\hat{f}^\epsilon\right\|_{L^2(\Omega^\epsilon)}^2.\end{aligned} \tag{3.13}$$

Then, from Korn's inequality, there exist a constant C_k independent of ϵ, such that

$$\begin{aligned}\Lambda_* C_k \epsilon \|\nabla w^\epsilon\|^2_{L^2(\Omega^\epsilon)} + \int_\omega k^\epsilon \left|\hat{w}^\epsilon\right| \; ds + \sqrt{2} \int_{\Omega^\epsilon} g^\epsilon \left(T^\epsilon\right) \left|D\left(\hat{w}^\epsilon\right)\right| dsd\kappa \\ \leq \frac{(\bar{h})^2}{4\left(\Lambda^* C_k\right)} \left\|\hat{f}\right\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.14}$$

So, from (3.14) we deduce (3.5), (3.6), (3.7), with $C = \left(\frac{\bar{h}}{2\Lambda_* C_k}\right)^2 \|\hat{f}\|^2_{L^2(\Omega)}$.
For (3.8) and (3.9), we use the same proof as in [19]. □

3.2. A priori estimates on the temperature

Theorem 3.2 *Assume that the assumptions of Theorem (3.1) are satisfied, assume also there exist three positive constants K^*, K_*, C_4 , such that*

$$0 < C_4 < K_* \leq \hat{K} \leq K^*, \text{ where } C_4 \text{ are determined in the proof .} \tag{3.15}$$

Then, there exists a positive constant C_1 independent of ϵ, such that

$$\sum_{i=1}^{2} \left\| \epsilon \frac{\partial \hat{T}^\epsilon}{\partial s_i} \right\|^2_{L^2(\Omega)} \leq C_1, \tag{3.16}$$

$$\left\| \frac{\partial \hat{T}^\epsilon}{\partial \kappa} \right\|^2_{L^2(\Omega)} \leq C_1, \tag{3.17}$$

Proof: choosing $\hat{\Phi} = \hat{T}^\epsilon$ in (3.4), we obtain

$$\int_\Omega \epsilon^2 \hat{K} \nabla \hat{T}^\epsilon \nabla \hat{T}^\epsilon dsd\kappa = \sum_{i=1}^{4} I_i, \tag{3.18}$$

where

$$I_1 = 2\int_\Omega \hat{\Lambda}\left(\hat{T}^\epsilon\right) \left|\widetilde{D}\left(\hat{w}^\epsilon\right)\right|^2 \hat{T}^\epsilon dsd\kappa, \qquad I_2 = \sqrt{2}\int_\Omega \hat{g}\left(\hat{T}^\epsilon\right) \left|\widetilde{D}\left(\hat{w}^\epsilon\right)\right| \hat{T}^\epsilon dsd\kappa,$$
$$I_3 = \int_\Omega \hat{\alpha}\left(\hat{T}^\epsilon\right) \hat{T}^\epsilon dsd\kappa, \qquad I_4 = \int_\Omega \epsilon^2 \hat{T}^\epsilon \nabla \hat{T}^\epsilon w^\epsilon dsd\kappa.$$

From (3.15), we have

$$\int_\Omega \epsilon^2 \hat{K}(\hat{T}^\epsilon) \nabla \hat{T}^\epsilon \nabla \hat{T}^\epsilon dsd\kappa \geqslant K_* \epsilon^2 \left\|\nabla \hat{T}^\epsilon\right\|^2_{L^2(\Omega)} \geqslant K_* \epsilon^2 \left\|\frac{\partial \hat{T}^\epsilon}{\partial s_i}\right\|^2_{L^2(\Omega)} + K_* \left\|\frac{\partial \hat{T}^\epsilon}{\partial \kappa}\right\|^2_{L^2(\Omega)}. \tag{3.19}$$

For I_1 by the cauchy-schwarz inequality, we give

$$|I_1| \leq \Lambda^* \left[\sum_{i,j=1}^{2} \frac{\epsilon^2}{2} \left\| \frac{\partial \hat{w}_i^\epsilon}{\partial s_j} + \frac{\partial \hat{w}_j^\epsilon}{\partial s_i} \right\|^2_{L^4(\Omega)} + \sum_{i=1}^{2} \left\| \frac{\partial \hat{w}_i^\epsilon}{\partial \kappa} + \epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial s_i} \right\|^2_{L^4(\Omega)} + 2\epsilon^2 \left\| \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \right\|^2_{L^4(\Omega)} \right] \left\| \hat{T}^\epsilon \right\|_{L^2(\Omega)}.$$

Using Young's inequality and the compact injection $H_1(\Omega)$ in $L^4(\Omega)$, there exists a constant $C_1(\Omega)$ independent of ϵ, such that

$$|I_1| \leqslant 2\Lambda^* C_1(\Omega) \left[\begin{array}{c} \sum_{i,j=1}^{2} \epsilon^2 \left\| \frac{\partial \hat{w}_i^\epsilon}{\partial s_j} \right\|^2_{H^1(\Omega)} + \sum_{i=1}^{2} \left\| \frac{\partial \hat{w}_i^\epsilon}{\partial \kappa} \right\|^2_{H^1(\Omega)} + \\ + \sum_{i=1}^{2} \epsilon^4 \left\| \frac{\partial \hat{w}_3^\epsilon}{\partial s_i} \right\|^2_{H^1(\Omega)} + \epsilon^2 \left\| \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \right\|^2_{H^1(\Omega)} \end{array} \right] \left\| \hat{T}^\epsilon \right\|_{L^2(\Omega)},$$

also, from (3.5), we get: $|I_1| \leqslant 2\Lambda^* C_1(\Omega) C \left\|\hat{T}^\epsilon\right\|_{L^2(\Omega)}$. Similarly,

$$|I_2| \leqslant \sqrt{2} g^* C \left\|| \hat{T}^\epsilon\right\|_{L^2(\Omega)} \text{ and } |I_3| \leqslant \alpha^* \bar{h} \left\|| \hat{T}^\epsilon\right\|_{L^2(\Omega)}. \tag{3.20}$$

The analog of the last inequality gives

$$\begin{aligned} |I_4| &\leqslant \epsilon^2 \left\|\hat{T}^\epsilon\right\|_{L^4(\Omega)} \|\hat{w}^\epsilon\|_{L^4(\Omega)} \left\|\nabla\hat{T}^\epsilon\right\|_{L^2(\Omega)}, \\ &\leqslant \epsilon^2 C_2 \left\|\hat{T}^\epsilon\right\|^{\frac{1}{2}}_{H_0^1(\Omega)} \left\|\hat{T}^\epsilon\right\|^{\frac{1}{2}}_{L^2(\Omega)} \|\hat{w}^\epsilon\|^{\frac{1}{2}}_{H_0^1(\Omega)} \|\hat{w}^\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} \left\|\nabla\hat{T}^\epsilon\right\|_{L^2(\Omega)}, \\ &\leqslant \epsilon^2 C_2 \left\|\hat{T}^\epsilon\right\|^{\frac{1}{2}}_{L^2(\Omega)} \|\nabla\hat{w}^\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} \|\hat{w}^\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} \left\|\nabla\hat{T}^\epsilon\right\|^{\frac{3}{2}}_{L^2(\Omega)}. \end{aligned}$$

By Young's inquality and from (3.5) we find

$$\begin{aligned} |I_4| &\leqslant \frac{3}{4} C_2 \epsilon^2 \left\|\nabla\hat{T}^\epsilon\right\|^2_{L^2(\Omega)} + \frac{1}{4}\epsilon^2 \left\|\hat{T}^\epsilon\right\|^2_{L^2(\Omega)} \|\nabla\hat{w}^\epsilon\|^2_{L^2(\Omega)} \|\hat{w}^\epsilon\|^2_{L^2(\Omega)}, \\ &\leqslant \frac{3}{4} C_2 \epsilon^2 \left\|\nabla\hat{T}^\epsilon\right\|^2_{L^2(\Omega)} + \frac{1}{4}\epsilon^2 C_3 \left\|\nabla\hat{T}^\epsilon\right\|^2_{L^2(\Omega)} \|\nabla\hat{w}^\epsilon\|^2_{L^2(\Omega)} \|\hat{w}^\epsilon\|^2_{L^2(\Omega)}, \\ &\leqslant (\frac{3}{4} C_2 + \frac{1}{4} C_3 C^2)\epsilon^2 \left\|\nabla\hat{T}^\epsilon\right\|^2_{L^2(\Omega)}. \end{aligned}$$

So,

$$|I_4| \leqslant C_4 \epsilon^2 \left\|\frac{\partial \hat{T}^\epsilon}{\partial s_i}\right\|^2_{L^2(\Omega)} + C_4 \left\|\frac{\partial \hat{T}^\epsilon}{\partial \kappa}\right\|^2_{L^2(\Omega)}, \tag{3.21}$$

where $C_4 = \frac{3}{4} C_2 + \frac{1}{4} C_3 C^2$.
By injecting $(3.15) - (3.16)$ in (3.14), and using (3.10), it becomes

$$(K_* - C_4)\epsilon^2 \left\|\frac{\partial \hat{T}^\epsilon}{\partial s_i}\right\|^2_{L^2(\Omega)} + (K_* - C_4) \left\|\frac{\partial \hat{T}^\epsilon}{\partial \kappa}\right\|^2_{L^2(\Omega)} \leqslant (2\Lambda^* C_1(\Omega) C + \sqrt{2} g^* C + \alpha^* \bar{h}) \left\|\hat{T}^\epsilon\right\|_{L^2(\Omega)}.$$

As: $\left\|\hat{T}^\epsilon\right\|_{L^2(\Omega)} \leqslant \bar{h} \left\|\frac{\partial \hat{T}^\epsilon}{\partial \kappa}\right\|_{L^2(\Omega)}$, we find

$$(K_* - C_4)\epsilon^2 \left\|\frac{\partial \hat{T}^\epsilon}{\partial s_i}\right\|^2_{L^2(\Omega)} + (K_* - C_4) \left\|\frac{\partial \hat{T}^\epsilon}{\partial \kappa}\right\|^2_{L^2(\Omega)} \leqslant C_5 \left\|\frac{\partial \hat{T}^\epsilon}{\partial \kappa}\right\|_{L^2(\Omega)}, \tag{3.22}$$

where $C_5 = (2\Lambda^* C_4(\Omega) C + \sqrt{2} g^* C + \alpha^* \bar{h})\bar{h}$.
According to (3.22) we deduce that : $\left\|\frac{\partial \hat{T}^\epsilon}{\partial \kappa}\right\|_{L^2(\Omega)} \leqslant C_5 (K_* - C_4)^{-1}$.
By injecting this last estimate in (3.22), we deduce (3.16) and (3.17).

□

Theorem 3.3 *Under the same assumptions as in Theorem* (3.1) *and Theorem* (3.2), *there exist* $w^\star = (w_1^\star, w_2^\star) \in V_\kappa$, $p^\star \in L_0^2(\Omega)$ *and* $T^* \in V_\kappa$ *such that:*

$$\begin{cases} \hat{w}_i^\epsilon \rightharpoonup w_i^\star, \ i=1,2 \quad weakly \ in \ V_\kappa, \\ \\ \epsilon \dfrac{\partial \hat{w}_i^\epsilon}{\partial s_j} \rightharpoonup 0, \ i,j=1,2 \ weakly \ in \ \ L^2(\Omega), \\ \\ \epsilon \dfrac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \rightharpoonup 0, \quad weakly \ in \ \ L^2(\Omega), \\ \\ \epsilon^2 \dfrac{\partial \hat{w}_3^\epsilon}{\partial s_i} \rightharpoonup 0, \ i=1,2 \ weakly \ in \ \ L^2(\Omega), \end{cases} \tag{3.23}$$

$$\epsilon \hat{w}_3^\epsilon \rightharpoonup 0, \quad weakly \ in \ \ L^2(\Omega), \tag{3.24}$$

$$\hat{p}^\epsilon \rightharpoonup p^\star, weakly \ in \ \ L^2(\Omega), \ p^\star \ depend \ only \ of \ s, \tag{3.25}$$

$$\begin{cases} \hat{T}^\epsilon \rightharpoonup T^*, \quad weakly \ in \ V_\kappa, \\ \\ \frac{\partial \hat{T}^\epsilon}{\partial s_i} \rightharpoonup 0, \ i=1,2 \ weakly \ in \ L^2(\Omega). \end{cases} \tag{3.26}$$

Proof: From the inequality (3.5) − (3.6) we find directly the convergence of (3.23), to prove (3.24) we use (3.5) and (3.7) Since $div(\hat{w}^\epsilon) = 0$, from (3.8) and (3.9) by choosing a particular test function, we get (3.25).
By inequality (3.17), we have

$$\left\|\hat{T}^\epsilon\right\|_{L^2(\Omega)} \leqslant \bar{h} \left\|\frac{\partial \hat{T}^\epsilon}{\partial \kappa}\right\|_{L^2(\Omega)} \leqslant \bar{h} C.$$

So, $\hat{T}^\epsilon$ is bounded in V_κ, which implies the existence of an element $\hat{T}^\epsilon$ in V_κ, such that $\hat{T}^\epsilon$ converges weakly to $\hat{T}^*$ in V_κ.
Moreover, inequality (3.16) shows that $\epsilon \left\|\frac{\partial \hat{T}^\epsilon}{\partial s_i}\right\|_{L^2(\Omega)} \leqslant C$, therefore $\epsilon \frac{\partial \hat{T}^\epsilon}{\partial s_i}$ converge to $\frac{\partial \hat{T}^*}{\partial s_i}$, and since $\hat{T}^\epsilon$ converge to $\hat{T}^*$ in V_κ, we have that $\epsilon \frac{\partial \hat{T}^*}{\partial s_i}$ converges to zero in V_κ.

□

4. Study of the limit problem

Theorem 4.1 *With the same assumptions of Theorem* (3.3), *the solution* (w^*, p^*, T^*) *satisfying the following relations*

$$\begin{aligned} \sum_{i=1}^{2} \int_\Omega \hat{\Lambda}(T^*) \frac{\partial (w_i^*)}{\partial \kappa} \frac{\partial (\hat{\varphi}_i - w_i^*)}{\partial \kappa} ds d\kappa - \int_\Omega p^*(s) \left(\frac{\partial \hat{\varphi}_1}{\partial s_1} + \frac{\partial \hat{\varphi}_2}{\partial s_2} \right) ds d\kappa \\ + \int_\Omega \hat{g}(T^*) \left(\left| \frac{\partial \hat{\varphi}}{\partial \kappa} \right| - \left| \frac{\partial w^*}{\partial \kappa} \right| \right) ds d\kappa + \int_\omega \hat{k} \left(|\hat{\varphi}| - |w^*| \right) ds \\ \geq \sum_{i=1}^{2} \int_\Omega \hat{f}_i (\hat{\varphi}_i - w_i^*) ds d\kappa, \forall \hat{\varphi} \in \Pi(V), \end{aligned} \tag{4.1}$$

and,

$$-\frac{\partial}{\partial \kappa} \left(\widehat{K} \frac{\partial T^*}{\partial \kappa} \right) = \sum_{i=1}^{2} \hat{\Lambda}(T^*) \left(\frac{\partial w_i^*}{\partial \kappa} \right)^2 + \sqrt{2} \hat{g} \left| \frac{\partial w^*}{\partial \kappa} \right| + \hat{\alpha}(T^*), \ in \ L^2(\Omega). \tag{4.2}$$

Moreover, if

$$\int_\Omega \left(\left(\hat{\varphi}_1(s,\kappa) \frac{\partial \theta}{\partial s_1}(s) + \hat{\varphi}_2(s,\kappa) \right) \frac{\partial \theta}{\partial s_2}(s) \right) ds d\kappa = 0, \quad \forall \theta \in C_0^1(\omega), \tag{4.3}$$

Then,

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}\left(T^{*}\right)\frac{\partial w_i^{\star}}{\partial\kappa}\frac{\partial\left(\hat{\varphi}_i-w_i^{\star}\right)}{\partial\kappa}dsd\kappa+\hat{g}\int_{\Omega}\left(\left|\frac{\partial\hat{\varphi}}{\partial\kappa}\right|-\left|\frac{\partial w^{\star}}{\partial\kappa}\right|\right)dsd\kappa + \int_{\omega}\hat{k}\left(|\hat{\varphi}|-|w^{\star}|\right)ds \geqslant \sum_{j=1}^{2}\left(\hat{f},\hat{\varphi}-w^{\star}\right). \tag{4.4}$$

Where, $\Pi(V)=\left\{\bar{\varphi}=(\hat{\varphi}_1,\hat{\varphi}_2)\in H^1(\Omega)^2:\exists\hat{\varphi}_3 \text{ such that } \hat{\varphi}=(\hat{\varphi}_1,\hat{\varphi}_2,\hat{\varphi}_3)\in V\right\}$.

Proof: We apply the $\lim\limits_{\epsilon\to 0}$ on the variational inequality (3.3), and using the convergence results of the Theorem (3.3), we deduce

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}(T^{*})\frac{\partial w_i^{*}}{\partial\kappa}\frac{\partial w_i^{*}}{\partial\kappa}dsd\kappa+\hat{g}\int_{\Omega}\left|\frac{\partial w_i^{*}}{\partial\kappa}\right|dsd\kappa+\int_{\omega}\hat{k}\left|u^{*}\right|ds\geqslant\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}(T^{*})\frac{\partial w_i^{*}}{\partial\kappa}\frac{\partial\hat{\varphi}}{\partial\kappa}dsd\kappa + \hat{g}\int_{\Omega}\left|\frac{\partial\hat{\varphi}}{\partial\kappa}\right|dsd\kappa+\int_{\omega}\hat{k}\left|\hat{\varphi}\right|ds+\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i(\hat{\varphi}_i-w_i^{*})dsd\kappa + \sum_{i=1}^{2}\int_{\Omega}p^{*}\frac{\partial\hat{\varphi}_i}{\partial s_i}dsd\kappa+\int_{\Omega}p^{*}\frac{\partial\hat{\varphi}_3}{\partial\kappa}dsd\kappa, \tag{4.5}$$

as $\int_{\Omega}p^{*}\frac{\partial\hat{\varphi}_3}{\partial\kappa}dsd\kappa=0$, because p^{*} is independent of κ, we find (4.1), and, if $\hat{\varphi}$ verifies condition (4.3), we deduce directly relation (4.4).

The same for (4.2) we apply the $\lim\limits_{\epsilon\to 0}$ on (3.4), we get

$$\int_{\Omega}\widehat{K}\frac{\partial T^{*}}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa=\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}\left(T^{*}\right)\left(\frac{\partial w_i^{*}}{\partial\kappa}\right)^{2}dsd\kappa+\sqrt{2}\hat{g}\int_{\Omega}\left|\frac{\partial w^{*}}{\partial\kappa}\right|dsd\kappa + \int_{\Omega}\hat{\alpha}\left(T^{*}\right)dsd\kappa, \tag{4.6}$$

by Green's formula, we obtain

$$-\frac{\partial}{\partial\kappa}\left(\widehat{K}\frac{\partial T^{*}}{\partial\kappa}\right)=\sum_{i=1}^{2}\hat{\Lambda}\left(T^{*}\right)\left(\frac{\partial w_i^{*}}{\partial\kappa}\right)^{2}+\sqrt{2}\hat{g}\left|\frac{\partial w^{*}}{\partial\kappa}\right|+\hat{\alpha}\left(T^{*}\right),\ \text{in } L^2(\Omega) \tag{4.7}$$

□

Theorem 4.2 *The variational inequality* (4.4) *is equivalent to the following system*

$$\int_{\Omega}\hat{\Lambda}\left(T^{*}\right)\left|\frac{\partial w^{*}}{\partial\kappa}\right|^{2}dsd\kappa+\int_{\Omega}\hat{g}\left(T^{*}\right)\left|\frac{\partial w^{*}}{\partial\kappa}\right|dsd\kappa+\int_{\omega}\hat{k}\left|w^{*}\right|ds=\int_{\Omega}\hat{f}w^{*}dsd\kappa, \tag{4.8}$$

and, $\forall\hat{\Phi}\in\Sigma\left(V\right)$,

$$\int_{\Omega}\hat{\Lambda}\left(T^{*}\right)\frac{\partial w^{*}}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa+\int_{\Omega}\hat{g}\left(T^{*}\right)\left|\frac{\partial\hat{\Phi}}{\partial\kappa}\right|dsd\kappa+\int_{\omega}\hat{k}\left|\hat{\Phi}\right|ds\geq\int_{\Omega}\hat{f}\hat{\Phi}dsd\kappa, \tag{4.9}$$

where,

$$\Sigma\left(V\right)=\left\{\hat{\Phi}\in\Pi(V):\hat{\varphi}\ \textit{satisfy}\ (4.3)\right\}.$$

Proof: According to [21, Lemma 5], we can choose $\hat{\varphi}=2w^{*}$ and $\hat{\varphi}=0$ respectively in (4.4), we find (4.8).

For (4.9), we choose $\hat{\Phi}=\hat{\varphi}-w^{*}$ for all $\hat{\Phi}\in\Sigma\left(V\right)$. □

Theorem 4.3 *Let us set*

$$\Sigma^* = \tilde{\Sigma}^* - \nabla P^* \text{ and } \tilde{\Sigma}^* = \hat{\Lambda}(T^*)\frac{\partial w^*}{\partial \kappa} + \hat{g}(T^*)\pi, \tag{4.10}$$

then

$$-\frac{\partial}{\partial \kappa}\left[\hat{\Lambda}(T^*)\frac{\partial w^*}{\partial \kappa} + \hat{g}(T^*)\frac{\frac{\partial w^*}{\partial \kappa}}{\left|\frac{\partial w^*}{\partial \kappa}\right|}\right] = \hat{f} - \nabla p^* \ , \text{ in } L^2(\Omega)^2. \tag{4.11}$$

Where $\pi \in L^\infty(\Omega)^2$ *and* $\|\pi\|_{\Omega,\infty} \leq 1$.

Proof: If $\frac{\partial w^*}{\partial \kappa} = 0$, from (4.10) we find $\left|\tilde{\Sigma}^*\right| < \hat{g}(T^*)$. For all $\hat{\Phi} \in \Sigma(K)$, choosing $\hat{\Phi} = \hat{\Phi}$, then $\hat{\Phi} = -\hat{\Phi}$ in (4.9), we obtain

$$\left|G\left(\hat{k}\hat{\Phi}, \frac{\partial \hat{\Phi}}{\partial \kappa}\right)\right| \leq \int_\omega \hat{k}\left|\hat{\Phi}\right| ds + \int_\Omega \hat{g}(T^*)\left|\frac{\partial \hat{\Phi}}{\partial \kappa}\right| ds d\kappa,$$

where

$$G\left(\hat{k}\hat{\Phi}, \frac{\partial \hat{\Phi}}{\partial \kappa}\right) = \int_\Omega \hat{\Lambda}(T^*)\frac{\partial w^*}{\partial \kappa}\frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa - \int_\Omega \hat{f}\hat{\Phi} ds d\kappa. \tag{4.12}$$

Now, by the Hanh-Banach theorem [4], then, $\exists (\chi, \pi) \in L^\infty(\omega)^2 \times L^\infty(\Omega)^2$, with $\|\chi\|_{\omega,\infty} \leq 1 \ \ \|\pi\|_{\Omega,\infty} \leq 1$, such that

$$G\left(\hat{k}\hat{\Phi}, \frac{\partial \hat{\Phi}}{\partial \kappa}\right) = -\int_\omega \chi\hat{k}\hat{\Phi} ds - \int_\Omega \pi\hat{g}(T^*)\frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa. \tag{4.13}$$

In particular, from (4.8) and (4.12) we find

$$\int_\omega \hat{k}\left|w^*\right| ds + \int_\Omega \hat{g}(T^*)\left|\frac{\partial w^*}{\partial \kappa}\right| ds d\kappa = \int_\omega \chi\hat{k}w^* ds + \int_\Omega \pi\hat{g}(T^*)\frac{\partial w^*}{\partial \kappa} ds d\kappa. \tag{4.14}$$

Moreover, from (4.12) and (4.13), we have

$$\int_\Omega \hat{\Lambda}(T^*)\frac{\partial w^*}{\partial \kappa}\frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa + \int_\omega \chi\hat{k}\hat{\Phi} ds + \int_\Omega \pi\hat{g}(T^*)\frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa - \int_\Omega \hat{f}\hat{\Phi} ds d\kappa = 0. \tag{4.15}$$

Next using (4.14), we have

$$\int_\omega \hat{k}\left(\left|w^*\right| - \chi w^*\right) ds + \int_{\frac{\partial w^*}{\partial \kappa} \neq 0} \hat{g}(T^*)\left(\left|\frac{\partial w^*}{\partial \kappa}\right| - \pi\frac{\partial w^*}{\partial \kappa}\right) ds d\kappa = 0.$$

As $\|\chi\|_{\omega,\infty} \leq 1$, $\|\pi\|_{\Omega,\infty} \leq 1$, we deduce

$$\left|\frac{\partial w^*}{\partial \kappa}\right| = \pi\frac{\partial w^*}{\partial \kappa} \text{ and } \left|w^*\right| - \chi w^*.$$

So, if $\left|\frac{\partial w^*}{\partial \kappa}\right| \neq 0$, by (4.10), we get

$$\tilde{\Sigma}^* = \hat{\Lambda}(T^*)\frac{\partial w^*}{\partial \kappa} + \hat{g}(T^*)\frac{\partial w^*/\partial \kappa}{\left|\partial w^*/\partial \kappa\right|}.$$

In this case, $\left|\tilde{\Sigma}^*\right| = \hat{\Lambda}\left(T^*\right)\left|\frac{\partial w^*}{\partial \kappa}\right| + \hat{g}\left(T^*\right) > \hat{g}\left(T^*\right),$
therefore, we can write

$$\hat{\Lambda}\left(T^*\right)\frac{\partial w^*}{\partial \kappa} = \begin{cases} 0, & \text{if } \left|\tilde{\Sigma}^*\right| \leq \hat{g}, \\ \tilde{\Sigma}^* - \hat{g}\left(T^*\right)\dfrac{\partial w^*/\partial \kappa}{\left|\partial w^*/\partial \kappa\right|}, & \text{if } \left|\tilde{\Sigma}^*\right| > \hat{g}, \end{cases} \tag{4.16}$$

Besides, from (4.15), there exists $p^* \in L^2\left(\Omega\right)^2$, such that

$$\begin{aligned} &\int_\Omega \hat{\Lambda}\left(T^*\right)\frac{\partial w^*}{\partial \kappa}\frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa + \int_\omega \chi \hat{k}\hat{\Phi} ds \\ &\qquad + \int_\Omega \pi \hat{g}\left(T^*\right)\frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa - \int_\Omega \hat{f}\hat{\Phi} ds d\kappa = -\int_\Omega \nabla p^* \hat{\Phi} ds d\kappa. \end{aligned} \tag{4.17}$$

Using (4.16) and (4.17) becomes

$$\int_\Omega \tilde{\Sigma}^* \frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa + \int_\omega \chi \hat{k}\hat{\Phi} ds = \int_\Omega \hat{f}\hat{\Phi} ds d\kappa - \int_\Omega \nabla p^* \hat{\Phi} ds d\kappa, \tag{4.18}$$

from which, (4.11) follows if we take $\hat{\Phi} \in H_0^1\left(\Omega\right)^2$ in (4.18). □

Theorem 4.4 *Under the assumptions of preceding theorems, u^* and p^* satisfy the following equality*

$$\begin{aligned} &\int_\omega \left[\frac{h^3}{12}\nabla p^* + \tilde{F} + \int_0^h \int_0^y \hat{\Lambda}\left(T^*\left(s,\zeta\right)\right)\frac{\partial w^*\left(s,\xi\right)}{\partial \xi} d\xi dy \right. \\ &+ \hat{g}\int_0^h \int_0^y \frac{\partial w^*/\partial \xi}{\left|\partial w^*/\partial \xi\right|}(s,\xi) d\xi dy - \frac{h}{2}\int_0^h \hat{\Lambda}\left(T^*\left(s,\zeta\right)\right)\frac{\partial w^*\left(s,\xi\right)}{\partial \xi} d\xi \\ &\left. + \frac{\hat{g}h}{2}\int_0^h \frac{\partial w^*/\partial \xi}{\left|\partial w^*/\partial \xi\right|}(s,\xi) d\xi \right].\nabla \varphi\left(s\right) ds = 0, \end{aligned} \tag{4.19}$$

for all $\varphi \in H^1\left(\omega\right)$, where

$$\tilde{F}\left(s\right) = \int_0^h F\left(s,y\right) dy - \frac{h}{2}F\left(s,h\right),\ F\left(s,y\right) = \int_0^h \int_0^\xi \hat{f}\left(s,t\right) dt d\xi.$$

Proof: To prove (4.19), we integrate twice (4.11) from 0 to κ, then taking $\kappa = h$, we obtain the requested result. □

For the uniqueness of the limit velocity and temperature, we let:

$$\begin{aligned} \mathcal{W}_\kappa &= \left\{ w \in V_\kappa : \tfrac{\partial^2 w}{\partial \kappa^2} \in L^2(\Omega) \right\}, \\ \mathcal{B}_c &= \left\{ w \in \mathcal{W}_\kappa \times \mathcal{W}_\kappa : \left\| \tfrac{\partial w}{\partial \kappa} \right\|_{V_\kappa} \leqslant c \right\}, \\ \tilde{\mathcal{W}}_\kappa &= \{ w \in \mathcal{W}_\kappa \times \mathcal{W}_\kappa : w \text{ satisfies condition (4.3)} \}. \end{aligned}$$

Theorem 4.5 *Under the assumptions* (2.11) − (2.12) *and if K_* is sufficiently large such that*

$$K_* > \left[1 + (\bar{h})^2\right] C_{\hat{\alpha}}.$$

Then, the solution (u^*, p^*, T^*) of the limit problem (4.2) *and* (4.8) − (4.9) *is unique $\left(\tilde{\mathcal{W}}_\kappa \cap \mathcal{B}_c\right) \times \mathcal{W}_\kappa$, for all*

$$0 < c < c_0 = \left(2C_{\hat{\Lambda}}\beta^4\right)^{-\frac{1}{2}} \left[K\left[1 + (\bar{h})^2\right]^{(-1)} - C_{\hat{\alpha}}\right]^{\frac{1}{2}}.$$

Where $\beta > 0, C_{\hat{\Lambda}} > 0, C_{\hat{\alpha}} > 0$.

Proof: We use the same techniques as in [4] to prove this theorem, Let $(w^{*,1}, p^{*,1}, T^{*,1})$, and $(w^{*,2}, p^{*,2}, T^{*,2})$ be two solutions of (4.2), and (4.8) – (4.9)

$$\begin{aligned}\int_\Omega \hat{K}\frac{\partial T^{*,1}}{\partial \kappa}\frac{\partial \hat{\Phi}}{\partial \kappa}dsd\kappa &= \sum_{i=1}^{2}\int_\Omega \hat{\Lambda}\left(T^{*,1}\right)\left(\frac{\partial w_i^{*,1}}{\partial \kappa}\right)^2\hat{\Phi}dsd\kappa \\ &+ \sqrt{2}\hat{g}\int_\Omega\left|\frac{\partial w_i^{*,1}}{\partial \kappa}\right|\hat{\Phi}dsd\kappa + \int_\Omega \hat{\alpha}\left(T^{*,1}\right)\hat{\Phi}dsd\kappa,\end{aligned} \tag{4.20}$$

$$\begin{aligned}\int_\Omega \hat{K}\frac{\partial T^{*,2}}{\partial \kappa}\frac{\partial \hat{\Phi}}{\partial \kappa}dsd\kappa &= \sum_{i=1}^{2}\int_\Omega \hat{\Lambda}\left(T^{*,2}\right)\left(\frac{\partial w_i^{*,2}}{\partial \kappa}\right)^2\hat{\Phi}dsd\kappa \\ &+ \sqrt{2}\hat{g}\int_\Omega\left|\frac{\partial w_i^{*,2}}{\partial \kappa}\right|\hat{\Phi}dsd\kappa + \int_\Omega \hat{\alpha}\left(T^{*,2}\right)\hat{\Phi}dsd\kappa.\end{aligned} \tag{4.21}$$

By subtraction and choosing $\hat{\Phi} = (T^{*,1} - T^{*,2}) \in H^1_{\Gamma_L \cup \Gamma_1} \in \Omega$, we get

$$\int_\Omega \hat{K}\left|\frac{\partial}{\partial \kappa}(T^{*,1} - T^{*,2})\right|^2 dsd\kappa = \sum_{i=1}^{4} N_i. \tag{4.22}$$

Where

$$N_1 = \sum_{j=1}^{2} N_1^j, N_1^j = \int_\Omega \hat{\Lambda}\left(T^{*,1}\right)\frac{\partial}{\partial \kappa}\left(w_i^{*,1} + w_i^{*,2}\right)\frac{\partial}{\partial \kappa}\left(w_i^{*,1} - w_i^{*,2}\right)\left(T^{*,1} - T^{*,2}\right)dsd\kappa,$$

$$N_2 = \sum_{j=1}^{2} N_2^j, N_2^j = \int_\Omega \left[\hat{\Lambda}\left(T^{*,1}\right) - \hat{\Lambda}\left(T^{*,2}\right)\right]\left(\frac{\partial w_i^{*,2}}{\partial \kappa}\right)^2\left(T^{*,1} - T^{*,2}\right)dsdz,$$

$$N_3 = \int_\Omega \left(\hat{\alpha}\left(T^{*,1}\right) - \hat{\alpha}\left(T^{*,2}\right)\right)\left(T^{*,1} - T^{*,2}\right)dsdz,$$

$$N_4 = \sqrt{2}\hat{g}\int_\Omega\left(\left(\frac{\partial w_i^{*,1}}{\partial \kappa}\right)^2 - \left(\frac{\partial w_i^{*,2}}{\partial \kappa}\right)^2\right)\left(T^{*,1} - T^{*,2}\right)dx'dz.$$

The increases of $N_i, i = 1, 2, 3$ are given by [4] as follows

$$N_1 = \left|\sum_{j=1}^{2} N_1^j\right| \leqslant 2\sqrt{2}\Lambda^*\beta^2 c\left\|w_i^{*,1} - w_i^{*,2}\right\|_{V_\kappa \times V_\kappa}\left\|T^{*,1} - T^{*,2}\right\|_{V_\kappa}, \tag{4.23}$$

$$\left|N_2^i\right| \leqslant C_{\hat{\Lambda}}\beta^4\left\|T^{*,1} - T^{*,2}\right\|^2_{V_\kappa}\left\|w_2^2\right\|^2_{W_\kappa}, \tag{4.24}$$

$$|N_3| \leqslant C_{\hat{\tau}}\left\|T^{*,1} - T^{*,2}\right\|^2_{V_\kappa}, \tag{4.25}$$

where $C_{\hat{\alpha}} > 0$ deducted from the assumption $\hat{\alpha}$ is $C_{\hat{\alpha}}$-Lipschitz continuous function on $\mathbb{R}$. Utilising the Cauchy-Schwartz inequality, we get:

$$|N_4| \leqslant 2g^*\left\|T^{*,1} - T^{*,2}\right\|^2_{V_\kappa}\left\|w_i^{*,1} - w_i^{*,2}\right\|_{V_\kappa \times V_\kappa}. \tag{4.26}$$

Injecting (4.23) – (4.26) in (4.22), we find:

$$\begin{aligned}K_*\left[1 + (h)^2\right]^{-1}\left\|T^{*,1} - T^{*,2}\right\|^2_{V_\kappa} &\leqslant 2\sqrt{2}\Lambda^*\beta^2 c\left\|w_i^{*,1} - w_i^{*,2}\right\|_{V_\kappa \times V_\kappa}\left\|T^{*,1} - T^{*,2}\right\|_{V_\kappa} \\ &+ C_{\hat{\Lambda}}\beta^4 c^2\left\|T^{*,1} - T^{*,2}\right\|^2_{V_\kappa} + C_{\hat{\tau}}\left\|T^{*,1} - T^{*,2}\right\|^2_{V_\kappa} \\ &+ 2g^*\left\|T^{*,1} - T^{*,2}\right\|^2_{V_\kappa}\left\|w^{*,2} - w^{*,1}\right\|_{V_\kappa \times V_\kappa},\end{aligned}$$

so,

$$\left\|T^{*,1}-T^{*,2}\right\|_{V_\kappa} \leqslant \left[K_*\left[1+(\bar{h})^2\right]^{-1}-2C_{\Lambda^*}\beta^4c^2-C_{\hat{\alpha}}\right]^{-1}\left[2\sqrt{2}\Lambda^*\beta^2c+2g^*\right]\left\|w^{*,2}-w^{*,1}\right\|_{V_\kappa\times V_\kappa}.$$

We assumed that:

$$0<c<c_0=\left(2C_{\hat{\Lambda}}\beta^4\right)^{-\frac{1}{2}}\left[K_*\left[1+(\bar{h})^2\right]^{(-1)}-C_{\hat{\alpha}}\right]^{\frac{1}{2}}.$$

Thus,

$$\left\|T^{*,1}-T^{*,2}\right\|_{V_\kappa} \leqslant \left[2\sqrt{2}\bar{\Lambda}^*\beta^2c+2g^*\right]\left(c_0^2-c^2\right)^{-1}\left\|w^{*,2}-w^{*,1}\right\|_{V_\kappa\times V_\kappa}. \tag{4.27}$$

We have also the two inequalities

$$\begin{aligned}&\sum_{i=1}^{2}\int_\Omega \hat{\Lambda}\left(T^{*,1}\right)\frac{\partial w_i^{*,1}}{\partial\kappa}\frac{\partial\left(\hat{\varphi}_i-w_i^{*,1}\right)}{\partial\kappa}dsd\kappa+\hat{g}\int_\Omega\left(\left|\frac{\partial\hat{\varphi}}{\partial\kappa}\right|-\left|\frac{\partial w^{*,1}}{\partial\kappa}\right|\right)dsd\kappa\\&+\int_\omega \hat{k}\left(|\hat{\varphi}|-\left|w^{*,1}\right|\right)ds\geqslant\sum_{j=1}^{2}\left(\hat{f},\hat{\varphi}-w^{*,1}\right).\end{aligned} \tag{4.28}$$

$$\begin{aligned}&\sum_{i=1}^{2}\int_\Omega \hat{\Lambda}\left(T^{*,2}\right)\frac{\partial w_i^{*,2}}{\partial\kappa}\frac{\partial\left(\hat{\varphi}_i-w_i^{*,2}\right)}{\partial\kappa}dsd\kappa+\hat{g}\int_\Omega\left(\left|\frac{\partial\hat{\varphi}}{\partial\kappa}\right|-\left|\frac{\partial w^{*,2}}{\partial\kappa}\right|\right)dsd\kappa\\&+\int_\omega \hat{k}\left(|\hat{\varphi}|-\left|w^{*,2}\right|\right)ds\geqslant\sum_{j=1}^{2}\left(\hat{f},\hat{\varphi}-w^{*,2}\right).\end{aligned} \tag{4.29}$$

We can take $\hat{\varphi}=w^{*,2}$ in (4.28) and $\hat{\varphi}=w^{*,1}$ in (4.29). Hence, we obtain

$$\sum_{i=1}^{2}\int_\Omega\left(\hat{\Lambda}\left(T^{*,1}\right)\frac{\partial w_i^{*,1}}{\partial\kappa}\frac{\partial\left(\hat{\varphi}_i-w_i^{*,1}\right)}{\partial\kappa}+\hat{\Lambda}\left(T^{*,2}\right)\frac{\partial w_i^{*,2}}{\partial\kappa}\frac{\partial\left(\hat{\varphi}_i-w_i^{*,2}\right)}{\partial\kappa}\right)dsd\kappa\geqslant 0,$$

then

$$\begin{aligned}&\sum_{i=1}^{2}\int_\Omega\hat{\Lambda}\left(T^{*,1}\right)\left|\frac{\partial}{\partial\kappa}\left(w_i^{*,1}-w_i^{*,1}\right)\right|^2dsd\kappa\\&\leqslant\sum_{i=1}^{2}\int_\Omega[\hat{\Lambda}\left(T^{*,1}\right)-\hat{\Lambda}\left(T^{*,2}\right)]\frac{\partial w_i^{*,1}}{\partial\kappa}\frac{\partial\left(w_i^{*,1}-w_i^{*,2}\right)}{\partial\kappa}dsd\kappa.\end{aligned} \tag{4.30}$$

As: $\hat{\Lambda}\geqslant\Lambda_*>0$ and by Poincaré's inequality, we find

$$\sum_{i=1}^{2}\int_\Omega\hat{\Lambda}\left(T^{*,1}\right)\left|\frac{\partial}{\partial\kappa}\left(w_i^{*,1}-w_i^{*,1}\right)\right|^2dsd\kappa\geqslant\Lambda_*\left[1+(\bar{h})^2\right]^{-1}\left\|w^{*,2}-w^{*,1}\right\|_{V_\kappa}^2. \tag{4.31}$$

Now, the analogous results of [3], is given by

$$\begin{aligned}&\left|\sum_{i=1}^{2}\int_\Omega[\hat{\Lambda}\left(T^{*,1}\right)-\hat{\Lambda}\left(T^{*,2}\right)]\;\frac{\partial w_i^{*,1}}{\partial\kappa}\frac{\partial\left(w_i^{*,1}-w_i^{*,2}\right)}{\partial\kappa}dsd\kappa\right|\\&\leqslant\sqrt{2}\beta^2C_{\hat{\Lambda}}c\left\|T^{*,2}-T^{*,1}\right\|_{V_\kappa}\left\|w^{*,2}-w^{*,1}\right\|_{V_\kappa}.\end{aligned}$$

Where, $\beta > 0, C_{\hat{\Lambda}} > 0$ and $c > 0$ are respectively deduced from, the embedding of V_κ in $L^4(\omega)$, the assumption $\hat{\Lambda}$ is $C_{\hat{\Lambda}}$-Lipschitz continuous function on $\mathbb{R}$, and $w^{*,i} \in B_c$. Therefore

$$\left\|w^{*,2} - w^{*,1}\right\|_{V_\kappa \times V_\kappa} \leqslant \sqrt{2}\beta^2 C_{\hat{\Lambda}} \Lambda_*^{-1} \left[1 + (\bar{h})^2\right] c \left\|T^{*,1} - T^{*,2}\right\|_{V_\kappa}. \tag{4.32}$$

And from (4.27), we deduce

$$\left(1 - \left(2\sqrt{2}\bar{\Lambda}\beta^2 c + 2g^*\right)(c_0^2 - c^2)^{-1}\sqrt{2}\beta^2 C_{\hat{\Lambda}} \Lambda_*^{-1}\left[1 + \bar{(h)}^2\right] c\right)\left\|T^{*,1} - T^{*,2}\right\|_{V_\kappa} \leqslant 0.$$

Assuming that

$$\left(1 - \left(2\sqrt{2}\Lambda^* \beta^2 c + 2g^*\right)(c_0^2 - c^2)^{-1}\sqrt{2}\beta^2 C_{\hat{\Lambda}} \Lambda_*^{-1}\left[1 + \bar{(h)}^2\right] c\right) > 0.$$

We have

$$\left\|T^{*,1} - T^{*,2}\right\|_{V_\kappa} = 0.$$

Then $T^{*,1} = T^{*,2}$ a.e in V_κ. From (4.32), we conclure $w^{*,2} = w^{*,1}$ a.e on $V_\kappa \times V_\kappa$, of uniqueness (w^*, T^*) implies that of p^*.

□

Acknowledgments

The authors are highly grateful to the anonymous referee for his/her valuable comments and suggestions for the improvement of the paper.

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