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On Some New Eisenstein Series Identities Involving the Broweins' Cubic Theta Function $a(q)$ and the Convolution Sum $\sum_{i+3j=m} \sigma(i)\sigma(j)$

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ABSTRACT: In this paper, employing the (p, k)-parametrization of Srinivasa Ramanujan's Eisenstein series $L(q), M(q)$ and the Borweins' cubic theta functions due to Alaca et. al [1,2,3], we derive some new Eisenstein series identities involving the Borweins' cubic theta function $a(q)$ with the help of Computer. Further, as an application of these, we deduce the convolution sum $\sum_{i+3j=m} \sigma(i)\sigma(j)$.

Key Words: Theta functions, Eisenstein series, Divisor function, Convolution sum.

Contents

1. Introduction

Throughout this paper, we assume that $q \in \mathbb{C}$ with $|q| < 1$.

In their papers [13,14] on cubic modular identities of Srinivasa Ramanujan, J. M. Borwein and P. M. Borwein introduced three functions $a(q), b(q)$ and $c(q)$ defined by:

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}$$

and

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{\left(m+\frac{1}{3}\right)^2+\left(m+\frac{1}{3}\right)\left(n+\frac{1}{3}\right)+\left(n+\frac{1}{3}\right)^2},$$

where $\omega = e^{\frac{2\pi i}{3}}$, which are now famously known as the Borweins' cubic theta functions. These in terms of infinite products is given by [13,14]:

$$b(q) = \frac{f_1^3}{f_3} \qquad \text{and} \qquad c(q) = 3q^{\frac{1}{3}} \frac{f_3^3}{f_1},$$

where

$$f_n := (q^n; q^n)_\infty \qquad \text{with} \qquad (a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n).$$

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In [13], Borwein's gave the following [32, p. 93] Eisenstein series representation for $a(q)$:

$$a(q)=1+6\sum_{m=0}^{\infty}\left(\frac{q^{3m+1}}{1-q^{3m+1}}-\frac{q^{3m+2}}{1-q^{3m+2}}\right)$$

and established the identity

$$a(q)=\varphi(q)\varphi(q^3)+4q\psi(q^2)\psi(q^6), \tag{1.1}$$

where

$$\varphi(q):=\sum_{m=-\infty}^{\infty}q^{m^2}=\frac{f_2^5}{f_1^2f_4^4} \qquad \text{and} \qquad \psi(q):=\sum_{m=0}^{\infty}q^{\frac{m(m+1)}{2}}=\frac{f_2^2}{f_1}. \tag{1.2}$$

Following Ramanujan [31, p. 136-162], we recall:

$$L(q) \;:=\; 1-24\sum_{m=1}^{\infty}\frac{mq^m}{1-q^m}, \tag{1.3}$$

$$M(q) \;:=\; 1+240\sum_{m=1}^{\infty}\frac{m^3q^m}{1-q^m} \tag{1.4}$$

and

$$N(q) \;:=\; 1-504\sum_{m=1}^{\infty}\frac{m^5q^m}{1-q^m}.$$

In [30, p. 257], Ramanujan recorded representations for $M(q)$, $M(q^3)$, $N(q)$ and $N(q^3)$ in terms of $a(q)$ and $c(q)$, for proofs of the same see [10], [15], [22,23]. Employing his theory of Eisenstein series, Ramanujan made [31, p. 136-162] significant contributions to the theory of divisor function, partition function and sums of squares. These have been studied by many mathematicians during recent years, for further reading, we here list few of them [9,11,16,17,23,24,36,12].

In their series of papers, Alaca et. al [1,2,3] expressed Borweins' cubic theta functions in terms of parameters p and k, employing which, Xia E. X. W. and Yao O. X. M. [39], Shruthi and B. R. Srivatsa Kumar [34,35] have recently obtained several Eisenstein series identities which associates $a(q)$, $b(q)$ and $c(q)$. These identities are a form of sum-to-product identities, using which, they derive representations for the convolution sums $\sum_{li+mj=n}\sigma(i)\sigma(j)$, $l,m\in\mathbb{N}$, for various values of l and m. Motivated by these, in this paper, we obtain some new Eisenstein series representations for product of the functions $a(q), a(q^2), a(q^4), a(-q)$ using (p,k)-parametrization and as an application of these, we extract a new representation for the convolution sum $\sum_{i+3j=m}\sigma(i)\sigma(j)$.

2. Preliminaries

Alaca and Williams [1,2,3] in terms of the parameters p and k given by :

$$p:=p(q)=\frac{\varphi^2(q)-\varphi^2(q^3)}{2\varphi^2(q^3)} \qquad \text{and} \qquad k:=k(q)=\frac{\varphi^3(q^3)}{\varphi(q)},$$

obtained parametric representations for difference of $L(q^{m_1})$ and $L(q^{m_2})$, $M(q^n)$, $a(q^i)$ and $a(-q^j)$ for various values of $m_1,m_2,n,i,j\in\mathbb{N}$, for details see [1,2,3]. We list here few of them for our further use:

$2L(q^2)-L(q)=k^2(1+4p+24p^2+14p^3+p^4)$,

$3L(q^3)-L(q)=k^2(2+16p+36p^2+16p^3+2p^4)$,

$4L(q^4) - L(q) = k^2(3 + 18p + 36p^2 + 24p^3),$

$6L(q^6) - L(q) = k^2(5 + 22p + 36p^2 + 22p^3 + 5p^4),$

$12L(q^{12}) - L(q) = k^2(11 + 34p + 36p^2 + 16p^3 + 2p^4),$

$3L(q^3) - 2L(q^2) = k^2(1 + 2p + 12p^2 + 2p^3 + p^4),$

$3L(q^3) - 8L(q^8) = -k^2(2 + 4p + 6p^2 + 13p^3 - \frac{5}{2}p^4),$

$4L(q^4) - 2L(q^2) = k^2(2 + 4p + 12p^2 + 10p^3 - p^4),$

$3L(q^6) - L(q^2) = k^2(2 + 4p + 6p^2 + 4p^3 + 2p^4),$

$2L(q^6) - L(q^3) = k^2(1 + 2p + 2p^3 + p^4),$

$M(q) = k^4(1 + 124p + 964p^2 + 2788p^3 + 3910p^4 + 2788p^5 + 964p^6 + 124p^7 + p^8),$

$M(q^2) = k^4(1 + 4p + 64p^2 + 178p^3 + 235p^4 + 178p^5 + 64p^6 + 4p^7 + p^8),$

$M(q^3) = k^4(1 + 4p + 4p^2 + 28p^3 + 70p^4 + 28p^5 + 4p^6 + 4p^7 + p^8),$

$M(q^4) = k^4\left(1 + 4p + 4p^2 - 2p^3 + 10p^4 + 28p^5 + \frac{31}{4}p^6 - \frac{29}{4}p^7 + \frac{1}{16}p^8\right),$

$M(q^6) = k^4(1 + 4p + 4p^2 - 2p^3 - 5p^4 - 2p^5 + 4p^6 + 4p^7 + p^8),$

$M(q^{12}) = k^4\left(1 + 4p + 4p^2 - 2p^3 - 5p^4 - 2p^5 + \frac{1}{4}p^6 + \frac{1}{4}p^7 + \frac{1}{16}p^8\right),$

$a(q) = k(1 + 4p + p^2),$

$a(q^2) = k(1 + p + p^2),$

$a(q^4) = k\left(1 + p - \frac{p^2}{2}\right)$

and

$a(-q) = k(1 - 2p - 2p^2).$

In the next Section, we follow the methodology employed in [39] to obtain our results. For completeness, we explain the procedure: Let $P(a^i(\pm q^j))$ be a function, now, utilizing the representation for $a(q), a(-q), a(q^2), a(q^4)$ in terms of p and k, we express $P(a^i(\pm q^j))$ (where $i, j \in \mathbb{N}$) as a function of p and k, denoted by, say $P(p, k)$. We then equate,

$$P(p, k) = \sum_{i=1}^{n} r_i Q_i, \tag{2.1}$$

where r_i's are rational numbers (to be found) and Q_i's are the difference of $L(q^n)$'s and or $M(q^m)$'s. Now, substituting the representation for Q_i's in terms of p and k, we see that both sides of (2.1) are expressions in p and k. Then, equating the coefficients of $p^i k^j$ on both sides of (2.1), we obtain a set of linear equations in r_i's. Finally, solving these employing computer, we determine rational values (if solution exists) for r_i's, substituting these values in (2.1), we derive the said Eisenstein series.

3. Some Eisenstein series identities involving $a(q)$

We in this Section, obtain some new Eisenstein series identities for the cubic functions involving $a(q), a(q^2), a(q^4)$ and $a(-q)$. The methodology adopted gives many more identities, however, we here list only few of them.

Theorem 3.1 *We have*

$$a^2(q) = 1 - 36\sum_{m=1}^{\infty}\frac{mq^{3m}}{1-q^{3m}} + 12\sum_{m=1}^{\infty}\frac{mq^{m}}{1-q^{m}}, \tag{3.1}$$

$$a^2(q^2) = 1 + 12\sum_{m=1}^{\infty}\frac{mq^{2m}}{1-q^{2m}} + 36\sum_{m=1}^{\infty}\frac{mq^{3m}}{1-q^{3m}} + 162\sum_{m=1}^{\infty}\frac{mq^{6m}}{1-q^{6m}},$$

$$\begin{aligned}a^2(-q) = 1 &+ 120\sum_{m=1}^{\infty}\frac{mq^{m}}{1-q^{m}} - 144\sum_{m=1}^{\infty}\frac{mq^{2m}}{1-q^{2m}} + 144\sum_{m=1}^{\infty}\frac{mq^{3m}}{1-q^{3m}}\\ &+ 96\sum_{m=1}^{\infty}\frac{mq^{4m}}{1-q^{4m}} - 144\sum_{m=1}^{\infty}\frac{mq^{6m}}{1-q^{6m}},\end{aligned}$$

$$a^2(q^4) = -3\sum_{m=1}^{\infty}\frac{mq^{m}}{1-q^{m}} + 3\sum_{m=1}^{\infty}\frac{mq^{2m}}{1-q^{2m}} + 9\sum_{m=1}^{\infty}\frac{mq^{3m}}{1-q^{3m}} - 9\sum_{m=1}^{\infty}\frac{mq^{6m}}{1-q^{6m}},$$

$$\begin{aligned}a(q)a(q^4) = 1 &- 12\sum_{m=1}^{\infty}\frac{mq^{m}}{1-q^{m}} + 90\sum_{m=1}^{\infty}\frac{mq^{2m}}{1-q^{2m}} - 72\sum_{m=1}^{\infty}\frac{mq^{3m}}{1-q^{3m}}\\ &- 48\sum_{m=1}^{\infty}\frac{mq^{4m}}{1-q^{4m}} + 18\sum_{m=1}^{\infty}\frac{mq^{6m}}{1-q^{6m}},\end{aligned}$$

$$\begin{aligned}4a(q)a(q^2) = 21 &+ 24\sum_{m=1}^{\infty}\frac{mq^{m}}{1-q^{m}} + 144\sum_{m=1}^{\infty}\frac{mq^{3m}}{1-q^{3m}} - 96\sum_{m=1}^{\infty}\frac{mq^{4m}}{1-q^{4m}}\\ &- 576\sum_{m=1}^{\infty}\frac{mq^{12m}}{1-q^{12m}},\end{aligned}$$

$$\begin{aligned}a(q)a(-q) = 1 &+ 12\sum_{m=1}^{\infty}\frac{mq^{m}}{1-q^{m}} + 132\sum_{m=1}^{\infty}\frac{mq^{2m}}{1-q^{2m}} - 108\sum_{m=1}^{\infty}\frac{mq^{3m}}{1-q^{3m}}\\ &- 96\sum_{m=1}^{\infty}\frac{mq^{4m}}{1-q^{4m}} + 36\sum_{m=1}^{\infty}\frac{mq^{6m}}{1-q^{6m}},\end{aligned}$$

(3.2)

$$a^4(q) = 1 + 24\sum_{m=1}^{\infty}\frac{m^3q^m}{1-q^m} + 216\sum_{m=1}^{\infty}\frac{m^3q^{3m}}{1-q^{3m}},$$

$$a^4(q^2) = 1 + 24\sum_{m=1}^{\infty}\frac{m^3q^{2m}}{1-q^{2m}} + 26\sum_{m=1}^{\infty}\frac{m^3q^{3m}}{1-q^{3m}} + 216\sum_{m=1}^{\infty}\frac{m^3q^{6m}}{1-q^{6m}},$$
$$a^4(q^4) = 1 + 24\sum_{m=1}^{\infty}\frac{m^3q^{4m}}{1-q^{4m}} + 216\sum_{m=1}^{\infty}\frac{m^3q^{12m}}{1-q^{12m}}$$

and

$$a^4(-q) = 1 - 24\sum_{m=1}^{\infty}\frac{m^3q^m}{1-q^m} + 432\sum_{m=1}^{\infty}\frac{m^3q^{2m}}{1-q^{2m}} - 216\sum_{m=1}^{\infty}\frac{m^3q^{3m}}{1-q^{3m}} - 384\sum_{m=1}^{\infty}\frac{m^3q^{4m}}{1-q^{4m}}$$
$$+ 3888\sum_{m=1}^{\infty}\frac{m^3q^{6m}}{1-q^{6m}} - 3456\sum_{m=1}^{\infty}\frac{m^3q^{12m}}{1-q^{12m}}.$$

Proof: (*of (3.1)*) [22] Following the methodology explained previously, we assume that

$$\begin{aligned}&A\cdot(2L(q^2)-L(q)) + B\cdot(3L(q^3)-2L(q^2)) + C\cdot(3L(q^3)-L(q))\\&+D\cdot(4L(q^4)-2L(q^2)) + E\cdot(3L(q^6)-L(q^2)) = a^2(q),\end{aligned}\tag{3.3}$$

where A, B, C, D, and E are rational numbers to be found. Next substituting the (p,k)-parametrization of $L(q^{m_i})$'s and $a^2(q)$ listed in Section 2, we equate the coefficients of $k^2, pk^2, p^2k^2, p^3k^2$ and p^4k^2 on both sides of the above, this yields:

$$\begin{aligned}&A + B + 2C + 2D + 2E = 1,\\&4A + 2B + 16C + 4D + 4E = 8,\\&24A + 12B + 36C + 12D + 6E = 18,\\&14A + 2B + 16C + 10D + 4E = 8\\\text{and}&\\&A + B + 2C - D + 2E = 1.\end{aligned}$$

On solving these, we have:

$$A = 0, \quad B = 0, \quad C = \frac{1}{2}, \quad D = 0, \quad E = 0.$$

Substituting the above values in (3.2), employing (1.3) and after some simplification we deduce (3.1). □

Similarly, assuming

$$A(2L(q^2)-L(q))+B(3L(q^3)-2L(q^2))+C(3L(q^3)-L(q))+D(2L(q^6)-L(q^3))+E(3L(q^6)-L(q^2))=a^2(q^2),$$

$$A(2L(q^2)-L(q))+B(3L(q^3)-2L(q^2))+C(3L(q^3)-L(q))+D(4L(q^4)-2L(q^2))+E(3L(q^6)-L(q^2))=a^2(-q),$$

$$A(2L(q^2)-L(q))+B(3L(q^3)-2L(q^2))+C(3L(q^3)-L(q))+D(4L(q^4)-2L(q^2))+E(3L(q^6)-L(q^2))=4a^2(q^4),$$

$$A(2L(q^2)-L(q))+B(3L(q^3)-2L(q^2))+C(3L(q^3)-L(q))+D(4L(q^4)-2L(q^2))+E(3L(q^6)-L(q^2))=2a(q)a(q^4),$$

$$A(2L(q^2)-L(q))+B(3L(q^3)-L(q))+C(4L(q^4)-L(q))+D(6L(q^6)-L(q))+E(12L(q^{12})-L(q))=a(q)a(q^2),$$

$$A(2L(q^2)-L(q))+B(3L(q^3)-2L(q^2))+C(3L(q^3)-L(q))+D(4L(q^4)-2L(q^2))+E(3L(q^6)-L(q^2))=a(q)a(-q),$$

$$A(4M(q^2)-M(q))+B(4M(q^3)-M(q))+C(16M(q^4)-4M(q))+D(4M(q^6)-M(q))+E(16M(q^{12})-4M(q))+F(4M(q^3)-M(q^2))+G(4M(q^6)-M(q^3))+H(4M(q^6)-M(q^2))+I(16M(q^{12})-4M(q^3))=a^4(q),$$

$$A(4M(q^2)-M(q))+B(4M(q^3)-M(q))+C(16M(q^4)-4M(q))+D(16M(q^{12})-4M(q^6))+E(16M(q^{12})-4M(q^2))+F(64M(q^{12})-16M(q^4))+G(4M(q^3)-M(q^2))+H(4M(q^6)-M(q^2))+I(16M(q^{12})-4M(q))=a^4(q^2),$$

$$A(4M(q^2)-M(q))+B(4M(q^3)-M(q))+C(16M(q^4)-4M(q))+D(16M(q^{12})-4M(q^6))+E(16M(q^{12})-4M(q^2))+F(64M(q^{12})-16M(q^4))+G(4M(q^3)-M(q^2))+H(4M(q^6)-M(q^2))+I(16M(q^{12})-4M(q))=16a^4(q^4),$$

$$A(4M(q^2)-M(q))+B(4M(q^3)-M(q))+C(16M(q^4)-4M(q))+D(16M(q^{12})-4M(q^6))+E(16M(q^{12})-4M(q^2))+F(64M(q^{12})-16M(q^4))+G(4M(q^3)-M(q^2))+H(4M(q^6)-M(q^2))+I(16M(q^{12})-4M(q))=a^4(-q)$$

and then employing the technique adopted for proving (3.1), we derive the other listed identities of Theorem 3.1.

Theorem 3.2 *We have*

$$a^4(q) = -819 - 240 \sum_{m=1}^{\infty} \left(\frac{343m^3q^m}{1-q^m} - \frac{1360m^3q^{2m}}{1-q^{2m}} - \frac{3033m^3q^{3m}}{1-q^{3m}} + \frac{12240m^3q^{6m}}{1-q^{6m}} \right)$$
$$+ 34 \left(5 - 24 \sum_{m=1}^{\infty} \left(\frac{6mq^{6m}}{1-q^{6m}} - \frac{mq^m}{1-q^m} \right) \right)^2 - 34 \left(1 - 24 \sum_{m=1}^{\infty} \left(\frac{3mq^{3m}}{1-q^{3m}} \right.\right.$$
$$\left.\left. - \frac{2mq^{2m}}{1-q^{2m}} \right) \right)^2 + 1 \left(2 - 24 \sum_{m=1}^{\infty} \left(\frac{3mq^{3m}}{1-q^{3m}} - \frac{mq^m}{1-q^m} \right) \right)^2.$$

Proof: We assume that

$$\begin{aligned} &A \cdot M(q) + B \cdot M(q^2) + C \cdot M(q^3) + D \cdot M(q^6) + E \cdot (4L(q^4) - L(q))^2 \\ &+ F \cdot (6L(q^6) - L(q))^2 + G \cdot (12L(q^{12}) - L(q))^2 + H \cdot (3L(q^3) - 2L(q^2))^2 \\ &+ I \cdot (3L(q^3) - L(q))^2 = a^4(q). \end{aligned} \tag{3.4}$$

Now, following the procedure given in the Proof of (3.1), and then equating the coefficients of k^4, pk^4, p^2k^4, $p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4$ and p^8k^4 on both sides of the above, we see that:

$$\begin{aligned} &A + B + C + D + 9E + 25F + 121G + H + 4I = 1, \\ &124A + 4B + 4C + 4D + 108E + 220F + 748G + 4H + 64I = 16, \\ &964A + 64B + 4C + 4D + 540E + 844F + 1948G + 52H + 1216I = 100, \\ &2788A + 178B + 28C - 2D + 1440E + 1804F + 2800G + 52H + 1216I = 304, \\ &3910A + 235B + 70C - 5D + 2160E + 2314F + 2428G + 154H + 1816I = 454, \\ &2788A + 178B + 28C - 2D + 1728E + 1804F + 1288G + 52H + 1216I = 304, \\ &964A + 64B + 4C + 4D + 576E + 844F + 400G + 28H + 400I = 100, \\ &124A + 4B + 4C + 4D + 220F + 64G + 4H + 64I = 16 \end{aligned}$$

and

$$A + B + C + D + 25F + 4G + H + 4I = 1.$$

Upon solving these using a computer, we obtain

$$A = -\frac{343}{10}, \quad B = 136, \quad C = \frac{3033}{10}, \quad D = -1224, \quad E = 0,$$
$$F = 34, \quad G = 0, \quad H = -34 \quad \text{and} \quad I = 1.$$

Substituting these values in (3.3), we have

$$\begin{aligned} &-343 \cdot M(q) + 1360 \cdot M(q^2) + 3033 \cdot M(q^3) - 12240 \cdot M(q^6) + 340 \cdot (6L(q^6) - L(q))^2 \\ &- 340 \cdot (3L(q^3) - 2L(q^2))^2 + 10 \cdot (3L(q^3) - L(q))^2 = 10 \cdot a^4(q). \end{aligned}$$

Next, employing definitions of $L(q)$ and $M(q)$ as in (1.3) and (1.4), we complete the proof of Theorem 3.2. □

4. A representation for the convolution sum $\sum_{i+3j=m} \sigma(i)\sigma(j)$

For $m, k \in \mathbb{N}$, the divisor function $\sigma_k(m)$ is defined by:

$$\sigma_k(m) = \begin{cases} \displaystyle\sum_{k|m} d^k, & \text{where } d \text{ runs over all the positive divisors of } m \\ 0, & \text{if } m = 0 \end{cases}.$$

We shall be denoting $\sigma(m)$ for $\sigma_1(m)$ for convenience.

The convolution sum $\sum_{li+kj=m} \sigma(i)\sigma(j)$ have been explicitly evaluated for various values of $l, k \in \mathbb{N}$, for details, see [1,2,3,4,5,6,7,8,18,19,20,21,25,26,27,28,29,33,34,35,37,38,39,40]. In this Section, we derive a representation for $\sum_{i+3j=m} \sigma(i)\sigma(j)$ employing the representations given for $\sum_{i+j=m} \sigma(i)\sigma(j)$, $\sum_{i+6j=m} \sigma(i)\sigma(j)$ and $\sum_{3i+2j=m} \sigma(i)\sigma(j)$ in [3].

Theorem 4.1 *If $m \in \mathbb{Z}^+$, we have*

$$\begin{aligned} 2^7 \cdot 3^3 \cdot \sum_{i+3j=m} \sigma(i)\sigma(j) = \; & 2^3 \cdot 3 \cdot 7 \cdot \sigma_3(m) - 2^4 \cdot 5 \cdot 17 \cdot 23 \cdot \sigma_3\left(\frac{m}{2}\right) - 3 \cdot 23 \cdot 43 \\ & \times \sigma_3\left(\frac{m}{3}\right) + 2^4 \cdot 3^2 \cdot 5 \cdot 17 \cdot 23 \cdot \sigma_3\left(\frac{m}{6}\right) + 2^4 \cdot 3^2 \\ & \times (1-2m) \cdot \sigma(m) - 2^7 \cdot 3 \cdot 17 \cdot (1-2m) \cdot \sigma\left(\frac{m}{2}\right) \\ & - 2^4 \cdot 3 \cdot 7 \cdot 41 \cdot \sigma\left(\frac{m}{3}\right) - A(m), \end{aligned}$$

where

$$\begin{aligned} 1 + \sum_{m=1}^{\infty} A(m) q^m = & \frac{f_2^{20} f_6^{20}}{f_1^8 f_3^8 f_4^8 f_{12}^8} + 16q \frac{f_2^{14} f_6^{14}}{f_1^6 f_3^6 f_4^4 f_{12}^4} + 96q^2 \frac{f_2^8 f_6^8}{f_1^4 f_3^4} \\ & + 256q^3 \frac{f_4^4 f_{12}^4 f_2^2 f_6^2}{f_1^2 f_3^2} + 256q^4 \frac{f_4^8 f_{12}^8}{f_2^4 f_6^4}. \end{aligned}$$

Proof: In Theorem 3.2, using (1.1) and (1.2), we see that

$$\frac{f_2^{20} f_6^{20}}{f_1^8 f_3^8 f_4^8 f_{12}^8} + 16q \frac{f_2^{14} f_6^{14}}{f_1^6 f_3^6 f_4^4 f_{12}^4} + 96q^2 \frac{f_2^8 f_6^8}{f_1^4 f_3^4} + 256q^3 \frac{f_4^4 f_{12}^4 f_2^2 f_6^2}{f_1^2 f_3^2} + 256q^4 \frac{f_4^8 f_{12}^8}{f_2^4 f_6^4}$$

$$
\begin{aligned}
&= 1 - 2^3\cdot 3\cdot 7^3\cdot \sum_{m=1}^{\infty}\sigma_3(m)q^m + 2^4\cdot 5\cdot 17\cdot \sum_{m=1}^{\infty}\sigma_3(m)q^{2m} + 3^2\cdot 337\\
&\times \sum_{m=1}^{\infty}\sigma_3(m)q^{3m} - 2^4\cdot 3^2\cdot 5\cdot 17\cdot \sum_{m=1}^{\infty}\sigma_3(m)q^{6m} + 2^6\cdot 3\cdot 43\cdot \sum_{m=1}^{\infty}\sigma(m)q^m\\
&- 2^6\cdot 3\cdot 17\cdot \sum_{m=1}^{\infty}\sigma(m)q^{2m} + 2^9\cdot 3^2\cdot \sum_{m=1}^{\infty}\sigma(m)q^{3m} - 2^6\cdot 3^2\cdot 5\cdot 17\\
&\times \sum_{m=1}^{\infty}\sigma(m)q^{6m} + 2^6\cdot 3^2\cdot 5\cdot 7\cdot \left(\sum_{m=1}^{\infty}\sigma(m)q^m\right)^2 - 2^9\cdot 3^3\cdot 17\\
&\times \left(\sum_{m=1}^{\infty}\sigma(m)q^{2m}\right)^2 - 2^6\cdot 3^2\cdot 5^2\cdot \left(\sum_{m=1}^{\infty}\sigma(m)q^{3m}\right)^2 + 2^9\cdot 3^4\cdot 17\\
&\times \left(\sum_{m=1}^{\infty}\sigma(m)q^{6m}\right)^2 - 2^9\cdot 3^3\cdot 17\cdot \sum_{m=1}^{\infty}\sigma(m)q^m\cdot \sum_{m=1}^{\infty}\sigma(m)q^{6m} + 2^9\cdot 3^3\cdot 17\\
&\times \sum_{m=1}^{\infty}\sigma(m)q^{2m}\cdot \sum_{m=1}^{\infty}\sigma(m)q^{3m} - 2^7\cdot 3^3\cdot \sum_{m=1}^{\infty}\sigma(m)q^m\cdot \sum_{m=1}^{\infty}\sigma(m)q^{3m}.
\end{aligned}
$$

Now on extracting the coefficients of q^m on both sides of the above, we have

$$
\begin{aligned}
A(m) = &-2^3\cdot 3\cdot 7^3\cdot \sigma_3(m) + 2^4\cdot 5\cdot 17\cdot \sigma_3\left(\frac{m}{2}\right) + 3^2\cdot 337\cdot \sigma_3\left(\frac{m}{3}\right)\\
&- 2^4\cdot 3^2\cdot 5\cdot 17\cdot \sigma_3\left(\frac{m}{6}\right) + 2^6\cdot 3\cdot 43\cdot \sigma(m) - 2^6\cdot 3\cdot 17\cdot \sigma\left(\frac{m}{2}\right)\\
&+ 2^9\cdot 3^2\cdot \sigma\left(\frac{m}{3}\right) - 2^6\cdot 3^2\cdot 5\cdot 17\cdot \sigma\left(\frac{m}{6}\right) + 2^6\cdot 3^2\cdot 5\cdot 7\cdot \sum_{i+j=m}\sigma(i)\sigma(j)\\
&- 2^9\cdot 3^2\cdot 17\cdot \sum_{i+j=\frac{m}{2}}\sigma(i)\sigma(j) - 2^6\cdot 3^2\cdot 5^2\cdot \sum_{i+j=\frac{m}{3}}\sigma(i)\sigma(j)\\
&+ 2^9\cdot 3^4\cdot 17\cdot \sum_{i+j=\frac{m}{6}}\sigma(i)\sigma(j) - 2^9\cdot 3^3\cdot 17\cdot \sum_{i+6j=m}\sigma(i)\sigma(j)\\
&+ 2^9\cdot 3^3\cdot 17\cdot \sum_{2i+3j=m}\sigma(i)\sigma(j) - 2^7\cdot 3^3\cdot \sum_{i+3j=m}\sigma(i)\sigma(j).
\end{aligned}
\tag{4.1}
$$

Next, from [3], we record the following representations:

$$
\sum_{i+j=m}\sigma(i)\sigma(j) = \frac{5}{12}\sigma_3(m) - \frac{m}{2}\sigma(m) + \frac{\sigma(m)}{12}, \tag{4.2}
$$

$$
\begin{aligned}
\sum_{i+6j=m}\sigma(i)\sigma(j) = &\frac{1}{120}\sigma_3(m) + \frac{1}{30}\sigma_3\left(\frac{m}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{m}{3}\right) + \frac{3}{10}\sigma_3\left(\frac{m}{6}\right)\\
&+ \left(\frac{1}{24} - \frac{m}{24}\right)\sigma(m) + \left(\frac{1}{24} - \frac{m}{4}\right)\sigma\left(\frac{m}{6}\right) - \frac{1}{120}\cdot C_6(m)
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
\sum_{2i+3j=m}\sigma(i)\sigma(j) = &\frac{1}{120}\sigma_3(m) + \frac{1}{30}\sigma_3\left(\frac{m}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{m}{3}\right) + \frac{3}{10}\sigma_3\left(\frac{m}{6}\right)\\
&+ \left(\frac{1}{24} - \frac{m}{12}\right)\sigma\left(\frac{m}{2}\right) + \left(\frac{1}{24} - \frac{m}{8}\right)\sigma\left(\frac{m}{3}\right) - \frac{1}{120}\cdot C_6(m),
\end{aligned}
\tag{4.4}
$$

where,

$$\sum_{m=1}^{\infty} C_6(m)q^m = q \cdot f_1^2 f_2^2 f_3^2 f_6^2.$$

Finally, on employing (4.2)-(4.4) in (4.1), we deduce Theorem 4.1. □

5. Conclusion

Using the methodology introduced by Xia. et. al. [39] which employs the Computer, one can derive numerous Eisenstein series. The Eisenstein series representations have been exclusively employed by various mathematicians in establishing congruence relations and convolution sums for several divisor functions and restricted divisor functions. We in this paper have made a similar effort using the Borweins' cubic theta function $a(q)$.

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