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Parseval Frames of Some Hilbert Spaces of Entire Vector Valued Functions via Naimark Dilation

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ABSTRACT: In this paper, we establish a matrix representation of a reproducing kernel corresponding to a de Branges matrix generated via matrix valued function multiplication, which generalises a well-known result. We utilise this technique to extend certain results on de Branges spaces of scalar valued functions to vector valued functions. By connecting the fundamental construction with Naimark dilation of frames, we discover certain existence and structural characteristics of Parseval frames of normalised reproducing kernels in vector valued de Branges spaces. The dilation arises when certain de Branges spaces of vector valued functions are embedded in a larger de Branges space of vector valued functions. In addition, the embedding translates the kernel functions associated with a frame sequence in the original space into a Riesz basis for the embedding space.

Key Words: de Branges Spaces, Entire vector valued functions, Naimark dilation, Parseval frames.

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1. Introduction

Louis de Branges [10] invented and thoroughly investigated spaces of entire vector valued functions, which were created in light of the model theory for linear transformations in Hilbert spaces. These spaces have been crucial in the implementation of canonical systems of differential and integral equations, as well as Dirac-Krein systems, to direct and inverse problems, see for example [2,4,5,7].

It is well known that if $\mathfrak{E} = [E_- \quad E_+]$ is an $n \times 2n$ de Branges matrix and $\psi(z)$ is a scalar entire function such that $\psi^{-1}\psi^{\#}$ is inner, then $\left[\psi^{\#}E_- \quad \psi E_+\right]$ is a de Branges matrix as well. In general, multiplying the matrix $\mathfrak{E}$ by another matrix valued function may not produce a de Branges matrix. So it is very natural to ask which matrices V can be multiplied by $\mathfrak{E}$ such that $\mathfrak{E}V$ is a de Branges matrix with certain properties.

It is proved in [6] that if V is a $2n \times 2n$ matrix such that $V^* j_n V = j_n$, where $j_n = \operatorname{diag}(I_n, -I_n)$, and I_n is the $n \times n$ identity matrix, then $\mathfrak{E}V$ is a de Branges matrix and the corresponding de Branges spaces are equal; $\mathcal{H}(\mathfrak{E}) = \mathcal{H}(\mathfrak{E}V)$. We expand this construction to produce certain de Branges matrices in the presented research.We also show how this technique can be used to generalize the results achieved in [1] on de Branges spaces of scalar valued functions to de Branges spaces of vector valued functions.

We consider the existence and structure properties of Parseval frames of normalized reproducing kernel functions in vector valued de Branges spaces. We will examine certain definitions and essential facts from the theory of reproducing kernel Hilbert spaces of vector valued functions in subsection 1.2, and as a particular instance of such spaces, the de Branges spaces of vector valued functions will be reviewed in section 1.3. Sections 2 and 3 are devoted to developing new results on the orthogonality of embedding within dilation of vector valued functions of de Branges spaces. We develop some necessary conditions for Parseval sequences for the spaces $\mathcal{H}(\mathfrak{E}_1)$ and $\mathcal{H}(\mathfrak{E}_2)$ by picking out the chief eastablishment with

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Naimark dilation of frames by immersing the de Branges space into a larger de Branges space $\mathcal{H}(\mathfrak{E})$ while embedding the kernel functions connected with a frame sequence into a Riesz basis for the immense space.

Some notations are necessary to describe the spaces we will consider here, see [7,9] for additional discussion and reading. $\mathbb{C}$ denotes complex plane, where $\mathbb{C}^+$ represents the open upper plane and $\mathbb{C}^-$ denotes lower half plane, $\mathbb{C}^n$ the complex $n \times 1$ vectors. The set of all $n \times m$ matrices with complex entries are denoted by $\mathbb{C}^{n\times m}$, here I_n represents the identity matrix of $\mathbb{C}^{n\times m}$. In a region Ω of the complex plane $\mathbb{C}$, a vector valued function $f(z)$ is called analytic in this region if for every choice of vector $u \in \mathbb{C}$ the complex valued function $u^* f(z)$ is analytic in the region. In a region Ω, a continuous matrix valued function $F(z)$, is called analytic in this region if for every choice of vectors u and v in $\mathbb{C}$ $u^* F(z) v$ is analytic in the region. The term "entire matrix valued function" refers to a matrix valued function having entries that are analytic throughout the whole complex plane. The Hermitian transpose of the matrix valued function $f(z)$, is denoted by $f^*(z)$, and we define the function $f^\#$ as $f^\#(z) := f^*(\bar{z})$. The space $\mathbb{H}_2^{n\times m}$ is the Hardy space of $n \times m$ matrix valued functions with entries in the classical Hardy space $\mathbb{H}_2$ with respect to $\mathbb{C}^+$, with norm

$$\|f\|_2^2 = \sup_{y>0} \int_{-\infty}^{\infty} \text{trace}\,\{f^*(x+iy)f(x+iy)\}\,dx < \infty$$

$\left(\mathbb{H}_2^{n\times m}\right)^\perp = \left\{f : f^\# \in \mathbb{H}_2^{m\times n}\right\}$ (When $\mathbb{H}_2^{n\times m}$ and $\left(\mathbb{H}_2^{n\times m}\right)^\perp$ are considered as subspaces of $L_2^{n\times m}$, the superscript $\perp$ indicates that they are orthogonal to each other). We shall use the symbol $\mathbb{H}_2^n$ for $\mathbb{H}_2^{n\times 1}$, and $(\mathbb{H}_2^m)^\perp$ for $\left(\mathbb{H}_2^{m\times 1}\right)^\perp$.

$\mathbb{H}_\infty^{n\times m}$ is the Hardy space of holomorphic $n \times m$ matrix valued functions in $\mathbb{C}^+$ with

$$\|f\|_\infty^2 = \sup\left\{\|f(z)\| : z \in \mathbb{C}^+\right\} < \infty$$

The Schur class $\mathcal{S}^{n\times n}$ is the class of $n \times n$ matrix valued functions $s(z)$ that are holomorphic and contractive in $\mathbb{C}^+$, i.e.,

$$I_n - s^*(z)s(z) \succeq 0, \text{ for } z \in \mathbb{C}^+$$

$\mathcal{S}_{in}^{n\times n}$ is the class of matrix valued functions $f \in \mathcal{S}^{n\times n}$ which are inner, i.e., $I_n - f^*(t)f(t) = 0$ for a.e. point $t \in \mathbb{R}$

1.1. Preliminaries

We start with some general preliminaries on frames, Parseval frames, and Riesz basis. A sequence $\{f_n\}_{n=1}^\infty$ is a frame for a separable Hilbert space $\mathcal{H}$ if there exists constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \le B\|f\|^2, \quad \text{for all} \quad f \in \mathcal{H} \tag{1.1}$$

Lower and higher frame limits are the constants A and B, respectively. If just the estimate on the right-hand side of (1.1) is assumed, it is referred to as a Bessel sequence.

Parseval frames are the tight frames for which $A = B = 1$. A Riesz basis is a frame that serves as a basis. It is obvious that for a Hilbert space $\mathcal{H}$, a Parseval frame $\{f_n\}_{n=1}^\infty$ is an orthonormal basis if and only if each f_n is a unit vector.

Let $\mathbb{F} = \{f_n\}_{n=1}^\infty$ be a Bessel sequence in $\mathcal{H}$. The analysis and synthesis operators, denoted respectively by $T_\mathbb{F} : \mathcal{H} \to \ell^2$ and $T_\mathbb{F}^* : \ell^2 \to \mathcal{H}$, are defined respectively by

$$T_\mathbb{F} : f \to (\langle f, f_n\rangle), \text{ and } T_\mathbb{F}^* : (c_n)_{n=1}^\infty \to \sum_{n=1}^{\infty} c_n f_n$$

The adjoint operator of the synthesis operator is the analysis operator. These operators are clearly defined and are constrained by (1.1). The frame operator, denoted by $S_\mathbb{F}$, is defined by $S_\mathbb{F} := T_\mathbb{F}^* T_\mathbb{F} : \mathcal{H} \to \mathcal{H}$, and is given by

$$S_\mathbb{F} f = \sum_{k=1}^{\infty} \langle f, f_n\rangle f_n, \forall f \in \mathcal{H} \tag{1.2}$$

If $\mathbb{F} = \{f_n\}$ is a frame, it is known that the series (1.2) converges unconditionally, the operator $S_{\mathbb{F}}$ is bounded, self adjoint, positive and has a bounded inverse, and the sequence $S_{\mathbb{F}}^{-1}(\mathbb{F}) := \left\{S_{\mathbb{F}}^{-1}(f_n)\right\}_{n=1}^{\infty}$ is a dual frame to $\mathbb{F}$, called the *canonical dual frame*. Thus we have the following frame expansions,

$$f = \sum_{n\in I} \langle f, f_n\rangle S_{\mathbb{F}}^{-1} f_n = \sum_{n\in I} \langle f, S_{\mathbb{F}}^{-1} f_n\rangle f_n \tag{1.3}$$

Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces. If $\mathbb{F} = \{f_n\}_{n=1}^{\infty} \subset \mathcal{H}_1$ and $\mathbb{G} = \{g_n\}_{n=1}^{\infty} \subset \mathcal{H}_2$ are two Parseval frames and orthogonal to each other, then for any $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$ we have

$$f = \sum_{n} \left(\langle f, f_n\rangle + \langle g, g_n\rangle\right) f_n$$

and

$$g = \sum_{n} \left(\langle f, f_n\rangle + \langle g, g_n\rangle\right) g_n$$

In other respects, the summed coefficients $\langle f, f_n\rangle + \langle g, g_n\rangle$ may be used to reconstruct both functions. This method is known as *multiplexing*, and it may be utilised in multiple access communications networks.

Let P denotes orthogonal projection from a Hilbert space $\mathcal{N}$ onto a closed subspace $\mathcal{H}$, and $\{f_n\}$ indicates a sequence in $\mathcal{N}$. Then $\{Pf_n\}$ is referred to as *orthogonal compression* of $\{f_n\}$ under P, and $\{f_n\}$ is referred to an *orthogonal dilation* of $\{Pf_n\}$. A Parseval frame in a Hilbert space $\mathcal{H}$ is an image of an orthonormal basis under an orthogonal projection of a larger Hilbert space $\mathcal{N} \supseteq \mathcal{H}$ onto $\mathcal{H}$, according to a classic fact on frame dilation credited to Han and Larson [14]. For positive operator valued measures, this conclusion may be thought of as a particular instance of Naimark's dilation theorem (see [16,17]). Han and Larson, in particular, established the following result.

Theorem 1.1 *Assume* $\{f_n\}_{n=1}^{\infty}$ *is a sequence in a Hilbert space* $\mathcal{H}$*. Then*

(i) $\{f_n\}$ *is a Parseval frame for* $\mathcal{H}$ *if and only if there exists a Hilbert space* $\mathcal{N} \supseteq \mathcal{H}$ *and an orthonormal basis* $\{e_n\}$ *for* $\mathcal{N}$ *such that if* P *is the orthogonal projection of* $\mathcal{N}$ *onto* $\mathcal{H}$ *then* $f_n = Pe_n$*, for all* $n \in \mathbb{N}$

(ii) $\{f_n\}$ *is a frame for* $\mathcal{H}$ *if and only if there exists a Hilbert space* $\mathcal{N} \supseteq \mathcal{H}$ *and a Riesz basis.* $\{u_n\}$ *for* $\mathcal{N}$ *such that if* P *is the orthogonal projection of* $\mathcal{N}$ *onto* $\mathcal{H}$ *then* $f_n = Pu_n$ *for all* $n \in \mathbb{N}$*.*

The following is the relationship between orthogonality of frames and Naimark dilation of frames (see [14,8]): If P is the projection onto $\mathcal{H} \subset \mathcal{N}$ and $\{u_n\}$ is a Riesz basis for $\mathcal{N}$, then $\{Pu_n\}$ and $\{(I-P)u_n\}$ are orthogonal frames for $\mathcal{H}$ and $\mathcal{H}^{\perp}$, respectively. Conversely, if $\mathbb{F} = \{f_n\}$ and $\mathbb{G} = \{g_n\}$ are orthogonal frames for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then $\{f_n + g_n\}$ is a frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$. It should be noted that the total of the frames does not have to be a foundation for the direct sum in general, but it will be supplied that

$$T_{\mathbb{F}}(\mathcal{H}_1) \oplus T_{\mathbb{G}}(\mathcal{H}_2) = \ell^2.$$

1.2. Reproducing Kernel Hilbert Spaces of Vector Valued Functions

This part fixes notations and reviews a number of facts about reproducing kernel Hilbert spaces of vector valued functions theory that will be required throughout this study; further information and accompanying proofs can be obtained in [3,4,5,7,13]. A reproducing kernel Hilbert space (RKHS) is defined as a Hilbert space $\mathcal{H}$ of $n \times 1$ vector valued functions defined on a subset Ω of $\mathbb{C}$, which contains an $n \times n$ matrix valued function $K_w(z)$ for $(z, w) \in \Omega \times \Omega$, called a RK, such that for every choice of $w \in \Omega, u \in \mathbb{C}^n$, and $f \in \mathcal{H}$:

1. $K_w(z)u \in \mathcal{H}$, as a vector valued function of z,
2. The reproducing kernel property

$$\langle f(.), K_w(.)u\rangle_{\mathcal{H}} = \langle f(w), u\rangle_{\mathbb{C}} = u^* f(w) \tag{1.4}$$

The Riesz representation theorem [11] guarantees the existence and uniqueness of the RK. The RK is important to the extent that

$$\sum_{i,j=1}^{r} u_j^* K_{w_i}(w_j) u_i \geq 0 \tag{1.5}$$

for every choice of points $w_1, \ldots, w_r \in \Omega$ and vectors $u_1, \ldots, u_r \in \mathbb{C}^n$ and every positive integer n. Consequently, the set $\{K_w(.)u : w \in \Omega, u \in \mathbb{C}^n\}$ is total in $\mathcal{H}$, that is

$$\mathcal{H} = \overline{\text{span}}\,\{K_w(.)u : w \in \Omega, u \in \mathbb{C}^n\}$$

The following RKHS characteristics are well known and easily tested; for additional information, see [12].

1. $\langle K_w(.)u_1, K_v(.)u_2 \rangle_{\mathcal{H}} = u_2^* K_w(v) u_1$, for all $w, v \in \mathbb{C}, u_1, u_2 \in \mathbb{C}^n$, and

$$\|K_w u\|_{\mathcal{H}}^2 = u^* K_w(w) u. \tag{1.6}$$

2. $\|f(w)\| \leq \|f\|_{\mathcal{H}} \|K_w(w)\|^{1/2}$, for all $w \in \Omega$ and $f \in \mathcal{H}$

We recommend the reader to [3] for a proof of the following theorem, which is a matrix variant of an Aronszajn theorem.

Theorem 1.2 *Let Ω be a subset of $\mathbb{C}$ and let the $n \times n$ matrix valued kernel $K_\omega(z)$ be positive on $\Omega \times \Omega$. Then there is a unique Hilbert space $\mathcal{H}$ of $n \times 1$ vector valued functions $f(z)$ on Ω such that*

$$K_\omega u \in \mathcal{H}, \quad \text{and} \quad \langle f, K_\omega u \rangle_{\mathcal{H}} = u^* f(\omega)$$

for every $\omega \in \Omega$, $u \in \mathbb{C}^n$ and $f \in \mathcal{H}$.

Example 1.1 ([7]) The Hardy space $\mathbb{H}_2^n$ is a RKHS of $n \times 1$ vector valued functions that are holomorphic in $\mathbb{C}^+$ with RK

$$K_\omega(z) = \frac{I_n}{-2\pi i(z - \bar{\omega})}, \text{ for } z, \omega \in \mathbb{C}^+$$

The following result is well known and will play a significant rule in the conclusions, see [15] for more details: Let $K_\omega^{(1)}(z)$ and $K_\omega^{(2)}(z)$ be two RK's, and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the corresponding RKHS's such that $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$. Then the RKHS $\mathcal{H}$ with RK $K^{(1)} + K^{(2)}$ is the orthogonal direct sum of the RKHS's $\mathcal{H}_1$ and $\mathcal{H}_2$;

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

and $\|f\|_{\mathcal{H}}^2 = \|f_1\|_{\mathcal{H}_1}^2 + \|f_2\|_{\mathcal{H}_2}^2$, where $f = f_1 + f_2$ for some $f_i \in \mathcal{H}_i, i = 1, 2$.

If $\{w_j\}_{j=1}^{\infty} \subset \mathbb{C}$ is a sequence of points and $\{\xi_j\}_{j=1}^{\infty} \in \mathbb{C}^n$ is a sequence of vectors, the set $\left\{K_{w_j}(.)\xi_j\right\}_{j=1}^{\infty}$ is a complete orthogonal set for the RKHS $\mathcal{H}$ with RK $K_w(z)$ if every $f \in \mathcal{H}$ can be expressed in the form

$$f(z) = \sum_{j=1}^{\infty} \left\langle f, K_{w_j}\xi_j \right\rangle_{\mathcal{H}} \frac{K_{w_j}(z)\xi_j}{\left\|K_{w_j}\xi_j\right\|^2} = \sum_{j=1}^{\infty} \xi_j^* f(w_j) \frac{K_{w_j}(z)\xi_j}{\left\|K_{w_j}\xi_j\right\|^2}.$$

1.3. de Branges Spaces of Vector Valued Functions

We will give a number of facts from the theory of de Branges spaces of vector valued functions in this part, which will be useful in the future. The majority of this data may be found in the articles [4,5,7,9].

An entire $n \times 2n$ matrix valued function $\mathfrak{E}(z) = [E_-(z) \quad E_+(z)]$ is called an entire *de Branges matrix* with $n \times n$ blocks $E_\pm(z)$ matrix valued entire functions, if

$$\det E_+(z) \not\equiv 0, \text{ in } \mathbb{C}, \quad \text{and} \quad \chi := E_+^{-1} E_- \in \mathcal{S}_{in}^{n \times n}. \tag{1.7}$$

An entire matrix valued function's determinant is an entire function. As a result, if the determinant of the whole matrix valued function $E_+(z)$ does not vanish identically at all but isolated points in the complex plane, the provided full matrix valued function has invertible values. Since $E_\pm(z)$ are entire matrix valued functions, the condition in (1.7) ensures that (see [12])

$$E_+(z)E_+^{\#}(z) = E_-(z)E_-^{\#}(z), \text{ for all } z \in \mathbb{C}. \tag{1.8}$$

Definition 1.1 Given a de Branges matrix $\mathfrak{E}$, the set of entire $n \times 1$ vector valued functions $f(z)$ satisfying

$$E_+^{-1}f \in \mathbb{H}_2^n \quad \text{and} \quad E_-^{-1}f \in (\mathbb{H}_2^n)^\perp \tag{1.9}$$

is a reproducing kernel Hilbert space with reproducing kernel

$$K_w^{\mathfrak{E}}(z) = \begin{cases} \frac{E_+(z)E_+^*(w)-E_-(z)E_-^*(w)}{2\pi i(\bar{w}-z)} & \text{, if } z \neq \bar{w} \\ \frac{E_+'(\bar{w})E_+^*(w)-E_-'(\bar{w})E_-^*(w)}{-2\pi i} & \text{, if } z = \bar{w} \end{cases} \tag{1.10}$$

with respect to the inner product

$$\langle f, g\rangle_{\mathcal{H}} = \langle E_+^{-1}f, E_+^{-1}g\rangle_{st} = \int_{-\infty}^{\infty} g^*(t)\Delta_{\mathcal{E}}(t)f(t)\,dt, \tag{1.11}$$

where

$$\Delta_{\mathfrak{E}}(t) = \{E_+(t)E_+^*(t)\}^{-1} = \{E_-(t)E_-^*(t)\}^{-1},$$

for all $t \in \mathbb{R}$ at which $\det E_\pm(z) \neq 0$.

The Hilbert space corresponding to the de Branges matrix $\mathfrak{E}$ is called the de Branges space $\mathcal{H}(\mathfrak{E})$; for every $w \in \mathbb{C}$, every $u \in \mathbb{C}^n$, and every $f \in \mathcal{H}(\mathfrak{E})$

1. $K_w^{\mathfrak{E}}u \in \mathcal{H}(\mathfrak{E})$ and
2. $\langle f, K_w^{\mathfrak{E}}u\rangle_{\mathcal{H}(\mathfrak{E})} = u^*f(w)$

Remark 1.1 If $E(z)$ is a scalar valued entire function which has no real zeros and $|E(z)| > |E(\bar{z})|$ for all $z \in \mathbb{C}^+$, then $\mathcal{H}(\mathfrak{E})$ with $\mathfrak{E} = [E^{\#}(z) \quad E(z)]$ is just the usual de Branges space corresponding to the de Branges function $E(z)$.

Example 1.2 ([12]) If $E_+^t(z) = e^{-izt}I_n$ and $E_-^t(z) = e^{izt}I_n$ for $t > 0$, then it is easy to see that $\mathfrak{E}_t(z) = [E_-^t(z) \quad E_+^t(z)]$ is an entire de Branges matrix, and the space $\mathcal{H}(\mathfrak{E}_t)$ is a vector Paley-Wiener space with RK

$$K_w^{\mathfrak{E}_t}(z) = \frac{\sin(z-\bar{w})t}{\pi(z-\bar{w})}I_n.$$

2. The de Branges Space $\mathcal{H}(\mathfrak{E})$

In the sequel, $\mathcal{U}_\alpha(\mathbb{C}^{n\times n})$ denotes the set of $n \times n$ scalar matrices with complex entries such that $\frac{\alpha}{2}AA^* = I_n$, where $\alpha > 0$. As a special case, $\mathcal{U}(\mathbb{C}^{n\times n})$ denotes the set of unitary matrices for the case where $\alpha = 2$. Note that if $P \in \mathcal{U}$ then there exists $A, B \in \mathcal{U}_\alpha$ such that $P = A^{-1}B$.

Let $\mathfrak{F} = [F_-(z) \quad F_+(z)]$, and $\mathfrak{E}_1(z) = \left[E_{1_-}(z) \quad E_{1_+}(z)\right]$ be two de Branges matrices. Fix two matrices $P, Q \in \mathcal{U}$. We define the following matrix valued functions

$$E_{2-}(z) = F_+^{-1}(z)F_-(z)E_{1_-}(z)P^{-1}, \quad E_{2+}(z) = F_+^{-1}(z)F_-(z)E_{1+}(z)Q^{-1} \tag{2.1}$$

$$G_{1-}(z) = F_-(z)E_{1-}(z), \quad G_{1+}(z) = F_-(z)E_{1+}(z), \tag{2.2}$$

and

$$G_{2-}(z) = F_+(z)E_{2-}(z), \quad G_{2+}(z) = F_+(z)E_{2+}(z). \tag{2.3}$$

We also define the $n \times 2n$ matrices $\mathfrak{E}_2, \mathfrak{G}_1$ and $\mathfrak{G}_2$ as follow:

(1) $\mathfrak{E}_2(z) := [E_{2-}(z) \quad E_{2+}(z)]$

(2) $\mathfrak{G}_1(z) := \left[G_{1_-}(z) \quad G_{1_+}(z)\right]$

(3) $\mathfrak{G}_2(z) := \left[G_{2_-}(z) \quad G_{2_+}(z)\right]$

It is straightforward to verify that these matrices are de Branges matrices.

Let $\alpha > 0$, and for the given P and Q in (2.1), we fix matrices $A_\pm, B_\pm \in \mathcal{U}_\alpha$ such that

$$P = A_-^{-1}B_-, \quad \text{and} \quad Q = A_+^{-1}B_+.$$

Lemma 2.1 *The $n \times 2n$ matrix $\mathfrak{E}$ defined by $\mathfrak{E}(z) := [E_-(z) \quad E_+(z)]$ where*

$$E_-(z) = G_{1_-}(z)B_-^{-1}, \quad E_+(z) = G_{2_+}(z)A_+^{-1} \tag{2.4}$$

for some $A_+, B_- \in \mathcal{U}_\alpha$, is a de Branges matrix.

Proof: From the definition of $\mathfrak{E}$ we know that $\det(E_\pm) \neq 0$. The only thing that needs to be checked is that $E_+^{-1}E_- \in S_{in}^{n\times n}$. First note that by (2.1), (2.2) and (2.3) we have

$$G_{1_+} = E_+B_+, G_{2_+} = E_+A_+, G_{1_-} = E_-B_-, G_{2_-} = E_-A_-.$$

Since $G_{1_+}^{-1}G_{1_-} \in S_{in}^{n\times n}$ then $(E_+B_+)^{-1} E_-B_- \in S_{in}^{n\times n}$. Hence

$$B_-^*E_-^*E_+^{*^{-1}}\left(B_+^{*^{-1}}B_+^{-1}\right)E_+^{-1}E_-B_- = I_p, \text{ a.e. on } \mathbb{R},$$

which implies that $E_-^*E_+^{*^{-1}}E_+^{-1}E_- = I_n$ a.e. on $\mathbb{R}$, as $B_+, B_- \in \mathcal{U}_\alpha$. Similar calculations shows that $E_+^{-1}E_- \in S_{in}^{n\times n}$. □

The proof of the next theorem follows directly from the discussion in the subsection 1.2.

Theorem 2.1 *Let $\mathfrak{E}(z) := [E_-(z) \quad E_+(z)]$ be the de Branges matrix defined in (2.4), then the corresponding RK is given by*

$$\begin{aligned} K_w^{\mathfrak{E}}(z) &:= [F_-(z) \quad F_+(z)] \begin{bmatrix} K_w^{\mathfrak{E}_1}(z) & 0 \\ 0 & K_w^{\mathfrak{E}_2}(z) \end{bmatrix} [F_-(w) \quad F_+(w)]^* \\ &= F_-(z)K_w^{\mathfrak{E}_1}(z)F_-^*(w) + F_+(z)K_w^{\mathfrak{E}_2}(z)F_+^*(w) \\ &= K_w^{(1)}(z) + K_w^{(2)}(z) \end{aligned} \tag{2.5}$$

where $K_w^{\mathfrak{E}_1}$ is a RK for the space $\mathcal{H}(\mathfrak{E}_1)$, $K_w^{\mathfrak{E}_2}$ is a RK for the space $\mathcal{H}(\mathfrak{E}_2)$, with respect to the inner product in $\mathcal{H}(\mathfrak{E})$, and $K_w^{(i)}(z)$ is the RK of the space $\mathcal{H}_i$, $i = 1, 2$, with

$$\begin{aligned} \mathcal{H}_1 &= F_-(z)\mathcal{H}(\mathfrak{E}_1) = \{F_-(z)f(z) : f \in \mathcal{H}(\mathfrak{E}_1)\} \\ \mathcal{H}_2 &= F_+(z)\mathcal{H}(\mathfrak{E}_2) = \{F_+(z)f(z) : f \in \mathcal{H}(\mathfrak{E}_2)\} \end{aligned}$$

3. Orthogonality in $\mathcal{H}(\mathfrak{E})$

In this section we prove that the space $\mathcal{H}(\mathfrak{E})$ is a direct sum of the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. First we prove that the spaces $\mathcal{H}(\mathfrak{E}_1)$ and $\mathcal{H}(\mathfrak{E}_2)$ can be embedded into the larger space $\mathcal{H}(\mathfrak{E})$. For simplicity we set $\alpha' = \sqrt{\frac{\alpha}{2}}$.

Proposition 3.1 *Let $\mathfrak{F}, \mathfrak{E}_1$ be any two de Branges matrices, and $\mathfrak{E}_2$ be the de Branges matrix defined in (2.1). The operator $\mathcal{I} : \mathcal{H}(\mathfrak{E}_2) \to \mathcal{H}(\mathfrak{E})$, defined by $\mathcal{I}(f) = \alpha' F_+ f$ is a linear isometry.*

Proof: If $f \in \mathcal{H}(\mathfrak{E}_2)$, then $E_{2+}^{-1} f \in \mathbb{H}_2^n$ and $E_{2-}^{-1} f \in (\mathbb{H}_2^n)^{\perp}$. Hence, $E_+^{-1}(\alpha' F_+ f) = \alpha' A_+ E_{2+}^{-1} f \in \mathbb{H}_2^n$. Moreover, $E_-^{-1}(\alpha' F_+ f) = \alpha' A_- E_{2-}^{-1} f \in (\mathbb{H}_2^n)^{\perp}$, that is, the operator $\mathcal{I}$ is well-defined. On the other hand, for any $f \in \mathcal{H}(\mathfrak{E}_2)$, we have

$$\begin{aligned} \langle \mathcal{I}(f), \mathcal{I}(f) \rangle_{\mathcal{H}(\mathfrak{E})} &= \left\langle E_+^{-1}(\alpha' F_+ f), E_+^{-1}(\alpha' F_+ f) \right\rangle_{\mathrm{st}} \\ &= \left\langle \alpha' A_+ E_{2+}^{-1} f, \alpha' A_+ E_{2+}^{-1} f \right\rangle \\ &= \left\langle E_{2+}^{-1} f, E_{2+}^{-1} f \right\rangle_{\mathrm{st}} = \langle f, f \rangle_{\mathcal{H}(\mathfrak{E}_2)} \end{aligned}$$

this completes the proof of the proposition.

□

The following result can also be shown in a straightforward way similar to Proposition 3.1:

Proposition 3.2 *Let $\mathfrak{F}, \mathfrak{E}_1$ be any two de Branges matrices, and $\mathfrak{E}_2$ be the de Branges matrix defined in (2.1). The operator $\mathcal{J} : \mathcal{H}(\mathfrak{E}_1) \to \mathcal{H}(\mathfrak{E})$, defined by $\mathcal{J}(f) = \alpha' F_- f$ is a linear isometry.*

Theorem 3.1 *The images of the operators $\mathcal{I}$ and $\mathcal{J}$ are orthogonal in $\mathcal{H}(\mathfrak{E})$.*

Proof: Let $f \in \mathcal{H}(\mathfrak{E}_1)$ and $g \in \mathcal{H}(\mathfrak{E}_2)$, then

$$\begin{aligned} \left\langle E_+^{-1}(\alpha' F_- f), E_+^{-1}(\alpha' F_+ g) \right\rangle &= \left\langle \alpha' B_+ E_{1_+}^{-1} f, \alpha' A_+ E_{2_+}^{-1} g \right\rangle \\ &= \left\langle \alpha' B_+ E_{1+}^{-1} f, \alpha' A_+ \left(E_{1+}^{-1} E_{1-} \right) E_{1_-}^{-1} E_{1+} E_{2+}^{-1} g \right\rangle = 0 \end{aligned}$$

because $f \in \mathcal{H}(\mathfrak{E}_1)$ if and only if $E_{1+}^{-1} f \in \mathbb{H}_2^n \ominus E_{1+}^{-} E_{1-} \mathbb{H}_2^n$, and $E_{1-}^{-1} E_{1+}$ is the inverse of a matrix valued inner function.

□

As a consequence of the orthogonality in Theorem 3.1 we get

Theorem 3.2 *Let $\mathfrak{F}, \mathfrak{E}_1$ be any two de Branges matrices, and $\mathfrak{E}_2$ be the de Branges matrix defined in (2.1). Then*

$$\mathcal{H}(\mathfrak{E}) = F_- \mathcal{H}(\mathfrak{E}_1) \oplus F_+ \mathcal{H}(\mathfrak{E}_2) \tag{3.1}$$

i.e., for any $h \in \mathcal{H}(\mathfrak{E})$, there exist a unique $f_1 \in \mathcal{H}(\mathfrak{E}_1)$ and $f_2 \in \mathcal{H}(\mathfrak{E}_2)$ such that $h = F_- f_1 + F_+ f_2$, and

$$\|h\|_{\mathcal{H}(\mathfrak{E})}^2 = \|f_1\|_{\mathcal{H}(\mathfrak{E}_1)}^2 + \|f_2\|_{\mathcal{H}(\mathfrak{E}_2)}^2$$

Proof: If there exists $h \in \mathcal{H}_1 \cap \mathcal{H}_2$, then $h = F_- f_1 = F_+ f_2$ for some $f_1 \in \mathcal{H}(\mathfrak{E}_1)$ and $f_2 \in \mathcal{H}(\mathfrak{E}_2)$. By Propositions 3.1 and 3.2

$$F_+ f_2 \in \mathrm{Image}(\mathcal{I}) \text{ and } F_- f_1 \in \mathrm{Image}(\mathcal{J})$$

which implies that $f_1 = f_2 \equiv 0$. That is, $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$. By orthogonality of $\mathcal{H}_1$ and $\mathcal{H}_2$ we obtain the direct sum in (3.1). Moreover, if $h = F_- f_1 + F_+ f_2 \in \mathcal{H}(\mathfrak{E})$, with $f_1 \in \mathcal{H}(\mathfrak{E}_1)$ and $f_2 \in \mathcal{H}(\mathfrak{E}_2)$ we have

$$\begin{aligned} \|h\|_{\mathcal{H}(\mathfrak{E})}^2 &= \langle h, h \rangle_{\mathcal{H}(\mathfrak{E})} \\ &= \langle F_- f_1 + F_+ f_2, F_- f_1 + F_+ f_2 \rangle_{\mathcal{H}(\mathfrak{E})} \\ &= \langle F_- f_1, F_- f_1 \rangle + \langle F_- f_1, F_+ f_2 \rangle + \langle F_+ f_2, F_- f_1 \rangle + \langle F_+ f_2, F_+ f_2 \rangle \\ &= \|F_- f_1\|_{\mathcal{H}(\mathfrak{E})}^2 + \|F_+ f_2\|_{\mathcal{H}(\mathfrak{E})}^2 \\ &= \|f_1\|_{\mathcal{H}(\mathfrak{E}_1)}^2 + \|f_2\|_{\mathcal{H}(\mathfrak{E}_2)}^2 \end{aligned}$$

□

We provide the important results of this section, which describes the deep link between the spaces $\mathcal{H}(\mathfrak{E}_1), \mathcal{H}(\mathfrak{E}_2)$, and $\mathcal{H}(\mathfrak{E})$. By embedding the de Branges spaces $\mathcal{H}(\mathfrak{E}_1)$ and $\mathcal{H}(\mathfrak{E}_2)$ into a larger de Branges space $\mathcal{H}(\mathfrak{E})$ while embedding the kernel functions associated with a frame sequence into a Riesz basis for the larger space. By associating the basic structure with Naimark dilation of frames, we provide essential conditions for Parseval sequences in these spaces.

Theorem 3.3 *If $\{\omega_j\} \subset \mathbb{C}$ and $\{u_j\} \subset \mathbb{C}^n$ such that $\left\{ \frac{K^{\mathfrak{E}}_{\omega_j}(.)u_j}{\sqrt{u_j^* K^{\mathfrak{E}}_{\omega_j}(\cdot)u_j}} \right\}$ is a complete orthonormal set for $\mathcal{H}(\mathfrak{E})$ then*

1. $\left\{ \frac{K^{\mathfrak{E}_1}_{\omega_j}(.)F_-^*(\omega_j)u_j}{\sqrt{u_j^* K^{\mathfrak{E}}_{\omega_j}(\cdot)u_j}} \right\}$ *is a Parseval frame for $\mathcal{H}(\mathfrak{E}_1)$, and for every $f \in \mathcal{H}(\mathfrak{E}_1)$*

$$f(z) = \sum_j u_j^* F_-(\omega_j) f(\omega_j) \frac{K^{\mathfrak{E}_1}_{\omega_j}(z)F_-^*(\omega_j)u_j}{u_j^* K^{\mathfrak{E}}_{\omega_j}(\omega_j)u_j} \tag{3.2}$$

2. $\left\{ \frac{K^{\mathfrak{E}_2}_{\omega_j}(.)F_+^*(\omega_j)u_j}{\sqrt{u_j^* K^{\mathfrak{E}}_{\omega_j}(\omega_j)u_j}} \right\}$ *is a Parseval frame for $\mathcal{H}(\mathfrak{E}_2)$, and for every $g \in \mathcal{H}(\mathfrak{E}_2)$*

$$g(z) = \sum_j u_j^* F_+(\omega_j) g(\omega_j) \frac{K^{\mathfrak{E}_2}_{\omega_j}(z)F_+^*(\omega_j)u_j}{u_j^* K^{\mathfrak{E}}_{\omega_j}(\omega_j)u_j} \tag{3.3}$$

Proof: By equation (2.5) we have

$$K^{\mathfrak{E}}_{\omega}(z)u = \mathcal{J}\left(K^{\mathfrak{E}_1}_{w}(z)F_-^*(w)u\right) + \mathcal{I}\left(K^{\mathfrak{E}_2}_{w}(z)F_+^*(w)u\right)$$

hence,

$$\frac{K^{\mathfrak{E}}_{\omega_j}(.)u_j}{\sqrt{u_j^* K^{\mathfrak{E}}_{\omega_j}(\omega_j)u_j}} = \frac{\mathcal{J}\left(K^{\mathfrak{E}_1}_{\omega_j}(.)F_-^*(\omega_j)u_j\right)}{\sqrt{u_j^* K^{\mathfrak{E}}_{\omega_j}(\omega_j)u_j}} + \frac{\mathcal{I}\left(K^{\mathfrak{E}_2}_{\omega_j}(.)F_+^*(\omega_j)u_j\right)}{\sqrt{u_j^* K^{\mathfrak{E}}_{\omega_j}(\omega_j)u_j}}$$

therefore,

$$P_{\mathfrak{E}}\left(\frac{K^{\mathfrak{E}}_{\omega_j}(.)u_j}{\sqrt{u_j^* K^{\mathfrak{E}}_{\omega_j}(\omega_j)u_j}}\right) = \frac{\mathcal{J}\left(K^{\mathfrak{E}_1}_{\omega_j}(.)F_-^*(\omega_j)u_j\right)}{\sqrt{u_j^* K^{\mathfrak{E}}_{\omega_j}(\omega_j)u_j}}$$

since $\left\{ \frac{K^{\mathfrak{E}}_{\omega_j}(.)u_j}{\sqrt{u_j^* K^{\mathfrak{E}}_{\omega_j}(\omega_j)u_j}} \right\}$ is an orthonormal set for $\mathcal{H}(\mathfrak{E})$ and $\mathcal{J}$ is an isometric from $\mathcal{H}(\mathfrak{E}_1)$ onto $\mathcal{J}(\mathcal{H}(\mathfrak{E}_1))$ then

$$\frac{\mathcal{J}\left(K^{\mathfrak{E}_1}_{\omega_j}(.)F_-^*(\omega_j)u_j\right)}{\sqrt{u_j^* K^{\mathfrak{E}}_{\omega_j}(\omega_j)u_j}} \tag{3.4}$$

is a Parseval frame for $\mathcal{J}(\mathcal{H}(\mathfrak{E}_1))$. Applying $\mathcal{J}^*$ to (3.4) we obtain the first claim. Consequently, given any $f \in \mathcal{H}(\mathfrak{E}_1)$ we have

$$\begin{aligned} f(z) &= \sum_j \left\langle f, \frac{K^{\mathfrak{E}_1}_{\omega_j}(.)F_-^*(\omega_j)u_j}{\sqrt{u_j^* K^{\epsilon}_{\omega_j}(\omega_j)u_j}} \right\rangle_{\mathcal{H}(\mathfrak{E}_1)} \frac{K^{\mathfrak{E}_1}_{\omega_j}(z)F_-^*(\omega_j)u_j}{\sqrt{u_j^* K^{\mathfrak{E}}_{\omega_j}(\omega_j)u_j}} \\ &= \sum_j u_j^* F_-(\omega_j) f(\omega_j) \frac{K^{\mathfrak{E}_1}_{\omega_j}(z)F^*_{-(\omega_j)u_j}}{u_j^* K^{\mathfrak{E}}_{\omega_j}(\omega_j)u_j} \end{aligned}$$

Using an analogous argument we obtain the second claim. □

The interesting aspect of the last result is that the Parseval frames for $\mathcal{H}(\mathfrak{E}_1)$ and $\mathcal{H}(\mathfrak{E}_2)$ given in Theorem 3.3 are orthogonal:

Theorem 3.4 *Assume the hypothesis of Theorem 3.3 then*

1. *For every $f \in \mathcal{H}(\mathfrak{E}_1)$*
$$\sum_j u_j^* F_-(\omega_j) f(\omega_j) \frac{K_{\omega_j}^{\mathfrak{E}_2}(.) F_+^*(\omega_j) u_j}{u_j^* K_{\omega_j}^{\mathfrak{E}}(.) u_j} = 0 \tag{3.5}$$

2. *For every $g \in \mathcal{H}(\mathfrak{E}_2)$*
$$\sum_j u_j^* F_+^*(\omega_j) g(\omega_j) \frac{K_{\omega_j}^{\mathfrak{E}_1}(.) F_-^*(\omega_j) u_j}{u_j^* K_{\omega_j}^{\mathfrak{E}}(.) u_j} = 0 \tag{3.6}$$

Proof: Let $f \in \mathcal{H}(\mathfrak{E}_1)$. Since $F_- f \in \mathcal{H}(\mathfrak{E})$ and $\left\{ \frac{K_{\omega_j}^{\mathfrak{E}}(.) u_j}{\sqrt{u_j^* K_{\omega_j}^{\mathfrak{E}}(.) u_j}} \right\}$ is a complete orthonormal set for $\mathcal{H}(\mathfrak{E})$ then

$$\begin{aligned}
\mathcal{I}(f)(z) &= \alpha' F_-(z) f(z) \\
&= \sum_j \left\langle \alpha' F_- f, \frac{K_{\omega_j}^{\mathfrak{E}}(.) u_j}{\sqrt{u_j^* K_{\omega_j}^{\mathfrak{E}}(\omega_j) u_j}} \right\rangle \frac{K_{\omega_j}^{\mathfrak{E}}(z) u_j}{\sqrt{u_j^* K_{\omega_j}^{\mathfrak{E}}(\omega_j) u_j}} \\
&= \sum_j \alpha' u_j^* F_-(\omega_j) f(\omega_j) \frac{K_{\omega_j}^{\mathfrak{E}}(z) u_j}{u_j^* K_{\omega_j}^{\mathfrak{E}}(\omega_j) u_j} \\
&= \sum_j \frac{\alpha' u_j^* F_-(\omega_j) f(\omega_j)}{u_j^* K_{\omega_j}^{\mathfrak{E}}(\omega_j) u_j} \left(F_-(z) K_{\omega_j}^{\mathfrak{E}_1}(z) F_-^*(\omega_j) u_j + F_+(z) K_{\omega_j}^{\mathfrak{E}_2}(z) F_+^*(\omega_j) u_j \right) \\
&= \sum_j \frac{\alpha' u_j^* F_-(\omega_j) f(\omega_j)}{u_j^* K_{\omega_j}^{\mathfrak{E}}(\omega_j) u_j} \left(\mathcal{J}\left(K_{\omega_j}^{\mathfrak{E}_1}(z) F_-^*(\omega_j) u_j \right) + \mathcal{I}\left(K_{\omega_j}^{\mathfrak{E}_2}(z) F_+^*(\omega_j) u_j \right) \right)
\end{aligned}$$

Applying $\mathcal{I}^*$ to the last line above, and using the fact that $\mathcal{I}^*(F_- f) = 0$ we obtain equation (3.5). similar argument applying $\mathcal{J}^*$ to $F_+ g$ yields equation (3.6). □

As a consequence of the last two theorems we get that under the assumptions of Theorem 3.3, it can be easily shown that if $f \in \mathcal{H}(\mathfrak{E}_1)$ and $g \in \mathcal{H}(\mathfrak{E}_2)$, we can encode both f and g into the *multiplexed samples*:

$$\left\{ u_j^* F_-(\omega_j) f(\omega_j) + u_j^* F_+^*(\omega_j) g(\omega_j) \right\}_j \tag{3.7}$$

which are transmitted in some fashion. The goal then is to recover f and g from these mixed samples. The proof of the next result follow immediately from Equations (3.2), (3.3), (3.5), and (3.6)

Corollary 3.1 *Assume the hypotheses of Theorem 3.3, $f \in \mathcal{H}(\mathfrak{E}_1)$ and $g \in \mathcal{H}(\mathfrak{E}_2)$. Given the samples $\{f(\omega_j)\}$ and $\{g(\omega_j)\}$, f and g can be reconstructed from the multiplexed samples in (3.7) as follows:*

$$f(z) = \sum_j \left(u_j^* F_-(\omega_j) f(\omega_j) + u_j^* F_+^*(\omega_j) g(\omega_j) \right) \frac{K_{\omega_j}^{\mathfrak{E}_1}(z) F_-^*(\omega_j) u_j}{u_j^* K_{\omega_j}^{\mathfrak{E}}(\omega_j) u_j}$$

$$g(z) = \sum_j \left(u_j^* F_-(\omega_j) f(\omega_j) + u_j^* F_+^*(\omega_j) g(\omega_j) \right) \frac{K_{\omega_j}^{\mathfrak{E}_2}(z) F_+^*(\omega_j) u_j}{u_j^* K_{\omega_j}^{\mathfrak{E}}(\omega_j) u_j}$$

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Conflict of interest

The author declare that he has no conflict of interest.

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