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Hyers-Ulam stability, exponential stability and exact admissibility of non-autonomous difference equations *

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ABSTRACT: In this manuscript we discuss the relation between Hyers-Ulam stability, uniform exponential stability and exact admissibility of non-autonomous difference equations in Banach spaces. We use the idea of discrete evolution semigroup in our proof. The same results for autonomous difference equations as an application of our results are also given in the form of corollaries. Also, at the end we provide some examples to support our theoretical results.

Key Words: Non-autonomous Discrete problems; discrete evolution families of bounded linear operators; discrete evolution semigroups; exact admissibility.

Contents

1. Introduction

In 1940, Ulam [1], put a problem before the mathematicians in a conference, the problem he ask was that "is there exists a linear operator which is near to a δ-linear operator on a metric group"?. After a year, Hyers [2] gave a positive answer to the question of Ulam, in the case of Banach space, which is then known by the Hyers-Ulam stability for a functional equation. The Hyers-Ulam stability was extended by Rassias [3] in 1978, and this stability is now referred to as the Hyers-Ulam-Rassias stability.

The asymptotic behavior of dynamical systems was entirely studied in the history. In 1963, Mircea Reghiş studied the linear system $\dot{y} = A(t)y, t \geq 0$, in his short description of research work ([4]) and is called it non-uniformly exponentially stability if there exist $f > 0, b \in \mathbb{R}$ and $L = L(f, b) > 0$ such that

$$||\Theta(\alpha)\Theta^{-1}(\beta)|| \leq Le^{-f\alpha}e^{b\beta} \text{ for every } \alpha \geq \beta \geq 0. \tag{1.1}$$

Here, $\Theta(\cdot)$ is the classical solution of the non-autonomous Cauchy Problem

$$\dot{Q}(s) = A(s)Q(s), s \geq 0 \quad Q(0) = \mathbf{I}. \tag{1.2}$$

We note that the relation (1.1) can also be written as:

$$\|\Theta(\alpha)\Theta^{-1}(\beta)\| \leq Le^{(c-d)\beta}e^{-d(\alpha-\beta)} \text{ for every } \alpha \geq \beta \geq 0. \tag{1.3}$$

The UES of the above family is a particular case of the nonuniform exponential stability by putting $d = c$ in (1.3).

A family $\mathcal{W} := \{W(t, s) : (t, s) \in \Delta\} \subset \mathcal{L}(Y)$, is said to be a strongly continuous evolution family if $W(t, t) = I, W(t, s)W(s, r) = W(t, r)$ for every $t \geq s \geq r$, and for each $x \in Y$ the map $(t, s) \mapsto W(t, s)x$

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is continuous on the set $\Delta := \{(t,s) : t \geq s \geq 0\}$. The family is called exponentially bounded (EB) if for some constants $\omega \in \mathbb{R}$ and $M \geq 1$ we have,

$$||W(t,s)|| \leq Me^{\omega(t-s)} \text{ for all } (t,s) \in \Delta. \tag{1.4}$$

An evolution family $\mathcal{W} := \{W(t,s) : t \geq s \geq 0\} \subset \mathcal{L}(Y)$ is said to be non-uniformly exponentially stable if there exists a positive real number μ and a map $H : \mathbb{R}_+ := [0,\infty) \to (0,\infty)$ such that

$$\|W(t,s)\| \leq H(s)e^{-\mu(t-s)} \quad \forall t \geq s \geq 0. \tag{1.5}$$

The theory of UES of evolution equations has been exhaustively analyzed in the last few decades. Equivalent characterizations for unifrom exponential stability with strong integral conditions, UES of the evolution semigroup related to the evolution family on certain Banach spaces of vector valued functions, or in a form of exact admissibility related to the spaces $C_{00}(\mathbb{R}_+, Y)$, $L^p(\mathbb{R}_+, Y)$ and others, were obtained in [7] [8] and [9].

The uniform exponential stability implies nonuniform exponential stability but the converse implications does not hold. Indeed, the evolution family $\{(\beta^2+1)/(\alpha^2+1) : \alpha \geq \beta \geq 0\}$ on $\mathbb{C}$ is non-uniformly strongly stable but is not non-uniformly exponentially stable, while the family $\{V(\alpha,\beta) := \exp[g(\beta) - g(\alpha)] : \alpha \geq \beta \geq 0\}$, with $g(\alpha) := (\alpha+1)[\sqrt{2} - \sin\ln(\alpha+1)], \alpha \geq 0$, is non-uniformly exponentially stable and u uniformly bounded (i.e. $\sup_{\alpha\geq\beta\geq 0} ||V(\alpha,\beta)|| < \infty$), but is not uniformly exponentially stable, [11], [12].

In this paper we will provide many results about stability and Hyer-Ulam stability, but mainly focus on the discrete case. The study of asymptotic behavior like exponential stability and Hyers-Ulam stability of time dependent difference systems $\mathbb{G}_{n+1} = A_n\mathbb{G}_n$ or $\mathbb{D}_{n+1} = A_n\mathbb{D}_n + f_n$ is too much complicated than the study of the autonomous case. For recent work in the field of Hyers-Ulam stability of difference equations of different type we refer to [15], [16], [17], [18] and [19].

From the structural point of view, the ideas of our main results are similar to [8], [22, Theorem 3 and 5], [13] and [9, Theorem 2.2] in literature. However, the path chosen by us for proof is different, at least at times, from that presented in [9] and [22]. For example, we show the equivalence of uniform exponentially stable for discrete evolution families and evolution semigroups on the spaces $E_p(\mathbb{Z}_+, Y)$ and $l^\infty(\mathbb{Z}_+, Y)$ in Theorem 3.1 and Theorem 2.1 respectively of this paper, which is not contained in [[22], Theorem 3] and [[9], Theorem 2.2].

In mathematics we usually observed that, many of the biological systems and models can be resolved by using differential equations. Differential equation have a lot of applications in various fields of natural sciences, economics, statistics and engineering. Although differential equations are too useful but sometime when we discussed a real life problem, we need to take the sample in discrete form, and showed the model in a form of difference equations for details see, [5,6]. The applications of difference equation have appeared recently in many fields of sciences.

We will present the relation between uniform exponential stability, Hyer-Ulam stability and exact admissibility of the discrete evolution family $\mathcal{W}$ on the space $l^\infty(\mathbb{Z}_+, Y)$. Furthermore equivalent characterization of uniformly exponentially stable for discrete evolution family in terms of evolution semigroup, invertibility of "infinitesimal generator", and exact admissibility of the space $c_0\mathbb{Z}_+, Y)$ and $l^p(\mathbb{Z}_+, Y)$, where $1 \leq p < \infty$ are also given in this paper. Similar result in discrete case on other spaces can also be found in [10], [14] and [21].

In our proof we are using the concept of discrete evolution semigroups. The continuous variant of these results was intensively studied in the last few decades. The concept of evolution semigroup was introduced by J. S. Howland in [20], and studied in 1976 by D. E. Evans [23]. Later, Y. Latushkin, T. Randolph, R. Nagel, R. Schnaubelt, F. Räbiger also work on this concept, see [24] and [25].

In the following, we present the scalar version of our result which seems to be nontrivial. Let M be a positive constant. A $(\mathbb{C} \setminus \{0\})$-valued sequence (x_n) is called *(absolutely valued)-M-nondecreasing* if $|x_n| \leq M|x_m|$ for all $n, m \in \mathbb{Z}_+, n \geq m$. An absolutely valued-$M$-non-decreasing sequence (x_n) decays exponentially if and only if for every sequence (f_n) belonging to $l^\infty(\mathbb{Z}_+, \mathbb{C})$, the "convolution" sequence $\left(\sum_{k=1}^n \frac{x_n}{x_k} g_k\right)$ belongs to $l^\infty(\mathbb{Z}_+, \mathbb{C})$.

2. Notations, basic definitions and preliminaries

By $\mathbb{Z}_+$, we denote the set all non-negative integer. Throughout the paper we will use some short notation for the following words: Uniform exponential stable by UES, Exact Admissible by EA, Uniformly bounded by UB, Hyers-Ulam by H-U.

A family $\mathcal{W} = \{W(l,r) : l, r \in \mathbb{Z}_+, l \geq r\} \subset \mathcal{L}(Y)$ is called *discrete evolution family on* $\mathbb{Z}_+$ if $W(l,l) = I$ for all $n \in \mathbb{Z}_+$ and $W(l,r)W(r,p) = W(l,p)$ for all $l, r, p \in \mathbb{Z}_+$ with $l \geq r \geq p$. Here I is the identity operator on Y.

A discrete evolution family has *exponential growth* or it is *exponentially bounded* if there exist a real number ω and $K = K_\omega \geq 1$ such that

$$\|W(n,j)\| \leq Ke^{\omega(n-j)} \text{ for all } n, j \in \mathbb{Z}_+, n \geq j.$$

A set $\mathbf{S} = \{S(j)\}_{j\in\mathbb{Z}_+}$ of bounded linear operators on the Banach space Y is called *discrete semigroup* on Y if $S(0) = I$ and $S(n+m) = S(n) \circ S(m)$ for all $n, m \in \mathbb{Z}_+$. Clearly, for each $n \in \mathbb{Z}_+$, $S(n) = S(1)^n$. The operator $S(1)$ ia said to be the algebraic generator of the discerete semigroup $\mathbf{S}$. The idea of infinitesimal generator in case of strongly continuous semigroups, leads to the the idea of "infinitesimal generator" for a discrete semigroup which is defined as $G := T(1) - I$.

The Banach spaces listed below will be extensively used in what follows.

- $l^p(\mathbb{Z}_+, Y) = \left\{(f_n)_{n\in\mathbb{Z}_+} : \quad \|(f_n)\|_p = \left(\sum_{n=0}^{\infty} \|f_n\|^p\right)^{1/p} < \infty\right\}$, $1 \leq p < \infty$;
- $l^\infty(\mathbb{Z}_+, Y) = \{(f_n)_{n\in\mathbb{Z}_+} : \quad \|(f_n)\|_\infty = \sup_{n\in\mathbb{Z}_+} \|f_n\| < \infty\}$;
- $c_0(\mathbb{Z}_+, Y) = \{(f_n)_{n\in\mathbb{Z}_+} : \quad \lim_{n\to\infty} \|f_n\| = 0\}$, endowed with $\|\cdot\|_\infty$.

Let us denote

$$E_p = E_p(\mathbb{Z}_+, Y) = \begin{cases} l^p(\mathbb{Z}_+, Y), & \text{if} \quad 1 \leq p \leq \infty \\ c_0(\mathbb{Z}_+, Y), & \text{if} \quad p = 0. \end{cases}$$

The boundedness, uniform exponential stability and $E_p(\mathbb{Z}_+, Y)$-admissibility for an evolution family $\mathcal{W}$ are presented as follows.

Definition 2.1 *The evolution family $\mathcal{W} = \{W(l,r) : l \geq r \in \mathbb{Z}_+\}$ is called:*

(i) UB, if for some $K > 0$, we have

$$\|W(l,r)\| \leq K \text{ for all } l, r \in \mathbb{Z}_+, l \geq r;$$

(ii) UES, if for some $\nu > 0$ the family

$$\{e^{\nu(l-r)}W(l,r) : l \geq r \in \mathbb{Z}_+\}$$

is uniformly bounded;

(iii) EA, if for every $f \in E_p$ the solution $\mathcal{W} * f$ belongs to E_p.

Here, $\mathcal{W} * f$ is the convolution sequence defined by

$$(\mathcal{W} * f)(n) = \sum_{k=0}^{n} W(n,k)f(k).$$

Obviously, the solutions of the discrete Cauchy Problems

$$\begin{cases} x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}_+ \\ x_0 = b, \end{cases} \tag{A_n, b}$$

and

$$\begin{cases} x_{n+1} &= A_n x_n + f_{n+1}, \quad n \in \mathbb{Z}_+ \\ x_0 &= b, \end{cases} \tag{A_n, f, b}$$

are given by $x_n = W(n,0)b$ and $x_n = W(n,0)b + (\mathcal{W} * f)(n)$, respectively, where $W(n,k) = A_{n-1}A_{n-2}...A_k$.

Using that equality, we can now present the significance of the concepts of E_p-EA. The family $\mathcal{W}$ is E_p-EA if and only if for every f in E_p, the solution of the Cauchy Problem (A, f, b) belongs to E_p.

Now we have to define the Hyers-Ulam stability of the system (A, f, b).

Definition 2.2 *A sequence ϕ is said to be an $\epsilon-$approximate solution of the system (A_n, b) if the following inequality hold: $\|\phi_{n+1} - A_n\phi_n\| \leq \epsilon$, for all n.*

Definition 2.3 *The system (A_n, b) is said to be Hyers-Ulam stable if there exists a positive real number L such that for every $\epsilon-$approximate solution $\{\phi_n\}$ there exists an exact solution $\{\xi_n\}$ such that $\|\phi_n - \xi_n\| \leq L\epsilon$, for all n.*

Now, we will present the discrete evolution semigroup concerning with a discrete evolution family on E_p. Let $\mathcal{W} = \{W(m,n) : m, n \in \mathbb{Z}_+, m \geq n\}$ be a discrete exponentially bounded evolution family on a Banach space Y. For each $r \in \mathbb{Z}_+$, the linear operator $\mathcal{T}_p(r)$, given by

$$(\mathcal{T}_p(r)f)(n) := \begin{cases} W(n, n-r)f(n-r), & \text{if } n \geq r \\ 0, & \text{otherwise,} \end{cases}$$

is well defined in $E_p(\mathbb{Z}_+, Y)$. Furthermore, it is a bounded operator and

$$\|\mathcal{T}_p(j)\|_{\mathcal{L}(E_p(\mathbb{Z}_+,Y))} \leq M \exp(\omega j).$$

The family $\mathcal{T}_p = \{\mathcal{T}_p(j)\}_{j\in\mathbb{Z}_+}$ is called the evolution semigroup related to $\mathcal{W}$.

Lemma 2.1 *(cf. [14]) Let $\mathcal{W} = \{W(n,j)_{n\geq j\geq 0}$, be an exponentially bounded discrete evolution family of bounded linear operators acting on a Banach space Y. If there exists a positive constant c such that*

$$(n - r + 1)\|W(n,r)\| \leq c \text{ for all } n \geq r \geq 0,$$

then $\mathcal{W}$ is uniformly exponentially stable.

Theorem 2.1 *[14] Let $\mathcal{W} := \{W(l,r) : n \geq m \geq 0\}$ be the same family as above, then the following are equivalent:*

- **(i)** *$\mathcal{W}$ is UES.*
- **(ii)** *$\mathcal{W}$ is $l^\infty(\mathbb{Z}_+, Y)$-EA.*

3. Main Results

In the following definition, we introduce the idea of discrete evolution semigroup associated to discrete semigroup on the space E_p.

Definition 3.1 *The evolution semigroup $\mathcal{S}_p = \{\mathbf{S}_p(j)\}_{j\in\mathbb{Z}_+}$ associated with $\mathbf{T}$ on E_p is defined by*

$$(\mathbf{S}_p(j)y)(n) := \begin{cases} T(r)y(n-r), & \text{if } n \geq r \\ 0, & \text{otherwise,} \end{cases} \tag{3.1}$$

The following lemma shows that evolution semigroup $\mathcal{S}_p = \{\mathbf{S}_p(j)\}_{j\in\mathbb{Z}_+}$ associated to discrete semigroup $\mathbf{T}$ act on E_p

Lemma 3.1 *Let $y \in E_p$ and $\mathbf{T} = T(j)_{j\in\mathbb{Z}_+}$ be the discrete semigroup of bounded linear operators on Y. Then sequence $\mathbf{S}_p(r)y$, given in (3.1) belongs to E_p for all $p = 0$ or $1 \leq 1 \leq \infty$.*

Proof: We will discuss three cases accordingly when $1 \le p < \infty$, $p = \infty$ and $p = 0$.
Case:1 We show that for each $y \in l^p(\mathbb{Z}_+, Y)$ and each fixed $r \in \mathbb{Z}_+$ the sequence $\mathbf{S}_p(j)y$ belongs to $l^p(\mathbb{Z}_+, Y)$.
if $n \ge r$, then

$$\begin{aligned} ||\mathbf{S}_p(r)y||^p_{l^p(\mathbb{Z}_+,Y)} &= \sum_{n=0}^{\infty} ||(\mathbf{S}_p(r)y)(n)||^p = \sum_{n=r}^{\infty} ||T(r)y(n-r)||^p \\ &\le M^p e^{p\omega r} \sum_{n=r}^{\infty} ||y(n-r)||^p, \end{aligned}$$

where $M > 1$. If $n < r$, then

$$||\mathbf{S}_p(r)y||^p_{l^p(\mathbb{Z}_+,Y)} = \sum_{n=0}^{\infty} ||(\mathbf{S}_p(r)y)(n)||^p = 0.$$

i.e. $||\mathbf{S}_p(r)y||_{l^p(\mathbb{Z}_+,Y)} \le Me^{\omega r}||f||_{l^p(\mathbb{Z}_+,Y)}$ for each $f \in l^p(\mathbb{Z}_+, Y)$.
Case:2 if $n \ge r$, then

$$\begin{aligned} ||\mathbf{S}_\infty(r)y||_{l^\infty(\mathbb{Z}_+,Y)} &= \sup_{n\in\mathbb{Z}_+} ||(\mathbf{S}_p(r)y)(n)|| \\ &= \sup_{n\ge r} ||T(r)y(n-r)|| \le Me^{\omega r}||y()||_{l^\infty(\mathbb{Z}_+,Y)}, \end{aligned}$$

if $n < r$, then $||\mathbf{S}_\infty(r)y||_{l^\infty(\mathbb{Z}_+,Y)} = \sup_{n\in\mathbb{Z}_+} ||(\mathbf{S}_p(r)y)(n)|| = 0$.
i.e. $||\mathbf{S}_\infty(r)y||_{l^\infty(\mathbb{Z}_+,Y)} \le Me^{\omega r}||f||_{l^\infty(\mathbb{Z}_+,Y)}$ for each $f \in l^\infty(\mathbb{Z}_+, Y)$.
The **Case 3** can be easily obtained by the same procedure. □

In the next theorem, we show the relation between uniform exponential stability, H-U stability and EA of the evolution semigroup on $E_p(\mathbb{Z}_+, Y)$, associated to a discrete, uniformly exponentially bounded family $\mathcal{W}$.

Theorem 3.1 *Let $p = 0$ or $1 \le p \le \infty$. For a discrete, exponentially bounded evolution family $\mathcal{W}$, of bounded linear operators acting on Y, the following five statements are equivalent:*

- **(i)** *The family $\mathcal{W}$ is UES.*
- **(ii)** *The evolution semigroup $\mathcal{T}_p$ associated to the family $\mathcal{W}$ on $E_p(\mathbb{Z}_+, Y)$ is UES.*
- **(iii)** *The "infinitesimal generator" $\mathcal{T}_p(1) - I$ of $\mathcal{T}_p$, is invertible.*
- **(iv)** *The family $\mathcal{W}$ is $E_p(\mathbb{Z}_+, Y)$-EA.*
- **(v)** *The system (A_n, b) is H-U stable.*

Proof:
The proof of implication (i) to (iv) is given in [21], we will only prove the implication (i) if and only if (v) . (i) $\Rightarrow$ (v) : Since the exact and ϵ-approximate solution of the system (i) are $\psi_n = W(n, 0)b$ and $X_n = W(n, 0)b + \sum_{k=0}^{n} W(n, k)f_k$ respectively.
Now consider
$\|X_n - \psi_n\| = \|W(n, 0)\xi - W(n, 0)\xi - \sum_{k=0}^{n} W(n, k)f_k\|$
$\le \epsilon \sum_{k=0}^{n} Me^{-k(n-k)}$
$= M\epsilon(e^{-\mu(n-1)} + e^{-\mu(n-2)} + ... + e^{-\mu(0)})$
$\le M\epsilon(1 + e^{-u} + e^{-2u} + ... + e^{-k(n-1)})$
$M\epsilon(1 + e^{-u} + ...)$
$= M\frac{e^u}{e^v-1}\epsilon = L\epsilon$, where $L = M\frac{e^u}{e^v-1}$.
That is the system (A_n, b) is Hyers-Ulam stable.

To prove the other implication we will prove that (v)⇒(iv) then as (iv) implies (i), so the result will follows. Let $f \in l_\infty(Z_+, X)$ then there exists $M > 0$ such that $\|f_n\| \leq M$ for all $x \in N$ that is $\|f\| \leq M$. Now using definition of Hyers-Ulam stability we have, for ϵ-approximate solution there exists $L > 0$ such that $\| \sum_{k=0}^{n} W(n,k)f(k)\| \leq LM$, for all $n \in Z_+$. That is $g(n) = \sum_{k=0}^{n} W(n,k)f(k)$ is bounded. This shows that the family W is exact admissible on $l_\infty(Z_+, X)$. Hence the results yields. Similarly if we have a sequence $\{f_n\}$ from l_p then again $\sum_{n=0}^{\infty} \|g(n)\|^p = \sum_{n=0}^{\infty} \| \sum_{k=0}^{n} W(n,k)f(k)\|^p \leq L\epsilon$, which shows that $g \in l_p$, that is the family l_p is exact admissible. Hence the result is proved for all E_p spaces, where $p = 0$ or $1 \leq p \leq \infty$. □

Remark 3.1 *If $\mathcal{W} := \{W(l,r) : l \geq r \in \mathbb{Z}_+\}$ is a 1-periodic discrete evolution family, then it obeys the relation $W(l,r) = W(l-r,0)$. Let*

$$S(n) := \{W(n,0) : n \in \mathbb{Z}_+\}.$$

Then clearly $\mathbf{S} = \{S(n)\}_{n\in\mathbb{Z}_+}$ become a discrete semigroup.

The same results for the autonomous version of Theorem 3.1 and Theorem 2.1 are presented as a corollaries.

Corollary 3.1 *Let $\mathbf{S} = \{S(n)\}_{n\in\mathbb{Z}_+}$ be a discrete semigroup and $\mathcal{S} := \{\mathcal{S}(j)\}_{j\in\mathbb{Z}_+}$, be the evolution semigroup related to $\mathbf{S}$ on $E_p(\mathbb{Z}_+, Y)$, where $p = 0$ or $1 \leq p \leq \infty$. The following five statements are equivalent:*

- **(i)** *The semigroup $\mathbf{S}$ is UES;*
- **(ii)** *The associated evolution semigroup $\mathcal{S}$ is UES;*
- **(iii)** *The generator $\mathcal{S}(1) - I$ of $\mathcal{S}$ is invertible;*
- **(iv)** *The discrete semigroup $\mathbf{S}$ is $E_p(\mathbb{Z}_+, Y)$-EA.*
- **(v)** *The system (A, b) with constant matrix, is Hyers-Ulam stable.*

Using Remark 3.1 and Theorem 3.2, the proof can be easily obtained.

Corollary 3.2 *Let $\mathbf{S} = \{S(n)\}_{n\in\mathbb{Z}_+}$ be a discrete semigroup and $\mathcal{S} := \{\mathcal{S}(r)\}_{r\in\mathbb{Z}_+}$, be the evolution semigroup related to $\mathbf{S}$ on $l^\infty(\mathbb{Z}_+, Y)$. The following statements are equivalent:*

- **(i)** *The semigroup $\mathbf{S}$ is UES;*
- **(ii)** *The discrete semigroup $\mathbf{S}$ is $l^\infty(\mathbb{Z}_+, Y)$-EA.*

The proof could be obtained by applying Remark 3.1 and Theorem 2.1.

4. Examples

In this section we will provide examples which will illustrate our main result, Theorem 3.1.

Example 1: Let $\{A_r\}_{r\in\mathbb{Z}_+}$ be a family of bounded linear operators on Y which is UB, i.e. $\sup_{r\in\mathbb{Z}_+}||A_r|| < \infty$. The solutions of the discrete Cauchy problems $(A_r, 0, b)$ and $(A_r, f_r, 0)$,

$$\begin{cases} G_{r+1} = A_r G_r, \quad r \in \mathbb{Z}_+, \quad r \geq k \\ G_0 = b, \qquad\qquad\qquad\qquad (A_r, 0, b) \end{cases}$$

and

$$\begin{cases} G_{r+1} = A_r G_r + f_{r+1}, \quad r \in \mathbb{Z}_+ \\ G_0 = b \qquad\qquad\qquad\qquad (A_r, f_r, b) \end{cases}$$

are $x_r = U(r,k)b$ and $y_r = \sum_{k=0}^{r} U(r,k)f(k)$, respectively. Here $U(r,k) := A_{r-1} \cdots A_k$ when $r > k$. By Theorem 2.1, the following statements are equivalent:

- For each $b \in Y$ the solution of $(A_r, 0, b)$ goes to zero exponentially or, equivalently, there are positive real numbers K and ν such that

$$\sup_{n>k}[Ke^{\nu(n-k)}||A_{n-1}\cdots A_k||] < \infty;$$

- For each $f_r \in l^\infty(\mathbb{Z}_+, Y)$ the solution of (A_r, f_r, b) belongs to $l^\infty(\mathbb{Z}_+, Y)$.

- The system $(A_r, 0, b)$ is Hyers-Ulam stable.

The next example gives the autonomous version of the above example, which is a concrete application for Corollary 3.2.

Example 2: Let A be a bounded linear operator on Y. The solutions of the discrete Cauchy problems, $(A, 0, b)$ and $(A, f_r, 0)$

$$\begin{cases} G_{r+1} = AG_r, \quad r \in \mathbb{Z}_+, \quad r \geq k \\ G_0 = b \end{cases} \qquad (A, 0, b)$$

and

$$\begin{cases} G_{r+1} = AG_r + f_{r+1}, \quad r \in \mathbb{Z}_+ \\ G_0 = b, \end{cases} \qquad (A, f_r, b)$$

are $G_r = S(r-k)b$ and $G_r = \sum_{k=0}^{r} S(r-k)f(k)$, respectively. Here $S(k) := A^k$.

By Corollary 3.2, the following statements are equivalent:

- For each $b \in Y$ the solution of $(A, 0, b)$ goes to zero exponentially or, equivalently, there are two positive real numbers K and ν such that

$$\|S(r)x\| \leq Ke^{-\nu r}\|x\| \text{ for all } x \in Y .$$

- For each $f_r \in l^\infty(\mathbb{Z}_+, Y)$ the solution of (A, f_r, b) belongs to $l^\infty(\mathbb{Z}_+, Y)$;

- The system (A,0,b) is Hyers-Ulam stable.

5. Conclusion

In this paper we provide a relation between Hyers-Ulam stability, uniform exponential stability and exact admissibility on the spaces E_p where $p = 0$ or $1 \leq p \leq \infty$. These results are new in a sense that it is proved via of evolution semigroup and also use the idea of exact admissibility. We also provide some examples which clearly show the applications of our theoretical work.

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