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# Certain additive mappings on semiprime rings and their characterization

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ABSTRACT: The objective of this article is to show that an additive mapping $\mathcal{H}:\mathcal{A}\to\mathcal{A}$ is a $\phi$-centralizer on $\mathcal{A}$ if it satisfies one of the following identities:

(*i*) $\mathcal{H}(a_1^p a_2^p + a_2^p a_1^p) = \mathcal{H}(a_1^p)\phi(a_2^p) + \phi(a_2^p)\mathcal{H}(a_1^p)$

(*ii*) $2\mathcal{H}(a_1^p a_2^p) = \mathcal{H}(a_1^p)\phi(a_2^p) + \phi(a_2^p)\mathcal{H}(a_1^p)$

for all $a_1, a_2 \in \mathcal{A}$, where $p \geq 1$ is a fixed integer, $\phi$ is a surjective endomorphism on a $p!$-torsion free semiprime ring $\mathcal{A}$. Some extensions of these results are also presented in the setting of ring with involution " $\star$ ". Furthermore, we also give the verity of examples that illustrate and enrich the subject matter.

Key Words: Semiprime ring, $\phi$-centralizer, $\phi^\star$-centralizer and algebraic identities.

## Contents



## 1. Introduction

Throughout $\mathcal{A}$ will represent an associative ring with identity $e$. A ring $\mathcal{A}$ is termed as $p$-torsion free, if $pa = 0$ implies $a = 0$ for all $a \in \mathcal{A}$, where $p > 1$ is a fixed integer. Note that $\mathcal{A}$ is known as a prime ring if $a_1\mathcal{A}a_2 = \{0\}$ implies $a_1 = 0$ or $a_2 = 0$, and is semiprime ring if $a\mathcal{A}a = \{0\}$ entails $a = 0$. If an additive mapping $\mathcal{G}:\mathcal{A}\to\mathcal{A}$ holds $\mathcal{G}(a_1a_2) = \mathcal{G}(a_1)a_2 + a_1\mathcal{G}(a_2)$ for all pairs $a_1, a_2 \in \mathcal{A}$, then it is termed a derivation, and if $\mathcal{G}(a_1^2) = \mathcal{G}(a_1)a_1 + a_1\mathcal{G}(a_1)$ is fulfilled for all $a_1 \in \mathcal{A}$, $\mathcal{G}$ is said to be a Jordan derivation. A mapping $\mathcal{G}$ is known as inner derivation if $\mathcal{G}(a_1) = aa_1 - a_1a$ for all $a_1$ in $\mathcal{A}$ and $a \in \mathcal{A}$ is fixed. Every derivation is a Jordan derivation, although the converse is not always true. A classical Herstein conclusion [9] asserts that every Jordan derivation with such a characteristic other than two is a derivation on a prime ring. If the second part from the right hand side in the definition of derivation and Jordan derivation is zero, then derivation and Jordan derivation is recognised as a left centralizer and a Jordan left centralizer respectively and if the first part from the right hand side is zero, then derivation and Jordan derivation is known as a right centralizer and Jordan right centralizer respectively. Historically, work on centralizers in Banach algebras was started by Helgosen [8]. Later, on commutative Banach algebras, Wang [15] investigated the idea of centralizers. Then, Johnson [10] introduced the concept of centralizers in rings as follows. Let $\mathcal{H}:\mathcal{A}\to\mathcal{A}$ be an additive mapping. $\mathcal{H}$ is known as a right (respectively left) centralizer if $\mathcal{H}(a_1a_2) = a_1\mathcal{H}(a_2)$ $\big($respectively $\mathcal{H}(a_1a_2) = \mathcal{H}(a_1)a_2\big)$ holds for all pairs $a_1, a_2 \in \mathcal{A}$ and is recognised as a Jordan right (respectively Jordan left) centralizer if $\mathcal{H}(a_1^2) = a_1\mathcal{H}(a_1)$ $\big($respectively $\mathcal{H}(a_1^2) = \mathcal{H}(a_1)a_1\big)$ holds for all $a_1 \in \mathcal{A}$. Any mapping which is right as well as left centralizer (Jordan) centralizer is called (Jordan) centralizer. Centralizers are often referred to as multipliers in this context (refer [16]). Additionally, Johnson presented the continuity of centralizers on Banach algebras and studied centralizers on topological algebras. (View this [11,12]). Some recent work related to the centralizer are present in [2,3,13]. Following Theorem 2.3.2 in [5], if $\mathcal{E}_\mathcal{C}$ is an extended centroid on a semiprime ring $\mathcal{A}$ and $\mathcal{H}:\mathcal{A}\to\mathcal{A}$ is a centralizer, then for all $a_1 \in \mathcal{A}$, $\mathcal{H}(a_1) = \beta a_1$, where $\beta$ is a fixed element in $\mathcal{E}_\mathcal{C}$. Zalar [17] demonstrated that every Jordan right centralizer on a 2-torsion free semiprime ring is a right centralizer. He achieved a similar outcome with Jordan left centralizer. Later,

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Vukman [14] has established an identical outcome that an additive mapping $\mathcal{H}:\mathcal{A}\to\mathcal{A}$ is a centralizer if $\mathcal{H}$ satisfies an algebraic equation $2\mathcal{H}(a_1^2)=\mathcal{H}(a_1)a_1+a_1\mathcal{H}(a_1)$ for all $a_1\in\mathcal{A}$, where $\mathcal{A}$ is 2-torsion free semiprime ring. An extension of the above result is given in [7]. Recently, E. Albas [1] introduced the following definitions, which are generalizations of the definitions of centralizer and Jordan centralizer. Let $\mathcal{A}$ be a ring, and $\phi$ be an endomorphism of $\mathcal{A}$. An additive mapping $\mathcal{H}:\mathcal{A}\to\mathcal{A}$ is recognised as a Jordan $\phi$-centralizer of $\mathcal{A}$ if $\mathcal{H}$ satisfies $\mathcal{H}(a_1a_2+a_2a_1)=\mathcal{H}(a_1)\phi(a_2)+\phi(a_2)\mathcal{H}(a_1)=\mathcal{H}(a_2)\phi(a_1)+\phi(a_1)\mathcal{H}(a_2)$ for all $a_1,a_2\in\mathcal{A}$. Equivalently, an additive mapping $\mathcal{H}:\mathcal{A}\to\mathcal{A}$ is termed as a right (respectively left) $\phi$-centralizer of $\mathcal{A}$ if $\mathcal{H}(a_1a_2)=\phi(a_1)\mathcal{H}(a_2)$ (respectively $\mathcal{H}(a_1a_2)=\mathcal{H}(a_1)\phi(a_2)$) for all $a_1,a_2\in\mathcal{A}$. If $\mathcal{H}$ is a left and right $\phi$-centralizer then we call $\mathcal{H}$ as an $\phi$-centralizer. In [1], Albas proved, under some conditions, every Jordan $\phi$-centralizer is a $\phi$-centralizer on a 2-torsion free semiprime ring $\mathcal{A}$. Motivated by such results, the authors offered some extensions of the above-mentioned results in the current work. The following results are required to establish the proof of the key theorems:

**Lemma 1.1** **([6, Lemma 1])** *Suppose that $\mathcal{A}$ is a $p!$-torsion free semiprime ring. If $\sum_{i=1}^{p}\lambda^i a_i=0$ for all $a_1,a_2,\cdots,a_p\in\mathcal{A}$ and $\lambda=1,2,\cdots,p$, then $a_i=0$ for all $i$.*

**Lemma 1.2** **([7, Theorem 1.2])** *Suppose that $\mathcal{A}$ is any 2 torsion free semiprime ring and $\mathcal{H}:\mathcal{A}\to\mathcal{A}$ is an additive mapping which satisfies the algebraic identity $2\mathcal{H}(a_1^2)=\mathcal{H}(a_1)\phi(a_1)+\phi(a_1)\mathcal{H}(a_1)$ for all $a_1\in\mathcal{A}$, where $\phi$ is a surjective endomorphism on $\mathcal{A}$. then $\mathcal{H}$ is a $\phi$-centralizer on $\mathcal{A}$.*

## 2. $\phi$-centralizer

**Theorem 2.1** *Suppose that $\mathcal{A}$ is any $p!$-torsion free semiprime ring with identity $e$. If an additive mapping $\mathcal{H}:\mathcal{A}\to\mathcal{A}$ satisfies $\mathcal{H}(a_1^pa_2^p+a_2^pa_1^p)=\mathcal{H}(a_1^p)\phi(a_2^p)+\phi(a_2^p)\mathcal{H}(a_1^p)$ for all $a_1,a_2\in\mathcal{A}$, where $\phi$ is a surjective endomorphism on $\mathcal{A}$ and $p\geq 1$ is a fixed integer, then $\mathcal{H}$ is a $\phi$-centralizer on $\mathcal{A}$.*

**Proof:** We have

$$\mathcal{H}(a_1^pa_2^p+a_2^pa_1^p)=\mathcal{H}(a_1^p)\phi(a_2^p)+\phi(a_2^p)\mathcal{H}(a_1^p) \text{ for all } a_1,a_2\in\mathcal{A}. \tag{2.1}$$

Replacing $a_1$ by $e$ (the identity element of $\mathcal{A}$) in the above equation, we get

$$2\mathcal{H}(a_2^p)=\mathcal{H}(e)\phi(a_2^p)+\phi(a_2^p)\mathcal{H}(e) \text{ for all } a_2\in\mathcal{A}. \tag{2.2}$$

Again, replace $a_2$ by $a_2+e$ in the above equation to get

$$\sum_{i=0}^{p}\binom{p}{i}[2\mathcal{H}(a_2^{p-i})-\mathcal{H}(e)\phi(a_2^{p-i})-\phi(a_2^{p-i})\mathcal{H}(e)]=0 \text{ for all } a_2\in\mathcal{A}.$$

Replace $a_2$ by $ka_2$ to obtain

$$\sum_{i=0}^{p}k^{p-i}\binom{p}{i}[2\mathcal{H}(a_2^{p-i})-\mathcal{H}(e)\phi(a_2^{p-i})-\phi(a_2^{p-i})\mathcal{H}(e)]=0 \text{ for all } a_2\in\mathcal{A}, k\in\mathbb{Z}^+.$$

Using Lemma 1.1 and for all $a_2\in\mathcal{A}$ and $i=1,2,3,\cdots,p-1$, we get

$$\binom{p}{i}[2\mathcal{H}(a_2^{p-i})-\mathcal{H}(e)\phi(a_2^{p-i})-\phi(a_2^{p-i})\mathcal{H}(e)]=0.$$

Particularly, take $i=p-1$, we obtain

$$p[2\mathcal{H}(a_2)-\mathcal{H}(e)\phi(a_2)-\phi(a_2)\mathcal{H}(e)]=0 \text{ for all } a_2\in\mathcal{A}.$$

We deduce the following from the fact that $\mathcal{A}$ is $p$-torsion free

$$2\mathcal{H}(a_2)=\mathcal{H}(e)\phi(a_2)+\phi(a_2)\mathcal{H}(e) \text{ for all } a_2\in\mathcal{A}. \tag{2.3}$$

Next, replace $a_1$ by $a_1+e$ in (2.1), we obtain

$$\begin{aligned}\tbinom{p}{0}[\mathcal{H}(a_1^pa_2^p+a_2^pa_1^p) &- \mathcal{H}(a_1^p)\phi(a_2^p)-\phi(a_2^p)\mathcal{H}(a_1^p)]\\ &+ \tbinom{p}{1}[\mathcal{H}(a_1^{p-1}a_2^p+a_2^pa_1^{p-1})-\mathcal{H}(a_1^{p-1})\phi(a_2^p)-\phi(a_2^p)\mathcal{H}(a_1^{p-1})]\\ &+ \tbinom{p}{2}[\mathcal{H}(a_1^{p-2}a_2^p+a_2^pa_1^{p-2})-\mathcal{H}(a_1^{p-2})\phi(a_2^p)-\phi(a_2^p)\mathcal{H}(a_1^{p-2})]+\cdots\\ &+ \tbinom{p}{p-1}[\mathcal{H}(a_1a_2^p+a_2^pa_1)-\mathcal{H}(a_2^p)\phi(a_1)-\phi(a_2^p)\mathcal{H}(a_1)]\\ &+ \tbinom{p}{p}[\mathcal{H}(2a_2^p)-\mathcal{H}(e)\phi(a_2^p)-\phi(a_2^p)\mathcal{H}(e)]=0.\end{aligned}$$

Using (2.1) and (2.2), we have

$$\begin{aligned}\tbinom{p}{1}[\mathcal{H}(a_1^{p-1}a_2^p+a_2^pa_1^{p-1}) &- \mathcal{H}(a_1^{p-1})\phi(a_2^p)-\phi(a_2^p)\mathcal{H}(a_1^{p-1})]\\ &+ \tbinom{p}{2}[\mathcal{H}(a_1^{p-2}a_2^p+a_2^pa_1^{p-2})-\mathcal{H}(a_1^{p-2})\phi(a_2^p)-\phi(a_2^p)\mathcal{H}(a_1^{p-2})]+\cdots\\ &+ \tbinom{p}{p-1}[\mathcal{H}(a_1a_2^p+a_2^pa_1)-\mathcal{H}(a_2^p)\phi(a_1)-\phi(a_2^p)\mathcal{H}(a_1)]=0.\end{aligned}$$

Using Lemma 1.1 after substituting $ka_1$ by $a_1$, we obtain

$$\tbinom{p}{i}[\mathcal{H}(a_1^{p-i}a_2^p+a_2^pa_1^{p-i}) \quad - \quad \mathcal{H}(a_1^{p-i})\phi(a_2^p)-\phi(a_2^p)\mathcal{H}(a_1^{p-i})]=0$$

for all $i=1,2,\cdots,p-1$. In particular put $i=p-1$ and using the fact that $\mathcal{A}$ is $p$-torsion free, we find that

$$\mathcal{H}(a_1a_2^p+a_2^pa_1)=\mathcal{H}(a_1)\phi(a_2^p)+\phi(a_2^p)\mathcal{H}(a_1) \text{ for all } a_1,a_2\in\mathcal{A}. \tag{2.4}$$

Again, replacing $a_2$ by $a_2+e$ in the above equation, and using (2.3), we have

$$\begin{aligned}&\tbinom{p}{1}[\mathcal{H}(a_1a_2^{p-1}+a_2^{p-1}a_1)-\mathcal{H}(a_1)\phi(a_2^{p-1})-\phi(a_2)\mathcal{H}(a_1^{p-1})]\\ &+\tbinom{p}{2}[\mathcal{H}(a_1a_2^{p-2}+a_2^{p-2}a_1)-\mathcal{H}(a_1)\phi(a_2^{p-2})-\phi(a_2)\mathcal{H}(a_1^{p-2})]\\ &+\cdots+\tbinom{p}{p-1}[\mathcal{H}(a_1a_2+a_2a_1)-\mathcal{H}(a_1)\phi(a_2)-\phi(a_2)\mathcal{H}(a_1)]=0.\end{aligned}$$

Replacing $a_2$ by $ka_2$ to arrive at

$$\sum_{i=1}^{p-i}\binom{p}{i}[\mathcal{H}(a_1a_2^{p-i}+a_2^{p-i}a_1)-\mathcal{H}(a_1)\phi(a_2^{p-i})-\phi(a_2)\mathcal{H}(a_1^{p-i})]=0 \text{ for all } a_1,a_2\in\mathcal{A}.$$

Using the same steps as we did earlier, we arrive at $\mathcal{H}(a_1a_2+a_2a_1)=\mathcal{H}(a_1)\phi(a_2)+\phi(a_2)\mathcal{H}(a_1)$ for all $a_1,a_2\in\mathcal{A}$. Replacing $a_2$ by $a_1$ and using Lemma 1.2, We achieve the desired outcome. □

**Theorem 2.2** *Suppose that $\mathcal{A}$ is a $p!$-torsion free semiprime ring with identity $e$. If an additive mapping $\mathcal{H}:\mathcal{A}\to\mathcal{A}$ satisfies an algebraic equation $2\mathcal{H}(a_1^pa_2^p)=\mathcal{H}(a_1^p)\phi(a_2^p)+\phi(a_2^p)\mathcal{H}(a_1^p)$ for all $a_1,a_2\in\mathcal{A}$, where $\phi$ is a surjective endomorphism on $\mathcal{A}$ and $p\geq 1$ is a fixed integer, then $\mathcal{H}$ is $\phi$-centralizer on $\mathcal{A}$.*

**Proof:** Given that

$$2\mathcal{H}(a_1^pa_2^p)=\mathcal{H}(a_1^p)\phi(a_2^p)+\phi(a_2^p)\mathcal{H}(a_1^p) \text{ for all } a_1,a_2\in\mathcal{A}. \tag{2.5}$$

Substitute $a_1$ by $e$ in the above equation to get

$$2\mathcal{H}(a_2^p)=\mathcal{H}(e)\phi(a_2^p)+\phi(a_2^p)\mathcal{H}(e) \text{ for all } a_2\in\mathcal{A}. \tag{2.6}$$

Replace $a_2$ by $a_2+e$ in the above equation to get

$$\sum_{i=0}^{p}\binom{p}{i}[2\mathcal{H}(a_2^{p-i})-\mathcal{H}(e)\phi(a_2^{p-i})-\phi(a_2^{p-i})\mathcal{H}(e)]=0 \text{ for all } a_2\in\mathcal{A}.$$

Replacing $a_2$ by $ka_2$, we obtain

$$\sum_{i=0}^{p} k^{p-i} \binom{p}{i} [2\mathcal{H}(a_2^{p-i}) - \mathcal{H}(e)\phi(a_2^{p-i}) - \phi(a_2^{p-i})\mathcal{H}(e)] = 0 \text{ for all } a_2 \in \mathcal{A}.$$

Use Lemma 1.1, for all $a_2 \in \mathcal{A}$ and for all $i = 1, 2, \cdots, p-1$ to get

$$\binom{p}{i} [2\mathcal{H}(a_2^{p-i}) - \mathcal{H}(e)\phi(a_2^{p-i}) - \phi(a_2^{p-i})\mathcal{H}(e)] = 0.$$

In particular, $i = p-1$ yields that

$$\binom{p}{p-1} [2\mathcal{H}(a_2) - \mathcal{H}(e)\phi(a_2) - \phi(a_2)\mathcal{H}(e)] = 0 \text{ for all } a_2 \in \mathcal{A}.$$

Since $\mathcal{A}$ is $p$-torsion free, then we get that

$$2\mathcal{H}(a_2) = \mathcal{H}(e)\phi(a_2) + \phi(a_2)\mathcal{H}(e) \text{ for all } a_2 \in \mathcal{A}. \tag{2.7}$$

Replacing $a_1$ by $a_1 + e$ in (2.5) and using (2.6) and (2.7), we have

$$\begin{aligned} \tbinom{p}{1}[2\mathcal{H}(a_1^{p-1}a_2^p) &- \mathcal{H}(a_1^{p-1})\phi(a_2^p) - \phi(a_2^p)\mathcal{H}(a_1^{p-1})] \\ &+ \tbinom{p}{2}[2\mathcal{H}(a_1^{p-2}a_2^p) - \mathcal{H}(a_1^{p-2})\phi(a_2^p) - \phi(a_2^p)\mathcal{H}(a_1^{p-2})] + \cdots \\ &+ \tbinom{p}{p-1}[2\mathcal{H}(a_1a_2^p) - \mathcal{H}(a_1)\phi(a_2^p) - \phi(a_2^p)\mathcal{H}(a_1)] = 0. \end{aligned}$$

Replacing $a_1$ by $ka_1$, we obtain

$$\begin{aligned} \tbinom{p}{1}k^{p-1}[2\mathcal{H}(a_1^{p-1}a_2^p) &- \mathcal{H}(a_1^{p-1})\phi(a_2^p) - \phi(a_2^p)\mathcal{H}(a_1^{p-1})] \\ &+ \tbinom{p}{2}k^{p-2}[2\mathcal{H}(a_1^{p-2}a_2^p) - \mathcal{H}(a_1^{p-2})\phi(a_2^p) - \phi(a_2^p)\mathcal{H}(a_1^{p-2})] + \cdots \\ &+ \tbinom{p}{p-1}k[2\mathcal{H}(a_1a_2^p) - \mathcal{H}(a_2^p)\phi(a_1) - \phi(a_2^p)\mathcal{H}(a_1)] = 0. \end{aligned}$$

Applying the same arguments, we arrive at

$$p[2\mathcal{H}(a_1a_2^p) - \mathcal{H}(a_1)\phi(a_2^p) - \phi(a_2^p)\mathcal{H}(a_1)] = 0 \text{ for all } a_1, a_2 \in \mathcal{A}.$$

Since $\mathcal{A}$ is $p$-torsion free, we find that

$$2\mathcal{H}(a_1a_2^p) = \mathcal{H}(a_1)\phi(a_2^p) + \phi(a_2^p)\mathcal{H}(a_1) \text{ for all } a_1, a_2 \in \mathcal{A}. \tag{2.8}$$

Again, replacing $a_2$ by $a_2 + e$ in the above equation, we obtain

$$\begin{aligned} &\tbinom{p}{1}[2\mathcal{H}(a_1a_2^{p-1}) - \mathcal{H}(a_1)\phi(a_2^{p-1}) - \phi(a_2)\mathcal{H}(a_1^{p-1})] \\ &+\tbinom{p}{2}[2\mathcal{H}(a_1a_2^{p-2}) - \mathcal{H}(a_1)\phi(a_2^{p-2}) - \phi(a_2)\mathcal{H}(a_1^{p-2})] \\ &+\cdots+\tbinom{p}{p-1}[2\mathcal{H}(a_1a_2) - \mathcal{H}(a_1)\phi(a_2) - \phi(a_2)\mathcal{H}(a_1)] = 0. \end{aligned}$$

Replacing $a_2$ by $ka_2$, we get

$$\sum_{r=1}^{p-i} k^{p-i} \binom{p}{i} [2\mathcal{H}(a_1a_2^{p-i}) - \mathcal{H}(a_1)\phi(a_2^{p-i}) - \phi(a_2^{p-i})\mathcal{H}(a_1)] = 0 \text{ for all } a_1, a_2 \in \mathcal{A}.$$

Using the same arguments as the above, we find $2\mathcal{H}(a_1a_2) = \mathcal{H}(a_1)\phi(a_2) + \phi(a_1)\mathcal{H}(a_2)$ for all $a_1, a_2 \in \mathcal{A}$. Replacing $a_2$ by $a_1$ and using Lemma 1.2, We obtain the desired conclusion. □

To illustrate the importance of semiprimeness in both theorems, we give the following example:

**Example 2.1** Consider a ring $\mathcal{A} = \left\{ \begin{pmatrix} \bar{i} & \bar{j} \\ \bar{0} & \bar{k} \end{pmatrix} \mid \bar{i}, \bar{j}, \bar{k} \in 2\mathbb{Z}_8 \right\}$. Define mappings $\mathcal{H}, \phi : \mathcal{A} \to \mathcal{A}$ by $\mathcal{H}\begin{pmatrix} \bar{i} & \bar{j} \\ \bar{0} & \bar{k} \end{pmatrix} = \begin{pmatrix} \bar{0} & \bar{j} \\ \bar{0} & \bar{0} \end{pmatrix}$ and $\phi\begin{pmatrix} \bar{i} & \bar{j} \\ \bar{0} & \bar{k} \end{pmatrix} = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{k} \end{pmatrix}$ for all $\bar{i}, \bar{j}, \bar{k} \in 2\mathbb{Z}_8$. It is clear that $\mathcal{H}$ and $\phi$ satisfy the identities (2.1) and (2.5) but $\mathcal{H}$ is not a centralizer on $\mathcal{A}$. Hence, semiprimeness hypothesis has significance for both key theorems.

## 3. $\phi^\star$-centralizer

This section is devoted to the study of $\phi^\star$-centralizer on a ring $\mathcal{A}$. A mapping $\star : \mathcal{A} \to \mathcal{A}$ is termed as an involution if it satisfies $(a_1 + a_2)^\star = a_1^\star + a_2^\star$, $(a_1a_2)^\star = a_2^\star a_1^\star$ and $(a_1^\star)^\star = a_1$ for all $a_1, a_2 \in \mathcal{A}$. A ring having an involution is known as a ring with involution or $\star$-ring. A mapping $\mathcal{H} : \mathcal{A} \to \mathcal{A}$ is a right (respectively left)$\star$-centralizer if it is additive and $\mathcal{H}(a_1a_2) = a_1^\star \mathcal{H}(a_2)$ (respectively $\mathcal{H}(a_1a_2) = \mathcal{H}(a_1)a_2^\star$) satisfies for all $a_1, a_2 \in \mathcal{A}$ and if it is right as well as left $\star$-centralizer, then it is known as $\star$-centralizer. $\mathcal{H}$ is termed as a right (respectively left) Jordan $\star$-centralizer if for all $a_1 \in \mathcal{A}$, $\mathcal{H}(a_1^2) = a_1^\star\mathcal{H}(a_1)$ (respectively $\mathcal{H}(a_1^2) = \mathcal{H}(a_1)a_1^\star$). $\mathcal{H}$ is a Jordan $\star$-centralizer of $\mathcal{A}$ if it is right as well as left Jordan $\star$-centralizer on $\mathcal{A}$. An additive mapping $\mathcal{H} : \mathcal{A} \to \mathcal{A}$ is known as a right (respectively left)$\phi^\star$-centralizer if $\mathcal{H}(a_1a_2) = \phi(a_1^\star)\mathcal{H}(a_2)$ (respectively $\mathcal{H}(a_1a_2) = \mathcal{H}(a_1)\phi(a_2^\star)$) holds for all $a_1, a_2 \in \mathcal{A}$ and particularly, a right (respectively left) Jordan $\phi^\star$-centralizer for all $a_1 \in \mathcal{A}$, $\mathcal{H}(a_1^2) = \phi(a_1^\star)\mathcal{H}(a_1)$ (respectively $\mathcal{H}(a_1^2) = \mathcal{H}(a_1)\phi(a_1^\star)$). The present research investigates potential applications of the findings that are demonstrated in the preceding section in the setting of a ring with involution "$\star$". In fact, it is shown that an additive mappings $\mathcal{H}$ on a $p!$-torsion free semiprime $\star$-ring $\mathcal{A}$ satisfying $2\mathcal{H}(a_1^pa_2^p) = \mathcal{H}(a_1^p)\phi((a_2^\star)^p)+\phi((a_2^\star)^p)\mathcal{H}(a_1^p)$ or $\mathcal{H}(a_1^pa_2^p+a_2^pa_1^p) = \mathcal{H}(a_1^p)\phi((a_2^\star)^p)+\phi((a_2^\star)^p)\mathcal{H}(a_1^p)$ for all $a_1, a_2 \in \mathcal{A}$, is a $\phi^\star$-centralizer of $\mathcal{A}$. We require the following lemma to explain our primary findings.

**Lemma 3.1 ( [4, Corollary 2.1])** *Suppose that $\mathcal{A}$ is a 2 torsion free semiprime ring with involution $\star$. If an additive mapping $\mathcal{H} : \mathcal{A} \to \mathcal{A}$ satisfies the condition $2\mathcal{H}(a_1^2) = \mathcal{H}(a_1)\phi(a_1^\star) + \phi(a_1^\star)\mathcal{H}(a_1)$ for all $a_1 \in \mathcal{A}$, then $\mathcal{H}$ is a $\phi^\star$-centralizer on $\mathcal{A}$.*

Next, start main result of this part.

**Theorem 3.1** *Suppose that $\mathcal{A}$ is any $p!$-torsion free semiprime ring with identity $e$ and involution $\star$. If an additive mapping $\mathcal{H} : \mathcal{A} \to \mathcal{A}$ satisfies $2\mathcal{H}(a_1^pa_2^p) = \mathcal{H}(a_1^p)\phi((a_2^\star)^p) + \phi((a_2^\star)^p)\mathcal{H}(a_1^p)$ for all $a_1, a_2 \in \mathcal{A}$, where $\phi$ is a surjective endomorphism on $\mathcal{A}$, then $\mathcal{H}$ is a $\phi^\star$-centralizer on $\mathcal{A}$, where $p \geq 1$ is a fixed integer.*

**Proof:** Since

$$2\mathcal{H}(a_1^pa_2^p) = \mathcal{H}(a_1^p)\phi((a_2^\star)^p) + \phi((a_2^\star)^p)\mathcal{H}(a_1^p) \text{ for all } a_1, a_2 \in \mathcal{A}, \tag{3.1}$$

then, replace $a_1$ by $e$ in the above equation to find

$$2\mathcal{H}(a_2^p) = \mathcal{H}(e)\phi((a_2^\star)^p) + \phi((a_2^\star)^p)\mathcal{H}(e) \text{ for all } a_2 \in \mathcal{A}. \tag{3.2}$$

Replace $a_2$ by $ka_2 + e$ in the above equation and use the fact that $e^\star = e^\star e = (ee^\star)^\star = (e^\star)^\star = e$ to get

$$\sum_{i=0}^{p} k^{p-i}\binom{p}{i}[2\mathcal{H}(a_2^{p-i}) - \mathcal{H}(e)\phi((a_2^\star)^{p-i}) - \phi((a_2^\star)^{p-i})\mathcal{H}(e)] = 0 \text{ for all } a_2 \in \mathcal{A}.$$

Using Lemma 1.1, then for all $a_2 \in \mathcal{A}$ and $i = 1, 2, \cdots, p-1$, we find

$$\binom{p}{i}[2\mathcal{H}(a_2^{p-i}) - \mathcal{H}(e)\phi((a_2^\star)^{p-i}) - \phi((a_2^\star)^{p-i})\mathcal{H}(e)] = 0.$$

For $i = p-1$, we obtain

$$\binom{p}{p-1}[2\mathcal{H}(a_2) - \mathcal{H}(e)\phi(a_2^\star) - \phi(a_2^\star)\mathcal{H}(e)] = 0 \text{ for all } a_2 \in \mathcal{A}.$$

Since $\mathcal{A}$ is $p$-torsion free, then we get that

$$2\mathcal{H}(a_2) = \mathcal{H}(e)\phi(a_2^\star) + \phi(a_2^\star)\mathcal{H}(e) \text{ for all } a_2 \in \mathcal{A}. \tag{3.3}$$

Next, replace $a_1$ by $a_1+e$ in (3.1) and using (3.1) and (3.2), we have

$$\begin{aligned}\binom{p}{1}[2\mathcal{H}(a_1^{p-1}a_2^p) &- \mathcal{H}(a_1^{p-1})\phi((a_2^\star)^p) - \phi((a_2^\star)^p)\mathcal{H}(a_1^{p-1})]\\ &+ \binom{p}{2}[2\mathcal{H}(a_1^{p-2}a_2^p) - \mathcal{H}(a_1^{p-2})\phi((a_2^\star)^p) - \phi((a_2^\star)^p)\mathcal{H}(a_1^{p-2})] + \cdots\\ &+ \binom{p}{p-1}[2\mathcal{H}(a_1a_2^p) - \mathcal{H}(a_1)\phi((a_2^\star)^p) - \phi((a_2^\star)^p)\mathcal{H}(a_1)] = 0.\end{aligned}$$

Replacing $a_1$ by $ka_1$, applying the same arguments, we arrive at

$$p[2\mathcal{H}(a_1a_2^p) - \mathcal{H}(a_1)\phi((a_2^\star)^p) - \phi((a_2^\star)^p)\mathcal{H}(a_1)] = 0.$$

We find the following using torsion restriction on $\mathcal{A}$

$$2\mathcal{H}(a_1a_2^p) = \mathcal{H}(a_1)\phi((a_2^\star)^p) + \phi((a_2^\star)^p)\mathcal{H}(a_1) \text{ for all } a_1,a_2\in\mathcal{A}. \tag{3.4}$$

Again, repeating the same process for $a_2$, we have

$$\sum_{r=1}^{p-i}\binom{p}{i}[2\mathcal{H}(a_1a_2^{p-i}) - \mathcal{H}(a_1)\phi((a_2^\star)^{p-i}) - \phi((a_2^\star)^{p-i})\mathcal{H}(a_1)] = 0 \text{ for all } a_1,a_2\in\mathcal{A}.$$

Applying the similar technique to find $2\mathcal{H}(a_1a_2) = \mathcal{H}(a_1)\phi(a_2^\star) + \phi(a_2^\star)\mathcal{H}(a_2)$ for all $a_1,a_2\in\mathcal{A}$. Replace $a_2$ by $a_1$ and use Lemma 3.1 to get the required result. □

**Theorem 3.2** *Suppose that $\mathcal{A}$ is any $p!$-torsion free semiprime ring with identity $e$ and involution $\star$. If an additive mapping $\mathcal{H}:\mathcal{A}\to\mathcal{A}$ satisfies*

$$\mathcal{H}(a_1^pa_2^p + a_2^pa_1^p) = \mathcal{H}(a_1^p)\phi((a_2^\star)^p) + \phi((a_2^\star)^p)\mathcal{H}(a_1^p) \textit{ for all } a_1,a_2\in\mathcal{A},$$

*where $\phi$ is a surjective endomorphism on $\mathcal{A}$, then $\mathcal{H}$ is a $\phi^\star$-centralizer on $\mathcal{A}$, where $p\geq 1$ is a fixed integer.*

**Proof:** Construct a mapping $\mathcal{T}:\mathcal{A}\to\mathcal{A}$ such that $\mathcal{T}(a_1) = \mathcal{H}(a_1^\star)$ for all $a_1\in\mathcal{A}$. Certainly, $\mathcal{T}$ is an additive mapping. Now, consider

$$\begin{aligned}\mathcal{T}(a_1^pa_2^p + a_2^pa_1^p) &= \mathcal{H}((a_1^pa_2^p + a_2^pa_1^p)^\star)\\ &= \mathcal{H}\big[(a_1^\star)^p(a_2^\star)^p + (a_2^\star)^p(a_1^\star)^p\big]\\ &= \mathcal{H}(a_1^\star)^p\phi(a_2^p) + \phi(a_2^p)\mathcal{H}(a_1^\star)^p\\ &= \mathcal{T}(a_1^p)\phi(a_2^p) + \phi(a_2^p)\mathcal{T}(a_1)^p \text{ for all } a_1,a_2\in\mathcal{A}.\end{aligned}$$

Using first theorem, we find that $\mathcal{T}$ is a $\phi$-centralizer on $\mathcal{A}$. Hence, $\mathcal{T}(a_1a_2) = \mathcal{T}(a_1)\phi(a_2) = \phi(a_1)\mathcal{T}(a_2)$ for all $a_1,a_2\in\mathcal{A}$. This implies that $\mathcal{H}(a_1^\star)^2 = \mathcal{H}(a_1^\star)\phi(a_1) = \phi(a_1)\mathcal{H}(a_1^\star)$ for all $a_1,a_2\in\mathcal{A}$. Now, replacing $a_1$ by $a_1^\star$ and $a_2$ by $a_2^\star$ and applying Lemma 3.1, one have the desired conclusion. □

**Example 3.1** Let $\mathcal{A} = \left\{\begin{pmatrix}\bar{i} & \bar{j}\\ \bar{0} & \bar{k}\end{pmatrix} \mid \bar{i},\bar{j},\bar{k}\in 2\mathbb{Z}_8\right\}$ is a ring with involution $\star:\mathcal{A}\to\mathcal{A}$ by $\begin{pmatrix}\bar{i} & \bar{j}\\ \bar{0} & \bar{k}\end{pmatrix}^\star = \begin{pmatrix}\bar{k} & -\bar{j}\\ \bar{0} & \bar{i}\end{pmatrix}$ for all $\bar{i},\bar{j},\bar{k}\in 2\mathbb{Z}_8$. Define mappings $\mathcal{H},\phi:\mathcal{A}\to\mathcal{A}$ by $\mathcal{H}\begin{pmatrix}\bar{i} & \bar{j}\\ \bar{0} & \bar{k}\end{pmatrix} = \begin{pmatrix}\bar{0} & \bar{j}\\ \bar{0} & \bar{0}\end{pmatrix}$ and $\phi\begin{pmatrix}\bar{i} & \bar{j}\\ \bar{0} & \bar{k}\end{pmatrix} = \begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{k}\end{pmatrix}$ for all $\bar{i},\bar{j},\bar{k}\in 2\mathbb{Z}_8$. It is clear that $\mathcal{H}$ satisfy the identities (3.1) and (3.2) but $\mathcal{H}$ is not a $\phi^\star$-centralizer on $\mathcal{A}$. Hence, semiprimeness hypothesis has significance for both main theorems.

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