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A study of the fuzzy multiplicative center in a special class of fuzzy near rings

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ABSTRACT: In this work, we describe the concept of the fuzzy multiplicative center of fuzzy near-rings which are defined from a binary operation and we make a theoretical study of their basic properties analogous to those of ordinary near-rings. The results obtained show that the domain of fuzzy near-rings is larger than that of classical near-rings.

Key Words: Prime near-rings, fuzzy group, fuzzy near-rings.

Contents

1. Introduction

In [13], Rosenfeld integrated fuzzy sets into the field of group theory, precisely he defined the notion of fuzzy subgroup of a group. Consequently, several studies have been conducted whose objective is to extend the notions of classical algebra to the fuzzy domain which is broader than the classical framework. In fact, a significant number of properties of classical group theory have been generalized, taking on meaning in the new framework (see [1,4,6,9,10]), where further references can be found. Also, other algebraic notions namely rings, fields, vector spaces are defined in the realm of fuzzy algebraic structures. It would therefore be interesting to find tools to study some of these more complicated fuzzy algebraic structures using fuzzy groups. In this sense, we cite the work of Liu [5] in which he integrated fuzzy sets in the field of ring theory, also the work of Yue [15], Mukherjee et al. [7], Dixit et al. [4] that they studied fuzzy rings. Thereby, some results similar to the cases of classical rings are shown. As for concepts of near-rings used here without mention, we consult for instance [2,3,11,12].
Recalling that the definition of fuzzy subgroups and fuzzy subring used by the researchers cited above is based on the fact that the subset of the group $\mathcal{G}$ and the ring $\mathcal{R}$ are fuzzy and that the binary operations on $\mathcal{R}$ are non-fuzzy in the classical sense. Clearly, this definition is different from another definition of which it is assumed that the set is non-fuzzy or classical and that the binary operation is fuzzy in the fuzzy sense. More recently, the concept of fuzzy group based on fuzzy binary operation has been introduced by Yuan and Lee in [14]. More in line with this latter approach, Aktaş and Çagman in [1] created a new kind of fuzzy ring based on Yuan and Lee's definition of fuzzy group.
Being motivated by their invaluable research, in [12] we was defined the concept of fuzzy near-ring and fuzzy multiplicative center as follows:

Definition 1.1 *for any nonempty set $\mathcal{X}$ with two fuzzy binary operations $\mathcal{T}$ and $\mathcal{L}$ is said fuzzy left near-ring if the following assertions hold:*

i) $(\mathcal{X}, \mathcal{T})$ *is a fuzzy group not necessarily commutative,*

ii) $\forall a, b, c, x_1, x_2 \in \mathcal{X}$, *we have* $(a * (b * c))(x_1) > \theta$ *and* $((a * b) * c)(x_2) > \theta \implies x_1 = x_2$,

iii) $\forall a, b, c, x_1, x_2 \in \mathcal{X}$, *we have* $(a * (b \circ c))(x_1) > \theta$ *and* $((a * b) \circ (a * c))(x_2) > \theta \implies x_1 = x_2$.

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Further, if we replace the last condition by:

$$\forall a, b, c, x_1, x_2 \in \mathcal{X},\ (b \circ c) * a)(x_1) > \theta \text{ and } ((b * a) \circ (c * a))(x_2) > \theta \implies x_1 = x_2,$$

then $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ *is called right fuzzy near-ring.*

Noting that $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ is called a prime fuzzy near-ring, if it has the property that $((x * y) * z)(e) > \theta$ for all $x, y, z \in \mathcal{X}$ implies that $x = e$ or $z = e$. Also,

$$Z_F(\mathcal{X}) = \{x \in \mathcal{X} \,/\, \mathcal{L}(x, y, z) > \theta \iff \mathcal{L}(y, x, z) > \theta,\ \forall y, z \in \mathcal{X}\}$$

denote the fuzzy multiplicative center of $\mathcal{X}$.
Our purpose in this work, is to continue to investigate in this line using Yuan and Lee's fuzzy group. Precisely, we use the definition of the fuzzy near-ring to get some fundamental results related to fuzzy multiplicative center.

2. Preliminary

In this section, we introduce some basic definitions and results which will be necessary in the rest of our paper. Note that the proofs of different results cited in Proposition 2.2 are developing by authors in [12, Theorem 1 & Theorem 2], respectively.

Definition 2.1 [14, Definition 2.2] *Let* $\mathcal{X}$ *be a nonempty set and* $\mathcal{T}$ *be a fuzzy subset of* $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ *and* $\theta \in [0, 1)$ *is a fixed number.* $\mathcal{T}$ *is called a fuzzy binary operation on* $\mathcal{X}$ *if the following conditions hold.*

(C_1) $\forall x, y \in \mathcal{X}, \exists z \in \mathcal{X}$ *such that* $\mathcal{T}(x, y, z) > \theta$.

(C_2) $\forall x, y, t_1, t_2 \in \mathcal{X},\ \mathcal{T}(x, y, t_1) > \theta$ *and* $\mathcal{T}(x, y, t_2) > \theta$ *implies* $t_1 = t_2$.

Let $\mathcal{T}$ and $\mathcal{L}$ be two fuzzy binary operations on $\mathcal{X}$, then we have the following mappings:

$$\begin{array}{rcl} \circ : \mathbb{F}(\mathcal{X}) \times \mathbb{F}(\mathcal{X}) & \longrightarrow & \mathbb{F}(\mathcal{X}) \\ (\mu, v) & \longmapsto & \mu \circ v \end{array} \quad \text{and} \quad \begin{array}{rcl} * : \mathbb{F}(\mathcal{X}) \times \mathbb{F}(\mathcal{X}) & \longrightarrow & \mathbb{F}(\mathcal{X}) \\ (\mu, v) & \longmapsto & \mu * v \end{array},$$

such that

$$\mathbb{F}(\mathcal{X}) = \{\mu \,/\, \mu : \mathcal{X} \longrightarrow [0, 1]\}$$

and

$$\begin{cases} \forall \mu, v \in \mathbb{F}(\mathcal{X}),\ (\mu \circ v)(z) = \bigvee_{x,y \in \mathcal{X}} \Big(\mu(x) \wedge v(y) \wedge \mathcal{T}(x, y, z)\Big), \\ \forall \mu, v \in \mathbb{F}(\mathcal{X}),\ (\mu * v)(z) = \bigvee_{x,y \in \mathcal{X}} \Big(\mu(x) \wedge v(y) \wedge \mathcal{L}(x, y, z)\Big). \end{cases}$$

Let $x, y \in \mathcal{X}$, $\mu = \{x\}$ and $v = \{y\}$, and let $\mu \circ v$ and $\mu * v$ be denoted by $x \circ y$ and $x * y$, respectively. Hence, the following assertions are verified

$$(x \circ y)(z) = \mathcal{T}(x, y, z)\ \forall z \in \mathcal{X},$$

$$(x * y)(z) = \mathcal{L}(x, y, z)\ \forall z \in \mathcal{X},$$

$$((x \circ y) \circ z)(t) = \bigvee_{h \in \mathcal{X}} \Big(\mathcal{T}(x, y, h) \wedge \mathcal{T}(h, z, t)\Big)\ \forall z, t \in \mathcal{X},$$

$$(x \circ (y \circ z))(t) = \bigvee_{h \in \mathcal{X}} \Big(\mathcal{T}(y, z, h) \wedge \mathcal{T}(x, h, t)\Big)\ \forall z, t \in \mathcal{X},$$

$$((x * y) * z)(t) = \bigvee_{h \in \mathcal{X}} \Big(\mathcal{L}(x, y, h) \wedge \mathcal{L}(h, z, t)\Big)\ \forall z, t \in \mathcal{X},$$

$$(x * (y * z))(t) = \bigvee_{h \in \mathcal{X}} \Big(\mathcal{L}(y, z, h) \wedge \mathcal{L}(x, h, t)\Big)\ \forall z, t \in \mathcal{X},$$

$$(x * (y \circ z))(t) = \bigvee_{h \in \mathcal{X}} \Big(\mathcal{T}(y, z, h) \wedge \mathcal{L}(x, h, t)\Big)\ \forall z, t \in \mathcal{X},$$

$$((x * y) \circ (x * z))(t) = \bigvee_{d,r \in \mathcal{X}} \Big(\mathcal{L}(x, y, d) \wedge \mathcal{L}(x, z, r) \wedge \mathcal{T}(d, r, t)\Big)\ \forall z, t \in \mathcal{X}.$$

Definition 2.2 [14, Definition 2.3] *Let $\mathcal{X}$ be a nonempty set and $\mathcal{T}$ a fuzzy binary operation on $\mathcal{X}$. $(\mathcal{X}, \mathcal{T})$ is called a fuzzy group if:*

(C_1) $\forall\, x, y, z, t_1, t_2 \in \mathcal{X}$, $((x \circ y) \circ z)(t_1) > \theta$ *and* $(x \circ (y \circ z))(t_2) > \theta$ *implies* $t_1 = t_2$,

(C_2) $\exists\, e_{\mathcal{T}} \in \mathcal{X}$ *such that* $(e_{\mathcal{T}} \circ x)(x) > \theta$ *and* $(x \circ e_{\mathcal{T}})(x) > \theta \;\forall\, x \in \mathcal{X}$,
$e_{\mathcal{T}}$ *is called the identity element of* $(\mathcal{X}, \mathcal{T})$ *and it's unique.*

(C_3) $\forall\, x \in \mathcal{X}$, $\exists y \in \mathcal{X}$ *such that* $(x\ o\ y)(e_{\mathcal{T}}) > \theta$ *and* $(y\ o\ x)(e_{\mathcal{T}}) > \theta$,
y is called the inverse element of x and denoted by x^{-1} and it's unique.

Proposition 2.1 [14, Proposition 2.1 (4)] *Let $(\mathcal{X}, \mathcal{T})$ be a fuzzy group, then*

$$(a \circ x)(y) > \theta \text{ and } (b \circ x)(y) > \theta \text{ implies } a = b.$$

Definition 2.3 [1, Definition 6] *Let $(\mathcal{X}, \mathcal{T})$ be a fuzzy group. $(\mathcal{X}, \mathcal{T})$ is called abelian fuzzy group if*

$$\mathcal{T}(x, y, z) > \theta \iff \mathcal{T}(y, x, z) > \theta \text{ for all } x, y, z \in \mathcal{X}.$$

Proposition 2.2 *i) Let $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ be a left fuzzy near-ring. Then,*
$forall x, y, z, t \in \mathcal{X},\ (x * (y \circ z))(t) > \theta \iff ((x * y) \circ (x * z))(t) > \theta.$

ii) Let $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ be a right fuzzy near-ring. Then,
$\forall x, y, z, t \in \mathcal{X},\ ((y \circ z) * x)(t) > \theta \iff ((y * x) \circ (z * x))(t) > \theta.$

iii) Let $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ be a fuzzy near-ring. Then,
$\forall x, y, t, z \in G,\ (x * (y * z))(t) > \theta \iff ((x * y) * z)(t) > \theta.$

3. Main results

This section is devoted to study the fuzzy multiplicative near-ring of a left fuzzy near-rings (respe. right fuzzy near-rings). According to Definition 2.1 (C_1), $\forall\ x, y \in \mathcal{X}$, $\exists z \in \mathcal{X}$ such that $\mathcal{T}(x, y, z) > \theta$; in what follows, we will use frequently this property without mentioning it each time. We begin with the following Lemma which is essential to prove our theorems.

Lemma 3.1 *Let $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ be a left fuzzy near-ring. Then,*

$$\forall x, y \in \mathcal{X},\ ((x * y) \circ (x * y^{-1}))(e) > \theta \quad \text{and} \quad ((x * y^{-1}) \circ (x * y))(e) > \theta.$$

Proof: Let $x, y \in \mathcal{X}$. In view of the property (i) in Proposition 2.2, we have

$$(x * (y \circ y^{-1}))(e) > \theta \iff ((x * y) \circ (x * y^{-1}))(e) > \theta.$$

Since,

$$\begin{aligned} (x * (y \circ y^{-1}))(e) &= \bigvee_{d \in \mathcal{X}} \Big(\mathcal{T}(y, y^{-1}, d) \wedge \mathcal{L}(x, d, e) \Big) \\ &\geq \mathcal{T}(y, y^{-1}, e) \wedge \mathcal{L}(x, e, e) \end{aligned}$$

and $\mathcal{L}(x, e, e) > \theta$, the previous equivalence gives $((x * y) \circ (x * y^{-1}))(e) > \theta$.
Similarly, we show that $((x * y^{-1}) \circ (x * y))(e) > \theta$ by checking that $(x * (y^{-1} \circ y))(e) > \theta$. □

Theorem 3.1 *Let $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ be a left fuzzy near-ring. Then,*

$$Z_F(\mathcal{X}) = \{x \in \mathcal{X} /\ \forall y \in \mathcal{X}, ((x * y) \circ (y * x^{-1}))(e) > \theta\}.$$

Proof: We suppose that $x \in Z_F(\mathcal{X})$ and prove that $((x*y)\circ(y*x^{-1}))(e) > \theta$ for all $y \in \mathcal{X}$. In fact, for each $y \in \mathcal{X}$, we have

$$((x*y)\circ(y*x^{-1}))(e) = \bigvee_{d_1,d_2\in\mathcal{X}}\Big(\mathcal{L}(y,x^{-1},d_2)\wedge\mathcal{L}(x,y,d_1)\wedge\mathcal{T}(d_1,d_2,e)\Big).$$

Let $t_1, t_2 \in \mathcal{X}$ such that $\mathcal{L}(x,y,t_1) > \theta$ and $\mathcal{L}(y,x^{-1},t_2) > \theta$ and using the fact that $x \in Z_F(\mathcal{X})$, we get $\mathcal{L}(y,x,t_1) > \theta$. From Lemma 3.1, we have $((y*x)\circ(y*x^{-1}))(e) > \theta$. But, since $((y*x)\circ(y*x^{-1}))(e) = \bigvee_{d_1,d_2\in\mathcal{X}}\Big(\mathcal{L}(y,x,d_1)\wedge\mathcal{L}(y,x^{-1},d_2)\wedge\mathcal{T}(d_1,d_2,e)\Big)$ then $\mathcal{T}(t_1,t_2,e) > \theta$. Consequently,

$$((x*y)\circ(y*x^{-1}))(e) > \theta.$$

Now, consider $x \in \mathcal{X}$ and suppose that $\forall\, r \in \mathcal{X}, ((x*r)\circ(r*x^{-1}))(e) > \theta$. Our main to prove that $x \in Z_F(\mathcal{X})$. Indeed, let $y, z \in \mathcal{X}$ and prove that $\mathcal{L}(x,y,z) > \theta \iff \mathcal{L}(y,x,z) > \theta$. Firstly, show that $\mathcal{L}(x,y,z) > \theta \Longrightarrow \mathcal{L}(y,x,z) > \theta$.
In view of condition (C_1) in Definition 2.1, there exist $t, t_1 \in \mathcal{X}$ verify $\mathcal{L}(y,x,t) > \theta$ and $\mathcal{L}(y,x^{-1},t_1) > \theta$. Taking $r = y$ in our assumption, we obtain

$$((x*y)\circ(y*x^{-1}))(e) > \theta. \tag{3.1}$$

Using (3.1) and taking account $\mathcal{L}(x,y,z) > \theta$ and $\mathcal{L}(y,x^{-1},t_1) > \theta$, we can see that

$$\mathcal{T}(z,t_1,e) > \theta. \tag{3.2}$$

Also, from Lemma 3.1 we have $((y*x)\circ(y*x^{-1}))(e) > \theta$; taking account $\mathcal{L}(y,x,t) > \theta$ and $\mathcal{L}(y,x^{-1},t_1) > \theta$, we conclude that

$$\mathcal{T}(t,t_1,e) > \theta. \tag{3.3}$$

Applying Proposition 2.1 to the two relations 3.2 and 3.3, we get $t = z$ and hence $\mathcal{L}(y,x,z) > \theta$.
Secondly, suppose that $\mathcal{L}(y,x,z) > \theta$ and sheck that $\mathcal{L}(x,y,z) > \theta$. For this purpose, let $t, t_1 \in \mathcal{X}$ such that $\mathcal{L}(x,y,t) > \theta$ and $\mathcal{L}(y,x^{-1},t_1) > \theta$. Since $((x*y)\circ(y*x^{-1}))(e) > \theta$, we find that

$$\mathcal{T}(t,t_1,e) > \theta. \tag{3.4}$$

According to Lemma 3.1, we have $((y*x)\circ(y*x^{-1}))(e) > \theta$, then since $\mathcal{L}(y,x,z) > \theta$ and $\mathcal{L}(y,x^{-1},t_1) > \theta$, we obtain

$$\mathcal{T}(z,t_1,e) > \theta. \tag{3.5}$$

In view of Proposition 2.1, the relations (3.4) and (3.5) give $t = z$ and consequently, $\mathcal{L}(x,y,z) > \theta$. □

Theorem 3.2 *Let $(\mathcal{X},\mathcal{T},\mathcal{L})$ be a left fuzzy near-ring. Then,*

$$Z_F(\mathcal{X}) = \{x \in \mathcal{X} /\ \forall y \in \mathcal{X}, ((y*x)\circ(x*y^{-1}))(e) > \theta\}.$$

Proof: By using Lemma 3.1 and similar arguments as in the Theorem 3.1, we obtain the required result. □

Theorem 3.3 *Let $(\mathcal{X},\mathcal{T},\mathcal{L})$ be a right fuzzy near-ring. Then,*

$$Z_F(\mathcal{X}) = \{x \in \mathcal{X} /\ \forall y \in \mathcal{X}, ((x*y)\circ(y^{-1}*x))(e) > \theta\}.$$

In order to give the proof of Theorem 3.3, we need the following Lemma:

Lemma 3.2 *Let $(\mathcal{X},\mathcal{T},\mathcal{L})$ be a right fuzzy near-ring. Then,*

$$\forall x, y \in \mathcal{X},\ ((y*x)\circ(y^{-1}*x))(e) > \theta \ \text{ and }\ ((y^{-1}*x)\circ(y*x))(e) > \theta.$$

Proof: Let $x, y \in \mathcal{X}$; in view of the property (ii) in Proposition 2.2, we have

$$((y \circ y^{-1}) * x))(e) > \theta \iff ((y * x) \circ (y^{-1} * x)(e) > \theta.$$

From

$$\begin{aligned} ((y \circ y^{-1}) * x)(e) &= \bigvee_{d \in \mathcal{X}} \Big(\mathcal{T}(y, y^{-1}, d) \wedge \mathcal{L}(d, x, e) \Big) \\ &\geq \mathcal{T}(y, y^{-1}, e) \wedge \mathcal{L}(e, x, e) \end{aligned}$$

and the fact that $\mathcal{L}(e, x, e) > \theta$, the previous equivalence shows that $((y * x) \circ (y^{-1} * x))(e) > \theta$. Similarly, we get $((y^{-1} * x) \circ (y * x))(e) > \theta$ by checking that $((y^{-1} \circ y) * x))(e) > \theta$. □

Proof of Theorem 3.3:
Consider $x \in Z_F(\mathcal{X})$ and show that $\forall y \in \mathcal{X}, ((x * y) \circ (y^{-1} * x))(e) > \theta$. In fact, for each $y \in \mathcal{X}$, we have

$$((x * y) \circ (y^{-1} * x))(e) = \bigvee_{d_1, d_2 \in \mathcal{X}} \Big(\mathcal{L}(x, y, d_1) \wedge \mathcal{L}(y^{-1}, x, d_2) \wedge \mathcal{T}(d_1, d_2, e) \Big).$$

Let $t_1, t_2 \in \mathcal{X}$ satisfying $\mathcal{L}(x, y, t_1) > \theta$ and $\mathcal{L}(y^{-1}, x, t_2) > \theta$ and using the fact $x \in Z_F(\mathcal{X})$, we conclude that $\mathcal{L}(y, x, t_1) > \theta$. On the other hand, Lemma 3.2 gives $((y * x) \circ (y^{-1} * x))(e) > \theta$, this yields $\mathcal{T}(t_1, t_2, e) > \theta$. Consequently,

$$((x * y) \circ (y^{-1} * x))(e) > \theta.$$

Conversely, suppose that $\forall\, k \in \mathcal{X}, ((x * k) \circ (k^{-1} * x))(e) > \theta$ and showing that $x \in Z_F(\mathcal{X})$. Let $y, z \in \mathcal{X}$ such that $\mathcal{L}(x, y, z) > \theta$ and prove that $\mathcal{L}(y, x, z) > \theta$.
Consider $t, t_1 \in \mathcal{X}$ verified $\mathcal{L}(y, x, t) > \theta$ and $\mathcal{L}(y^{-1}, x, t_1) > \theta$, and by our step assumption, we have $((x * y) \circ (y^{-1} * x))(e) > \theta$. It follows that

$$\mathcal{T}(z, t_1, e) > \theta. \tag{3.6}$$

In view of Lemma 3.2, we have $((y * x) \circ (y^{-1} * x))(e) > \theta$, which implies that

$$\mathcal{T}(t, t_1, e) > \theta. \tag{3.7}$$

From (3.6) and (3.7), because of Proposition 2.1, we find that $t = z$ and therefore $\mathcal{L}(y, x, z) > \theta$.
Now, we suppose that $\mathcal{L}(y, x, z) > \theta$ and check that $\mathcal{L}(x, y, z) > \theta$. For this purpose, taking $t, t_1 \in \mathcal{X}$ such that $\mathcal{L}(x, y, t) > \theta$ and $\mathcal{L}(y, x^{-1}, t_1) > \theta$ and using the fact that $((x * y) \circ (y^{-1} * x))(e) > \theta$, we arrive at

$$\mathcal{T}(t, t_1, e) > \theta. \tag{3.8}$$

Once again, by Lemma 3.2, we have $((y * x) \circ (y^{-1} * x))(e) > \theta$, which implies that

$$\mathcal{T}(z, t_1, e) > \theta. \tag{3.9}$$

Invoking Proposition 2.1 and using (3.8) and (3.9), we conclude that $t = z$; and hence $\mathcal{L}(x, y, z) > \theta$.

Theorem 3.4 *Let $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ be a right fuzzy near-ring. Then,*

$$Z_F(\mathcal{X}) = \{x \in \mathcal{X} /\ \forall y \in \mathcal{X}, ((y * x) \circ (x^{-1} * y))(e) > \theta\}.$$

Proof: we prove the theorem by using Lemma 3.2 and similar proof as used in the previous theorem. □

Proposition 3.1 *Let $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ be a left fuzzy near-ring. Then,*

$$\forall x, y, z \in \mathcal{X}, \quad \big(((x * y) * z) \circ ((x * y) * z^{-1})\big)(e) > \theta.$$

Proof: Let $x, y, z \in \mathcal{X}$, we have

$$\big(((x*y)*z)\circ((x*y)*z^{-1})\big)(e) = \bigvee_{d_1,d_2\in\mathcal{X}}\Big(((x*y)*z)(d_1)\wedge((x*y)*z^{-1})(d_2)\wedge\mathcal{T}(d_1,d_2,e)\Big).$$

There exists $t_1, h_1, h_2 \in \mathcal{X}$ satisfying $\mathcal{L}(x,y,t_1) > \theta$, $\mathcal{L}(t_1,z,h_1) > \theta$ and $\mathcal{L}(t_1,z^{-1},h_2) > \theta$, which implies that $((x*y)*z)(h_1) > \theta$ and $((x*y)*z^{-1})(h_2) > \theta$. From Lemma 3.1, we have $((t_1*z)\circ(t_1*z^{-1}))(e) > \theta$. Hence, $\mathcal{T}(h_1,h_2,e) > \theta$ and therefore

$$\big(((x*y)*z)\circ((x*y)*z^{-1})\big)(e) > \theta.$$

□

Proposition 3.2 *Let $(\mathcal{X},\mathcal{T},\mathcal{L})$ be a right fuzzy near-ring. Then,*

$$\forall x,y,z\in\mathcal{X},\quad \big((x*(y*z))\circ(x^{-1}*(y*z))\big)(e) > \theta.$$

Proof: Let $x, y, z \in \mathcal{X}$. By the condition (C_1) in Definition 2.1, there exist $t_1, h_1, h_2 \in \mathcal{X}$ such that $\mathcal{L}(y,z,t_1) > \theta$, $\mathcal{L}(x,t_1,h_1) > \theta$ and $\mathcal{L}(x^{-1},t_1,h_2) > \theta$ which allowed us to deduce that $(x*(y*z))(h_1) > \theta$ and $(x^{-1}*(y*z))(h_2) > \theta$. Also, as an application of Lemma 3.2, we have $((x*t_1)\circ(x^{-1}*t_1))(e) > \theta$, that is, $\mathcal{T}(h_1,h_2,e) > \theta$. Accordingly,

$$\begin{aligned}\big((x*(y*z))\circ(x^{-1}*(y*z))\big)(e) &= \bigvee_{d_1,d_2\in\mathcal{X}}\Big((x*(y*z))(d_1)\wedge(x^{-1}*(y*z))(d_2)\wedge\mathcal{T}(d_1,d_2,e)\Big)\\ &\geq ((x*(y*z))(h_1)\wedge(x^{-1}*(y*z))(h_2)\wedge\mathcal{T}(h_1,h_2,e)\\ &> \theta.\end{aligned}$$

□

As an application of the Propositions 3.1 and 3.2, we get the following result.

Theorem 3.5 *Let $(\mathcal{X},\mathcal{T},\mathcal{L})$ be a prime left fuzzy near-ring (resp. prime right fuzzy near-ring) and let $x\in Z_F(\mathcal{X})^*$. If $\forall\, t\in\mathcal{X}$, $\exists h\in\mathcal{X}$ such that $((x*z)*t)(h) > \theta$ and $(t*(x*z))(h) > \theta$, then $z\in Z_F(\mathcal{X})$.*

To develop our theorem, we start by stating the following two lemmas as follows:

Lemma 3.3 *Let $(\mathcal{X},\mathcal{T},\mathcal{L})$ be a left fuzzy near-ring and $x,z\in\mathcal{X}$. If $\forall a\in\mathcal{X}$, $\exists h\in\mathcal{X}$ such that $((x*z)*a)(h) > \theta$ and $(a*(x*z))(h) > \theta$, then $\forall\, t\in\mathcal{X}$ $\big(((x*z)*t)\circ(t*(x*z^{-1}))\big)(e) > \theta$.*

Proof: Let $t\in\mathcal{X}$. By hypothesis, there exists $h_1\in\mathcal{X}$ such that $((x*z)*t)(h_1) > \theta$ and $(t*(x*z))(h_1) > \theta$. Taking $h_2\in\mathcal{X}$ satifying $(t*(x*z^{-1}))(h_2) > \theta$ and combining Proposition 3.1 and Proposition 2.2(iii), we obtain

$$\big((t*(x*z))\circ(t*(x*z^{-1}))\big)(e) > \theta$$

which implies that $\mathcal{T}(h_1,h_2,e)$. And therefore,

$$\begin{aligned}\big(((x*z)*t)\circ(t*(x*z^{-1}))\big)(e) &\geq ((x*z)*t)(h_1)\wedge(t*(x*z^{-1}))(h_2)\wedge\mathcal{T}(h_1,h_2,e)\\ &> \theta.\end{aligned}$$

□

Lemma 3.4 *Let $(\mathcal{X},\mathcal{T},\mathcal{L})$ be a right fuzzy near-ring and $x,z\in\mathcal{X}$. If $\forall\, a\in\mathcal{X}$, $\exists h\in\mathcal{X}$ such that $((x*z)*a))(h) > \theta$ and $(a*(x*z)))(h) > \theta$, then $\forall t\in\mathcal{X}$ $\big(((x*z)*t)\circ((t^{-1}*(x*z)\big)(e) > \theta$.*

Proof: Exchange Proposition 3.1 by Proposition 3.2 in the proof of Lemma 3.3 and using the similar arguments, we get the required result. $\square$

Proof of Theorem 3.5:
• Firstly, assume that $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ is a left fuzzy near-ring and let $x, z \in \mathcal{X}$ such that $x \in Z_F(\mathcal{X})^*$ and $\forall t \in \mathcal{X}$, $\exists h \in \mathcal{X}$ satisfying the both conditions $((x*z)*t)(h) > \theta$ and $(t*(x*z))(h) > \theta$. Our goal is to show that $z \in Z_F(\mathcal{X})$ by showing the result of Theorem 3.1.i.e, $\forall t \in \mathcal{X}, ((z*t) \circ (t*z^{-1}))(e) > \theta$.
Let t be an arbitrary element in $\mathcal{X}$; they exist $h_1, h_2, h \in \mathcal{X}$ satisfy $\mathcal{L}(z, t, h_1) > \theta$, $\mathcal{L}(t, z^{-1}, h_2) > \theta$ and $\mathcal{T}(h_1, h_2, h) > \theta$ which implies that

$$((z*t) \circ (t*z^{-1}))(h) > \theta. \tag{3.10}$$

Since $t \in \mathcal{X}$, then Lemma 3.3 gives $\big(((x*z)*t) \circ (t*(x*z^{-1}))\big)(e) > \theta$, which means that there are $a, b \in \mathcal{X}$ verify $((x*z)*t)(a) > \theta$, $(t*(x*z^{-1}))(b) > \theta$ and $\mathcal{T}(a, b, e) > \theta$. Since Proposition 2.2(iii) expresses the associative property of " $*$ ", then

$$((x*z)*t)(a) > \theta \iff (x*(z*t))(a) > \theta \tag{3.11}$$

and

$$(t*(x*z^{-1}))(b) > \theta \iff ((t*x)*z^{-1})(b) > \theta. \tag{3.12}$$

On the other hand, in virtue of $x \in Z_F(\mathcal{X})^*$, we have

$$((t*x)*z^{-1})(b) > \theta \iff ((x*t)*z^{-1})(b) > \theta \iff (x*(t*z^{-1})(b) > \theta. \tag{3.13}$$

From equations (3.12) and (3.13), it follows that

$$(t*(x*z^{-1}))(b) > \theta \iff (x*(t*z^{-1})(b) > \theta. \tag{3.14}$$

Combining (3.11) and (3.14) and as $\mathcal{T}(a, b, e) > \theta$, we infer that

$$\big((x*(z*t)) \circ (x*(t*z^{-1}))\big)(e) > \theta. \tag{3.15}$$

Now, taking $t_1, t_2 \in \mathcal{X}$ satisfy $\mathcal{L}(x, h_1, t_1) > \theta$ and $\mathcal{L}(x, h_2, t_2) > \theta$. Thus $(x*(t*z^{-1}))(t_2) > \theta$ and $(x*(z*t))(t_1) > \theta$ which, in view of (3.15), imply that $\mathcal{T}(t_1, t_2, e) > \theta$.
Since,

$$((x*h_1) \circ (x*h_2))(e) \geq \mathcal{L}(x, h_1, t_1) \wedge \mathcal{L}(x, h_2, t_2) \wedge \mathcal{T}(t_1, t_2, e) > \theta$$

then, by using Proposition 2.2(i), we obtain $(x*(h_1 \circ h_2))(e) = \bigvee_{d \in \mathcal{X}} \Big((\mathcal{L}(x, d, e) \wedge \mathcal{T}(h_1, h_2, d) \Big) > \theta$. As $\mathcal{T}(h_1, h_2, h) > \theta$, the preceding result shows that $\mathcal{L}(x, h, e) > \theta$.
Let $s, k \in \mathcal{X}$ satisfying $\mathcal{L}(t, x, s) > \theta$ and $\mathcal{L}(s, h, k) > \theta$ and check that $k = e$. For this, it suffices to show that $\mathcal{L}(s, h, e) > \theta$.
We have $((t*x)*h)(k) \geq \mathcal{L}(t, x, s) \wedge \mathcal{L}(s, h, k) > \theta$ and $(t*(x*h))(e) \geq \mathcal{L}(x, h, e) \wedge \mathcal{L}(t, e, e) > \theta$ which forces $\mathcal{L}(x, h, e) > \theta$. Taking account $\mathcal{L}(s, h, k) > \theta$ and applying (C_2) of Definition 1.1, we obtain $k = e$. Consequently, $\mathcal{T}(s, h, e) > \theta$. Since, $x \in Z_F(\mathcal{X})^*$ and $\mathcal{L}(t, x, s) > \theta$, then $\mathcal{L}(t, x, s) > \theta$, which implies that,

$$((x*t)*h)(e) \geq \mathcal{L}(x, t, s) \wedge \mathcal{L}(s, h, e) > \theta.$$

By the primeness of $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ and the fact that $x \in Z_F(\mathcal{X})^*$, we conclude that $h = e$ and therefore (3.10) reduces to $((z*t) \circ (t*z^{-1}))(e) > \theta$ for all $t \in \mathcal{X}$, then $z \in Z_F(\mathcal{X})$ by Theorem 3.1.
• Secondly, suppose that $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ is a right fuzzy near-ring and let $x \in Z_F(\mathcal{X})^*$. By hypothesis given, we have

$$\forall t \in \mathcal{X},\ \exists h \in \mathcal{X} \text{ such that } ((x*z)*t)(h) > \theta \text{ and } (t*(x*z))(h) > \theta.$$

Prove that $z \in Z_F(\mathcal{X})$; by applying Theorem 3.3, it suffices to show that $((z*t)\circ(t^{-1}*z))(e) > \theta$ for all $t \in \mathcal{X}$.
Indeed, let $t \in \mathcal{X}$ and using the hypothesis of the statement together Lemma 3.4, we arrive at $\big(((x*z)*t)\circ(t^{-1}*(x*z))\big)(e) > \theta$ which implies the existence of $a, b \in \mathcal{X}$ such that $((x*z)*t)(a) > \theta$, $(t^{-1}*(x*z))(b) > \theta$ and $\mathcal{T}(a,b,e) > \theta$. Using the fact that $x \in Z_F(\mathcal{X})^*$ and by Proposition 2.2(iii), we get the following equivalences

$$((x*z)*t)(a) > \theta \iff ((z*x)*t)(a) > \theta \iff (z*(x*t))(a) > \theta \iff ((z*t)*x)(a) > \theta. \quad (3.16)$$

Similarly, we get

$$(t^{-1}*(x*z))(b) > \theta \iff ((t^{-1}*z)*x)(b) > \theta. \quad (3.17)$$

In the light of (3.16), (3.17) and $\mathcal{T}(a,b,e) > \theta$, we conclude that

$$\big(((z*t)*x)\circ((t^{-1}*z)*x)\big)(e) > \theta. \quad (3.18)$$

Also, there exist $h_1, h_2, h \in \mathcal{X}$ satisfying $\mathcal{L}(z,t,h_1) > \theta$, $\mathcal{L}(t^{-1},z,h_2) > \theta$ and $\mathcal{T}(h_1,h_2,h) > \theta$, which implies that

$$((z*t)\circ(t^{-1}*z))(h) > \theta. \quad (3.19)$$

Now, we consider $t_1, t_2 \in \mathcal{X}$ such that $\mathcal{L}(h_1,x,t_1) > \theta$ and $\mathcal{L}(h_2,x,t_2) > \theta$. Thus, $((z*t)*x))(t_1) > \theta$ and $((t^{-1}*z)*x)(t_2) > \theta$. Therefore, $\mathcal{T}(t_1,t_2,e) > \theta$ from (3.18).
As

$$((h_1*x)\circ(h_2*x))(e) \geq \mathcal{L}(h_1,x,t_1) \wedge \mathcal{L}(h_2,x,t_2) \wedge \mathcal{T}(t_1,t_2,e) > \theta,$$

then in view of Proposition 2.2(ii), the preceding relation gives $((h_1 \circ h_2)*x)(e) > \theta$ and thus $\mathcal{L}(h,x,e) > \theta$.
On the other hand, let $s, k \in \mathcal{X}$ satisfying $\mathcal{L}(x,t,s) > \theta$ and $\mathcal{L}(h,s,k) > \theta$.
We have $(h*(x*t))(k) \geq \mathcal{L}(x,t,s) \wedge \mathcal{L}(h,s,k) > \theta$ and $((h*x)*t))(e) \geq \mathcal{L}(h,x,e) \wedge \mathcal{L}(e,t,e) > \theta$.
In virtue of the condition (C_2) in Definition 1.1, we conclude that $k = e$, that is, $\mathcal{L}(h,s,e) > \theta$. Since, $x \in Z_F(\mathcal{X})^*$ and $\mathcal{L}(x,t,s) > \theta$, then $\mathcal{L}(t,x,s) > \theta$. Hence,

$$(h*(t*x))(e) \geq \mathcal{L}(t,x,s) \wedge \mathcal{L}(h,s,e) > \theta.$$

By the primeness of $(\mathcal{X}, \mathcal{T}, \mathcal{L})$ and the fact that $x \in Z_F(\mathcal{X})^*$, we obtain $h = e$. Thus, (3.19) yields $((z*t)\circ(t^{-1}*z))(e) > \theta$ and hence $z \in Z_F(\mathcal{X})$ by Theorem 3.3. This completes the proof of our theorem.

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