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A further result related to Brück conjecture and linear differential polynomial

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ABSTRACT: In connection to the conjecture of R. Brück we improve a uniqueness problem for entire function that share a polynomial with linear differential polynomial.

Key Words: Brück conjecture, entire function, polynomial sharing, uniqueness.

Contents

1. Introduction, Definitions and Results

For an entire function f, $M(r,f) = \max_{|z|=r} |f(z)|$ denotes the *maximum modulus function* of f. Then the *order* $\sigma(f)$ and *lower order* $\lambda(f)$ of f are defined respectively by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \quad \text{and} \quad \lambda(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Also the *hyper order* $\sigma_2(f)$ and *lower hyper order* $\lambda_2(f)$ of f are defined respectively by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r} \quad \text{and} \quad \lambda_2(f) = \liminf_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}.$$

Let f, g and a be entire functions in the open complex plane $\mathbb{C}$. If $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, then we say that f and g share the function a CM (counting multiplicities). If, in particular, a is a constant, then we say that f and g share the value a CM.

In 1977 L. A Rubel and C. C. Yang [10] were the first to consider the relation between an entire function f and its first derivative f' when they share two finite values CM and proved the following result.

Theorem 1.1 [10] *Let f be a nonconstant entire function and a, b be two distinct complex numbers. If f and $f^{(1)}$ share the values a, b CM, then $f \equiv f^{(1)}$.*

This work of Rubel and Yang inspired a lot of researchers and initiated a stream of research on a new branch of uniqueness theory. In this direction, in 1996 R. Brück [2] proposed the following conjecture.
Brück's Conjecture: Let f be a nonconstant entire function such that $\sigma_2(f) < \infty$ and $\sigma_2(f) \notin \mathbb{N}$. If f and $f^{(1)}$ share a finite value a CM, then $f^{(1)} - a = c(f-a)$, where c is a nonzero constant.

The conjecture for $a = 0$, Brück himself resolved it, but the case $a \neq 0$ is not completely resolved in its full generality.

For an entire function of finite order, G. G. Gundersen and L. Z. Yang [5] generalised the conjecture in the following manner.

Theorem 1.2 [5] *Let f be a nonconstant entire function of finite order. If f and $f^{(1)}$ share one finite value a CM, then $f^{(1)} - a = c(f-a)$ for some nonzero constant c.*

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For an entire function of finite order, L. Z. Yang [12] and J. P. Wang [11] resolved and generalised Brück conjecture for higher order derivatives and proved the following results.

Theorem 1.3 [12] *Let f be a nonconstant entire function of finite order. If f and $f^{(k)}$ share one finite value a CM, then $f^{(k)} - a = c(f - a)$ for some nonzero constant c and $k(\geq 1)$ is an integer.*

Theorem 1.4 [11] *Let f be a nonconstant entire function of finite order and a be a nonconstant polynomial. If f and $f^{(k)}$ share a CM, then $f^{(k)} - a = c(f - a)$ for some nonzero constant c and $k(\geq 1)$ is an integer.*

Afterwards Z. X. Chen and K. H. Shon [3] and I. Lahiri and S. Das [6] extended Theorem 1.2 to a class of entire functions of unrestricted order and proved the following theorems.

Theorem 1.5 [3] *Let f be a nonconstant entire function with $\sigma_2(f) < \frac{1}{2}$. If f and $f^{(1)}$ share a finite value a CM, then $f^{(1)} - a = c(f - a)$, where c is a nonzero constant.*

Theorem 1.6 [6] *Let f be a nonconstant entire function with $\lambda_2(f) < \frac{1}{2}$ and $\sigma_2(f) < \infty$. Suppose that $a = a(z)$ is a polynomial. If f and $f^{(k)}$ share a CM, then $f^{(k)} - a = c(f - a)$, where c is a nonzero constant and $k(\geq 1)$ is an integer.*

In the paper, the aim is to improve and generalise the above theorems by considering the following problems:

(i) Replacement of shared value by shared polynomial;

(ii) Replacement of higher derivatives by linear differential polynomial with polynomial coefficients.

Let f be an entire function. We consider a differential polynomial of the form

$$L(f) = a_p(z)f^{(p)} + a_{p-1}(z)f^{(p-1)} + \cdots + a_1(z)f^{(1)} + a_0(z)f, \tag{1.1}$$

where p is a positive integer and $a_0(z), a_1(z), \ldots, a_p(z)$ are polynomials.

Further, let $\chi = 1 + \max\limits_{0\leq j\leq p} \chi_j$, where $\chi_j = \max\left\{\frac{\deg a_j - \deg a_p}{p-j}, 0\right\}$.

We now state the main result of the paper.

Theorem 1.7 *Let f be a nonconstant entire function such that $\sigma(f) \notin [1, \chi]$, $\lambda_2(f) < \frac{1}{2}$ and $\sigma_2(f) < \infty$. Suppose that $L(f)$ is given by (1.1).*

If f and $L(f)$ share a polynomial $a = a(z)$ CM, then $L(f) - a = c(f - a)$, where c is a nonzero constant.

If all $a_i(z)$'s $1 \leq i \leq p$ are constants, then we obtain the following corollary.

Corollary 1.1 *Let f be a nonconstant entire function such that $\sigma(f) \neq 1, \lambda_2(f) < \frac{1}{2}$ and $\sigma_2(f) < \infty$. If f and $L(f)$ share a polynomial $a = a(z)$ CM, then $L(f) - a = c(f - a)$, where c is a nonzero constant.*

The following examples show that the condition $\sigma(f) < 1$ and $\sigma(f) > \chi$ in Theorem 1.7 is best possible.

Example 1.1 *Let $f(z) = e^z + z, a(z) = z$ and $L(f) = f^{(2)} - 2f^{(1)} + f$. Then $\chi = 1 + \max\limits_{1\leq j\leq 3}\{\chi_j, 0\} = 1$ and f and $L(f)$ share z CM but $L(f) - z = -2e^{-z}(f - z)$, where f satisfies $\sigma(f) = 1$.*

Example 1.2 [9] *Let $f = e^{-\frac{z^2}{2}} + z^2, a(z) = z^2$ and $P(f) = \frac{1}{3}f^{(2)} + \frac{z}{3}f^{(1)} + \frac{1}{3}f$. Then $\chi = 1 + \max\limits_{1\leq j\leq 3}\{\chi_j, 0\} = 2$ and f and $L(f)$ share z^2 CM but $P(f) - z^2 = \frac{2}{3}e^{\frac{z^2}{2}}(f - z^2)$, where f satisfies $\sigma(f) = 2$.*

2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1 {p.5 [7]} *Let $g : (0, +\infty) \to \mathbb{R}$ and $h : (0, +\infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite logarithmic measure. Then for any $\delta > 1$, there exists $R > 0$ such that $g(r) \leq h(r^\delta)$ holds for $r > R$.*

Lemma 2.2 {p.9 [7]} *Let $P(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0 (b_n \neq 0)$ be a polynomial of degree n. Then for every $\epsilon (> 0)$ there exists $R(> 0)$ such that for all $|z| = r > R$ we get*

$$(1 - \epsilon)|b_n| r^n \leq |P(z)| \leq (1 + \epsilon)|b_n| r^n.$$

Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ be an entire function. Then the power series $\sum\limits_{n=0}^{\infty} |a_n| r^n$ converges for every $r > 0$ and so for any given $r > 0$, we have $\lim\limits_{r\to\infty} |a_n| r^n = 0$. Hence the maximum term $\mu(r, f) = \max\limits_{n\geq 0} |a_n| r^n$ is well defined. Also we define $\nu(r, f)$, the *central index* of f, as the greatest exponent m such that $\mu(r, f) = |a_m| r^m$.

Lemma 2.3 [8] *For an entire function f*

$$\mu(f) = \liminf_{r\to\infty} \frac{\log \nu(r, f)}{\log r} \quad \textit{and} \quad \mu_2(f) = \liminf_{r\to\infty} \frac{\log \log \nu(r, f)}{\log r}.$$

Lemma 2.4 [4] *For an entire function f*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log \nu(r, f)}{\log r} \quad \textit{and} \quad \sigma_2(f) = \limsup_{r\to\infty} \frac{\log \log \nu(r, f)}{\log r}.$$

Lemma 2.5 {p.51 [7]} *Let f be a transcendental entire function. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure such that for $|z| = r \notin [0, 1] \cup E$ and $|f(z)| = M(r, f)$ we get*

$$\frac{f^{(k)}(z)}{f(z)} = (1 + o(1)) \left\{ \frac{\nu(r, f)}{z} \right\}^k$$

for $k = 1, 2, 3, \ldots, n$, where n is a positive integer.

Let $h(z)$ be a nonconstant function subharmonic in the open complex plane $\mathbb{C}$ and let

$$A(r) = A(r, h) = \inf_{|z|=r} h(z) \quad \text{and} \quad B(r) = B(r, h) = \sup_{|z|=r} h(z).$$

Then the *order* $\sigma(h)$ and the *lower order* $\lambda(h)$ of h are defined respectively by

$$\sigma(h) = \limsup_{r\to\infty} \frac{\log B(r, h)}{\log r}$$

and

$$\lambda(h) = \liminf_{r\to\infty} \frac{\log B(r, h)}{\log r}.$$

The *upper logarithmic density* and the *lower logarithmic density* of $E \subset [1, \infty)$ are respectively defined by

$$\overline{\operatorname{logdens}}(E) = \limsup_{r\to\infty} \frac{\int_1^r \frac{\chi_E(t)}{t} dt}{\log r}$$

and

$$\overline{\text{logdens}}(E) = \liminf_{r\to\infty} \frac{\int_1^r \frac{\chi_E(t)}{t}dt}{\log r},$$

where χ_E be the *characteristic function* of E.

The quantity $\lim_{r\to\infty}\int_1^r \frac{\chi_E(t)}{t}dt$ defines the *logarithmic measure* of E. It is easy to note that if $\overline{\text{logdens}}(E) > 0$, then E has infinite logarithmic measure.

Lemma 2.6 [1] *Let $h(z)$ be a nonconstant subharmonic function in the open complex plane $\mathbb{C}$ of lower order $\lambda, 0 \le \lambda < 1$. If $\lambda < \beta < 1$, then*

$$\overline{\text{logdens}}\{r : A(r) > (\cos\beta\pi)B(r)\} \ge 1 - \frac{\lambda}{\beta},$$

where $A(r) = \inf_{|z|=r} h(z)$ and $B(r) = \sup_{|z|=r} h(z)$.

3. Proof of Theorem 1.7

Proof: Since f and $L(f)$ share a CM, there exists an entire function A such that

$$\frac{L(f)-a}{f-a} = e^A. \tag{3.1}$$

If A is a constant, then the result holds clearly. So we suppose that A is a nonconstant entire function and consider the following two cases.

Case 1. Let $\sigma(f) < \infty$. Then from (3.1) we get that A is a polynomial.

Since $\sigma(f) \notin (1,\chi)$, then either $\sigma(f) < 1$ or $\sigma(f) > \chi$.

If $\sigma(f) < 1$, then (3.1) implies that A is a constant. So $\sigma(f) > \chi \ge 1$ and therefore f is a transcendental entire function.

Now we suppose that A is a nonconstant polynomial.

Again, for any z with $|f(z)| = M(r,f)$ we get by Lemma 2.2 (choosing $\epsilon = \frac{1}{2}$)

$$\left|\frac{a(z)}{f(z)}\right| \le \frac{M(r,a)}{M(r,f)} \le \frac{\frac{3}{2}|\alpha| r^{\deg a}}{M(r,f)} \to 0 \tag{3.2}$$

as $r \to \infty$, where α is the leading coefficient of the polynomial $a(z)$.

Now by Lemma 2.5 there exists $E \subset [1,\infty)$ with finite logarithmic measure such that for $|z| = r \notin E \cup [0,1]$ and $|f(z)| = M(r,f)$ we get

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^j (1+o(1)), \tag{3.3}$$

for $j = 1,2,\dots p$, where p is a positive integer.

Now for all z with $|z| = r \notin E\cup[0,1]$ and $|f(z)| = M(r,f)$ we get by (3.3)

$$\begin{aligned}\frac{L(f)}{f} &= a_0(z) + \sum_{j=1}^{p} a_j(z)\left(\frac{\nu(r,f)}{z}\right)^j (1+o(1))\\ &= a_0(z) + \frac{a_p(z)}{z^p}\left\{\nu(r,f)^p + \sum_{j=1}^{p-1}\frac{a_j(z)}{a_p(z)} z^{p-j}(\nu(r,f))^j\right\}(1+o(1)).\end{aligned} \tag{3.4}$$

Let $d_j = \deg a_j$ for $j = 1,2,\dots,p$. Since $\sigma = \sigma(f) > 1 + \frac{d_j - d_p}{p-j}$ for $j = 1,2,\dots,p-1$, we can choose an ϵ such that

$$0 < \epsilon < \min_{1\le j<p} \frac{1}{2}\left\{(\sigma-1) + \frac{d_j - d_p}{p-j}\right\}.$$

Since $\sigma > 1 + \frac{d_j - d_p}{p-j} + \epsilon$ for $1 \leq j < p$, we get by Lemma 2.4, we get for all sufficiently large values of r

$$\nu(r,f) > r^{\left\{1+\frac{d_j-d_p}{p-j}+\epsilon\right\}}, \tag{3.5}$$

for $j = 1, 2, \ldots, p-1$.

So by Lemma 2.2 and (3.5) we get for all sufficiently large values of r and $j = 1, 2, \ldots, p-1$

$$\begin{aligned} & \frac{\left|\frac{a_j(z)}{a_p(z)} z^{p-j} (\nu(r,f))^j\right|}{(\nu(r,f))^p} \\ \leq \quad & M_1 r^{d_j - d_p + p - j} (\nu(r,f))^{-(p-j)} \\ < \quad & M_1 r^{\{d_j - d_p + p - j - p + j - d_j + d_p - \epsilon(p-j)\}} \\ = \quad & M_1 r^{-\epsilon(p-j)} \to 0 \text{ as } |z| = r \to \infty, \end{aligned}$$

where $M_1 (> 0)$ is a suitable constant.

Hence

$$\frac{a_j(z)}{a_p(z)} z^{p-j} (\nu(r,f))^j = o\left(\nu(r,f)^p\right), \tag{3.6}$$

as $r \to \infty$.

By the similar argument, we can show that

$$a_0(z) = o\left(a_p(z) \left(\frac{\nu(r,f)}{z}\right)^p\right), \tag{3.7}$$

as $r \to \infty$.

So for sufficiently large $|z| = r \notin E \cup [0,1]$ with $|f(z)| = M(r,f)$ we get by (3.4), (3.6) and (3.7)

$$\frac{L(f(z))}{f(z)} = a_p(z) \left(\frac{\nu(r,f)}{z}\right)^p (1 + o(1)). \tag{3.8}$$

From (3.1) we get

$$e^A = \frac{\frac{L(f)}{f} - \frac{a}{f}}{1 - \frac{a}{f}}. \tag{3.9}$$

Now for all z with $|z| = r \notin E \cup [0,1]$ and $|f(z)| = M(r,f)$, we get by (3.2), (3.8) and (3.9)

$$e^A = a_p(z) \left(\frac{\nu(r,f)}{z}\right)^p (1 + o(1)). \tag{3.10}$$

Now from (3.10) we get for all large $|z| = r \notin [0,1] \cup E$ with $|f(z)| = M(r,f)$

$$\begin{aligned} |A(z)| &= |\log e^{A(z)}| \\ &= \left|\log a_p(z) \left(\frac{\nu(r,f)}{z}\right)^p\right| + o(1) \\ &= |\log a_p(z) + p \log \nu(r,f) - p \log z| + o(1) \\ &\leq d_p \log r + p \log \nu(r,f) + p \log r + 8p\pi \\ &< \{d_p + 2p\sigma + p\} \log r + 8p\pi. \end{aligned} \tag{3.11}$$

Also by Lemma 2.2 (choosing $\epsilon = \frac{1}{2}$) we obtain for all large $|z| = r$

$$\frac{1}{2} |\alpha| r^{\deg A} \leq |A(z)|, \tag{3.12}$$

where α is the leading coefficient of A.

Now the equations (3.11) and (3.12) together imply $\deg A = 0$ and so A is a constant, which is a contradiction.

Case 2. Let $\sigma(f) = \infty$. We now consider the following two subcases.

Subcase 2.1. Let A be a nonconstant polynomial. Then from (3.10) we get for all large $|z| = r \notin [0,1] \cup E$ with $|f(z)| = M(r, f)$

$$|A(z)| \leq d_p \log r + p \log \nu(r, f) + p \log r + 8p\pi. \tag{3.13}$$

Then from (3.12) and (3.13) we obtain for all large $|z| = r \notin [0,1] \cup E$ with $|f(z)| = M(r, f)$

$$\frac{1}{2}|\alpha| r^{\deg A} \leq d_p \log r + p \log \nu(r, f) + p \log r + 8p\pi. \tag{3.14}$$

Hence by Lemma 2.1 for given δ, $1 < \delta < \frac{3}{2}$ and (3.14), we get for all large values of r

$$\frac{1}{2}|\alpha| r^{\deg A} \leq p \log \nu(r^\delta, f) + (p + d_p)\delta \log r + 8p\pi$$

and so

$$r^{\deg A}\left(\frac{1}{2}|\alpha| - \frac{(p + d_p)\delta \log r}{r^{\deg A}}\right) \leq p \log \nu(r^\delta, f) + 8p\pi.$$

This implies $\deg A \leq \delta\lambda_2(f) < \frac{\delta}{2} < \frac{3}{4} < 1$, a contradiction. Therefore A is a constant.

Subcase 2.2. Let A be a transcendental entire function. Since for an entire function $A(z)$, $h(z) = \log|A(z)|$ is a subharmonic function in $\mathbb{C}$, and also from (3.1) we get $\lambda(h) = \lambda_2(A) \leq \lambda_2(f) < \frac{1}{2}$.

Suppose that $H = \{r : A(r) > (\cos\beta\pi)B(r)\}$, where $A(r) = \inf_{|z|=r} \log|f(z)|$, $B(r) = \sup_{|z|=r} \log|f(z)|$ and $\beta \in \left(\lambda_2(A), \frac{1}{2}\right)$.

Then by Lemma 2.6 we see that $\overline{\text{logdens}}\, H > 0$, i.e., H has infinite logarithmic measure. Also by Lemma 2.2 for $|z| = r \in H \backslash \{[0,1] \cup E\}$ with $|f(z)| = M(r, f)$ we get

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu(r, f)}{z}\right)^k (1 + o(1)), \tag{3.15}$$

where k is a positive integer.

Now by (3.2), (3.10) and (3.15) we get for all large $|z| = r \in H \backslash \{[0,1] \cup E\}$ with $|f(z)| = M(r, f)$

$$e^{A(z)} = a_p(z)\left(\frac{\nu(r, f)}{z}\right)^p (1 + o(1))$$

and so

$$\begin{aligned} |A(z)| &= \left|\log e^{A(z)}\right| \\ &= \left|\log a_p(z)\left(\frac{\nu(r, f)}{z}\right)^p\right| + o(1) \\ &= |d_p \log z + p \log \nu(r, f) - p \log z| + o(1) \\ &\leq d_p \log r + p \log \nu(r, f) + p \log r + 8p\pi \\ &< 2pr^{\sigma_2(f)+1}. \end{aligned} \tag{3.16}$$

Now by Lemma 2.1, there exists a constant c, $0 < c < 1$ such that for all z satisfying $|z| = r \in H \backslash \{[0,1] \cup E\}$ with $|f(z)| = M(r, f)$, we have

$$(M(r, A))^c < |A(z)|. \tag{3.17}$$

Now by (3.16) and (3.17), we get

$$\frac{(M(r,A))^c}{r^{\sigma_2(f)+1}} < 2p. \tag{3.18}$$

This is impossible because A is transcendental and so $\frac{(M(r,A))^c}{r^{\sigma_2(f)+1}} \to \infty$ as $r \to \infty$. This proves the theorem. □

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References

1. P. D. Barry, *On a theorem of Kjellberg*, Quart. J. Math. Oxford (2), 15(1964), 179-191.
2. R. Brück, *On entire functions which share one value CM with their first derivative*, Result. Math., 30(1996), 21-24.
3. Z. X. Chen and K. H. Shon, *On conjecture of R. Brück concerning the entire function sharing one value CM with its derivative*, Taiwanese J. Math., 8(2)(2004), 235-244.
4. Z. X. Chen and C. C. Yang, *Some further results on the zeros and growths of entire solutions of second order linear differential equations*, Kodai Math. J., 22(1999), 273-285.
5. G. G. Gundersen and L. Z. Yang, *Entire functions that share one value with one or two of their derivatives*, J. Math. Anal. Appl., 223(1998), 88-95.
6. I. Lahiri and S. Das, *A note on a conjecture of R. Brück*, Appl. Math. E-Note., 21(2021), 152-156.
7. I. Laine, *Nevanlinna Theory and Complex Differential Equations*, De Gruyter, Berlin, New York (1993).
8. X. M. Li and H. X. Yi, *Some results on the regular solutions of a linear differential equation*, Comp. Math. Appl., 56(2008), 2210-2221.
9. Z. Mao, *Uniqueness theorems on entire functions and their linear differential polynomials*, Results Math., 55(2009), 447-456.
10. L. A. Rubel and C. C. Yang, *Values shared by an entire function and its derivative*, Lecture Notes in Math., Vol. 599, Springer-Verlag, Berlin (1977), 101-103.
11. J. P. Wang, *Entire functions that share a polynomial with one of their derivatives*, Kodai Math. J., 27(2004), 144-151.
12. L. Z. Yang, *Solution of a differential equation and its applications*, Kodai Math. J., 22(1999), 458-464.

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