Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.68089

Transitivity in QTAG-Modules: A study

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ABSTRACT: In this paper, we explore the characteristics that ensure a $QTAG$-module M is transitive or fully transitive, whenever $H_\beta(M)$ is transitive or fully transitive for any ordinal β. We extend the concept of transitivity to incorporate strong transitivity and projective full transitivity for QTAG-modules, and prove that a strongly transitive $QTAG$-module M is fully transitive. We also demonstrate that the full transitivity of M is equivalent to the full transitivity, transitivity, and projective full transitivity of $\bigoplus_I M$ for any index set I. Finally, we pose some open problems for further research, including the characterization of projectively fully transitive modules in terms of their endomorphism rings and the determination of the structure of projectively fully transitive modules over certain classes of rings.

Key Words: QTAG Modules, h-pure submodule, h-reduced module, tansitive modules, fully transitive modules.

Contents

1. Introduction

Several concepts that were originally developed for groups, such as purity, projectivity, injectivity, and height, have been extended to modules. However, to achieve results that are specific to groups and not applicable to modules in general, certain conditions have been imposed either on modules or the underlying rings. In this study, the condition was imposed on the modules such that every finitely generated submodule of any homomorphic image of the module is a direct sum of uniserial modules, while the rings are associative with unity. With these conditions in place, several elegant results of groups can be proven for QTAG-modules that are not true in general.

Transitivity for $QTAG$-modules, analogous to the transitivity of abelian groups, was defined by Sikander [5] and Hasan [4]. Although some generalizations of transitivity and Ulm-supports were investigated, still a large area remained unexplored. While $QTAG$-modules are not typically transitive or fully transitive, it was found that h-reduced $QTAG$-modules exhibit both transitive and fully transitive properties.

The structure of the paper is outlined as follows:

Section 2 provides a brief review of QTAG-module theory and important related concepts. Section 3 focuses on investigating the conditions necessary for a $QTAG$-module M to be transitive or fully transitive when $H_\beta(M)$ is transitive or fully transitive. This analysis is performed for any arbitrary ordinal β. Furthermore, it is demonstrated that if $M/H_\beta(M)$ is a direct sum of countably generated modules and

* Corresponding author.
2010 *Mathematics Subject Classification:* 20K10.
Submitted May 03, 2023. Published December 04, 2025

$H_\beta(M)$ is fully transitive, then M is also fully transitive. This result holds true for transitivity when β is a countable ordinal.

Section 4 deals with the generalizations of the transitivity, where we study strong transitive modules and projectively transitive modules. We establish that a $QTAG$-module M is fully transitive if it is strongly transitive. Additionally, we demonstrate that for any index set I, the full transitivity of M is equivalent to the full transitivity of $\bigoplus_I M$, the transitivity of $\bigoplus_I M$, and the projectively full transitivity of $\bigoplus_I M$.

2. Preliminaries

The investigation of the structure of QTAG-modules was initiated by Singh [11], and subsequent research by Khan, Mehdi, Abbasi, among others, generalized various group concepts for the QTAG-modules [7,9]. These scholars introduced various ideas and structures for QTAG-modules that were inspired by group structures and yielded some exciting findings. However, several concepts still require further generalization for modules. In the present study, we aim to extend the scope of QTAG-module theory to include full transitivity and projectively full transitivity. The literature on abelian p-groups has explored various concepts of transitivity, as documented in [1] and [2]. This present study extends several findings from the paper [1].

According to the definition provided by Singh [12], a module M over an associative ring R with unity is classified as a $QTAG$-module if each finitely generated submodule of any homomorphic image of M can be expressed as a direct sum of uniserial modules. It should be noted that all rings R under consideration in this context are associative and have unity, and the modules M are regarded as unital $QTAG$-modules. An element $x \in M$ is said to be uniform if xR is a non-zero uniform module (and thus uniserial). Furthermore, for any R-module M with a unique composition series, we use the notation $d(M)$ to represent its decomposition length. According to the paper by Khan [8], the exponent and height of x in M are denoted by $e(x)$ and $H_M(x)$ respectively, and are defined as follows: $e(x) = d(xR)$ and $H_M(x) = \sup\{d(yR/xR) \mid y \in M, x \in yR, \text{ and } y \text{ is uniform}\}$.

The submodule of M generated by the elements of height at least k is denoted by $H_k(M)$, while the submodule of M generated by the elements of exponents at most k is denoted by $H^k(M)$. In particular, a module M is called h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$, as defined in [7]. On the other hand, M is referred to as h-reduced if it does not contain any h-divisible submodule. In other words, it does not contain any element of infinite height. A $QTAG$-module M is called separable if its intersection with all submodules of height at least one is zero. A submodule N of M is said to be σ-pure if for every ordinal β with $\beta \leq \sigma$, we have $H_\beta(M) \cap N = H_\beta(N)$ [10]. A $QTAG$-module M is *transitive* (*fully transitive*) if for $x, y \in M$ such that $U(x) \leq U(y)$, there exists an automorphism (endomorphism) f of M such that $f(x) = y$.

Throughout the paper we will use following abbreviation:

Countably Generated module	*CGmodule*
Direct sum of countably generated modules	*DSCGmodule*

3. Transitive $QTAG$-modules

Let M be a CG *h-reduced QTAG-* module. Then for any ordinal β, every automorphism of $H_\beta(M)$ is induced by an automorphism of M.

Now we may infer the following:

Consider two isomorphic CG $QTAG$-modules M, M' and an ordinal β. Let $x \in M, y \in M'$ such that $U(x) = U(y), \bar{x} = x + H_\beta(M), \bar{y} = y + H_\beta(M'), d(\bar{x}R) = n$. If f is an isomorphism from $H_\beta(M)$ onto $H_\beta(M')$ such that $f(x_1) = y_1$ then f extends to an isomorphism $\bar{f}$ of M onto $M'$with $\bar{f}(x) = y$, here $d\left(\frac{xR}{x_1R}\right) = d\left(\frac{yR}{y_1R}\right) = n$.

To establish our main result, we must demonstrate the following lemmas:

Lemma 3.1 *Let M be a $QTAG$-module and let β be an ordinal. If N is a h-pure submodule of $H_\beta(M)$ and if K is a submodule of M such that $\frac{M}{N} = \frac{K}{N} + \frac{H_k(M)}{N}$, then K is β-pure in M.*

Proof: Suppose the result holds good for all ordinals $\mu < \alpha$, where $\alpha \leq \beta$. If α is a limit ordinal then $H_\alpha(K) = \bigcap_{\mu<\alpha} (H_\mu(K)) = \bigcap_{\mu<\alpha} (H_\mu(M) \cap K) = (\bigcap_{\mu<\alpha} H_\mu(M) \cap K) = H_\alpha(M) \cap K$. Therefore we may assume that $\alpha - 1$ is defined. Let $x \in H_{\alpha-1}(M)$ and $d\left(\frac{xR}{x_1R}\right) = 1$ for some $x_1 \in K$. Since $M = K + H_\beta(M), x = y + z$ for some $y \in K, z \in H_\beta(M)$. Thus $y = x - z \in H_{\alpha-1}(M) \cap K = H_{\alpha-1}(K)$. Now $y_1 = x_1 - z_1 \in K \cap H_\beta(M) = N$, here $d\left(\frac{yR}{y_1R}\right) = d\left(\frac{zR}{z_1R}\right) = 1$. Thus $y_1 = u_1, d\left(\frac{uR}{u_1R}\right) = 1$ and $u \in H_\beta(M) \cap K \subseteq H_{\alpha-1}(M) \cap K = H_{\alpha-1}(K)$. Now $x_1 = y_1 + u_1 \in H_\alpha(K)$, therefore $H_\alpha(M) \cap K = H_\alpha(K)$. □

Lemma 3.2 *Let M be a $QTAG$-module such that $M/H_\beta(M)$ is a DSCG-module and β is a countable ordinal. Suppose $H_\beta(M)$ is transitive and there is an automorphism f of M such that $f(x) = f(y)$, whenever $U(x) = U(y)$ and $x_1 = y_1$ where $d\left(\frac{xR}{x_1R}\right) = d\left(\frac{yR}{y_1R}\right) = n = e(x + H_\beta(M))$. Then M is transitive.*

Proof: Let $x, y \in M$ such that $U(x) = U(y)$ and $e(x + H_\beta(M)) = n$. If $d\left(\frac{xR}{x_1R}\right) = n = d\left(\frac{yR}{y_1R}\right)$ then $x_1 \in H_\beta(M)$ and $U(x_1) = U(y_1)$. Since $H_\beta(M)$ is transitive, there is an automorphism ϕ of $H_\beta(M)$ such that $\phi(x_1) = y_1$. Again $\frac{M}{H_\beta(M)}$ is a DSCG module and β is a countable ordinal, therefore ϕ can be extended to an automorphism $\bar{\phi}$ of M. If $\bar{\phi}(x) = z$ and $d\left(\frac{zR}{z_1R}\right) = n$ then $z_1 = y_1$ and $U(z) = U(x) = U(y)$ because $\bar{\phi}$ is an automorphism. Therefore z and y satisfy the conditions of the statement and we have an automorphism f such that $f(z) = y$. If we put $\psi = f\bar{\phi}$ then ψ is an automorphism such that $\psi(x) = y$. □

Now we are able to prove the following :

Theorem 3.1 *Let M be a $QTAG$-module and let β be a countable ordinal. If $M/H_\beta(M)$ is a DSCG modules and $H_\beta(M)$ is transitive then M is transitive.*

Proof: Let $x, y \in M$ such that $U(x) = U(y)$ and $e(x + H_\beta(M)) = e(\bar{x}) = n$. By Lemma 3.2, $x_1 = y_1$ where $d\left(\frac{xR}{x_1R}\right) = d\left(\frac{yR}{y_1R}\right) = n$. There is a *h-pure* and *h-dense* submodule N of $H_\beta(M)$ such that $x_1 = y_1 \in N$, and N is a DSCG module. We may select submodules P, Q of M such that $x \in P, y \in Q$ and $\frac{M}{N} = \frac{P}{N} + \frac{H_\beta(M)}{N} = \frac{Q}{N} + \frac{H_\beta(M)}{N}$. By Lemma 3.1 $H_\beta(P) = H_\beta(M) \cap P = N = Q \cap H_\beta(M) = H_\beta(Q)$. Since M is the DSCG submodules, $H_\beta(P) = H_\beta(Q)$, $\frac{P}{N} \simeq \frac{M}{H_\beta(M)} \simeq \frac{Q}{N}$ and β is a countable ordinal, the identity automorphism of N can be extended to an isomorphism ϕ from P onto Q. As P and Q are DSCG modules, with the help of ϕ we may decompose P and Q such that (i)$P = P_1 + P_2, Q =$

$Q_1+Q_2, x \in P_1, y \in Q_1$, (ii) $P_1 \simeq Q_1$ and $P_2 \simeq Q_2$ under the isomorphism ϕ (iii) $H_\beta(P_1) = H_\beta(Q_1)$ and $H_\beta(P_2) = H_\beta(Q_2)$ (iv) $g(P_1) = g(Q_1) \leq \aleph_0$.

Now for all $\alpha \leq \beta$, $H_\alpha(M) \cap P = H_\alpha(P)$ and $H_\alpha(M) \cap Q = H_\alpha(Q)$ and $x_1 = y_1$ where $e(x+H_\beta(M)) = e(y+H_\beta(M)) = n$ and the *U-sequences* of x and y in M are same. Therefore the *U-sequences* of x and y are same in P and Q respectively. Since $H_\beta(P_1) = H_\beta(Q_1)$ we extend this identity isomorphism to obtain an isomorphism ψ from P_1 onto Q_1 such that $\psi(x) = y$. Again P_2 and Q_2 are isomorphic DSCG modules such that $H_\beta(P_2) = H_\beta(Q_2)$ and β is a countable ordinal, therefore there is an isomorphism $\eta : P_2 \to Q_2$ such that the restriction of η on $H_\beta(P_2)$ is identity.
Now we define a map $\sigma(x+z) = \phi(x)+\eta(z)$ where $x \in P_1, z \in P_2$. Now σ is an isomorphism from P onto Q such that the restriction of σ on $H_\beta(P)$ is identity and $H_\beta(P) = H_\beta(Q) = N$ such that $\sigma(x) = y$. If we define $f : M \to M$ such that $f(u+x) = \sigma(u)+x$ for $u \in P$ and $x \in H_\beta(M)$ then f is an automorphism and we are done. □

A *QTAG*-module M is fully transitive if for any $x, y \in M$ with $U(x) \leq U(y)$, there exists an endomorphism f of M such that $f(x) = y$. This motivates us to define a system of fully transitive modules. Here we define and characterize these systems.

Now we need the following:

Theorem 3.2 *A QTAG-module M is fully transitive if and only if its h-*reduced *submodule is fully transitive.*

Proof: If M is fully transitive, then its *h-reduced* submodule is also fully transitive.

For the converse we express M as the direct sum of a *h-divisible* submodule D and *h-reduced* submodule N i.e. $M = N \oplus D$. Let $x, y (\neq 0) \in M$ such that $U(x) \leq U(y)$. We may write $x = x_1 + x_2, y = y_1 + y_2, x_1, y_1 \in N, x_2, y_2 \in D$. Now there may be four cases.
Case(i) $x_1, y_1 \neq 0$, thus $U(x) = U(x_1), U(y) = U(y_1)$ i.e. $U(x_1) \leq U(y_1)$. Therefore there exists an endomorphism ϕ of N such that $\phi(x_1) = x_2$. Now this ϕ can be extended to an endomorphism $\bar{\phi}$ of M such that $\bar{\phi}(z) = \phi(z)$ if $z \in N$ and $\bar{\phi}(z) = 0$ if $z \in D$.
Case(ii) $x_1 \neq 0, y_1 = 0$. We may define a map $\phi : xR \to D$ such that $\phi(x) = y$. Let $\rho : xR \to M$ be an embedding. Being *h-divisible* D is injective, therefore there exists a homomorphism $\psi : M \to D$ such that $\psi\rho = \phi$ i.e. $\psi(x) = y$.
Case(iii) $x_1 = 0, y_1 \neq 0$ is not possible as $U(x_1) \leq U(y_1)$.
Case(iv) $x_1, y_1 = 0$ i.e. $x, y \in D$. Therefore xR, yR are the submodules of D. Consider the homomorphism $\phi : xR \to yR$ such that $\phi(x) = y$ and the embedding $\sigma : xR \to D$. Since D is *h-divisible* it is injective and there exists an endomorphism ψ of D such that $\psi\sigma = \phi$. This ψ can be extended to an endomorphism of M. Therefore M is fully transitive. □

Now we define the following:

Definition 3.1 *A family of QTAG-modules $\{M_i\}_{i \in I}$ is said to be fully transitive if for each pair of modules M_i, M_j and $x \in M_i$, $y \in M_j$,$U(x) \leq U(y)$, there exists a homomorphism f from M_i into M_j mapping x onto y.*

Definition 3.2 *A system of QTAG-modules $\{M_i\}_{i \in I}$ satisfies the monotonicity condition if for each nonzero element $x_j \in M_j$ the relations*
(i) $inf(U(x_{i1}), ...U(x_{ik})) \leq U(x_j)$ where $x_{il} \in M_{il}$ $il \in I$, $l = 1, ...k$ $il \neq it$ if $t \neq i$
(ii)$U(x_j) \ngeq U(x_l)$ for all $l = 1, ...j$ imply the existence of the elements $x_{j_1}, ...x_{j_t} \in M_j$ such that
(1) $x_{j1} + ... + x_{jt} = x_j$
(2) For each element x_{jk} $(k = 1, 2, ..., t)$, there exists an element x_{iu} $(u = 1, 2, ...k)$ such that $U(x_{iu}) \leq U(x_{jk})$.

Now we are able to prove the following:

Theorem 3.3 *Let M be a QTAG-module which is the direct sum of QTAG-modules M_i such that $M = \bigoplus_{i\in I} M_i$. Then M is fully transitive if and only if the system of modules $\{M_i\}$ is fully transitive and satisfies the monotonicity condition.*

Proof: Let M be a fully transitive $QTAG$-module. Consider any two arbitrary modules M_i, M_j from the system and $x_i \in M_i$, $x_j \in M_j$ such that $H(x_i) \leq H(x_j)$. If ρ_i, ρ_j are the injective maps from M_i, M_j into M then $H(\rho_i x_i) \leq H(\rho_j x_j)$. Since M is fully transitive there exists an endomorphism ϕ of M such that $\phi(\rho_i x_i) = (\phi\rho_i)x_i = \rho_j x_j$.
If $\pi_j : M \to M_j$ is the projection map then $(\pi_j\phi\rho_i)x_i = \pi_j\rho_j x_j = x_j$. In order to prove the monotonicity condition consider a nonzero element $x_j \in M_j$ such that $inf(U(x_{i1}, x_{i2}, ...x_{it})) \leq U(x_j)$, $U(x_j) \ngeq U(x_k)$ for all $k = 1, 2, ...t$. For every k consider the injective maps $\rho_{ik} : M_{ik} \to M$ and $\rho_j : M_j \to M$. Then $U(\rho_{i1}(x_{i1}) + ... + \rho_{it}(x_{it})) \leq U(\rho_j(x_j))$. Since M is fully transitive, there exists an endomorphism ϕ of M such that $\phi(\rho_{i1}(x_{i1}) + ... + \rho_{it}(x_{it})) = \rho_j x_j$. By operating the projection map $\pi_j : M \to M_j$ we get $(\pi_j\phi\rho_{i1})x_{i1} + ... + (\pi_j\phi\rho_{it})x_{it} = x_j$. Suppose $x_{jk} = (\pi_j\phi\rho_{ik})x_{ik}$ for all $k = 1, 2....t$. Then $x_j = x_{j1} + ... + x_{jt}$ and for each element x_{jk} there exists x_{ik} such that $U(x_{ik}) \leq U(x_{jk})$.

For the converse let $x, y \in M$ with $U(x) \leq U(y)$. Suppose $x = x_{i1} + ... + x_{im}, y = y_{j1} + .. + y_{jn}$ where $x_{it} \in M_{it}$, $y_{jk} \in M_{jk}$. If for y_{jk}, there exists x_{it} such that $U(y_{jk}) \geq U(x_{it})$ then there exists a homomorphism $\psi_k : M_{it} \to M_{jk}$ such that $\psi_k(x_{it}) = y_{jk}$ because the system is fully transitive. We define $\phi_k = \psi_k\pi_{it}$ where π_{it} is the projection of M onto M_{ij}. Suppose there is no element x_{it} such that $U(y_{jk}) \geq U(x_{it})$. Since $U(y_{jk}) \geq infU(x_{it})$ and the system satisfies monotonicity condition there exist elements $y_{jku} \in M_{jk}$ such that $y_{jk} = y_{jk1} + ... + y_{jkl}$ and for each y_{jkv} there are x_{itv} such that $U(y_{jku}) \geq U(x_{itv})$. Now the full transitivity of the system $\{M_i\}$ ensures the existence of $\psi_{kv} : M_{itv} \to M_{jk}$ such that $\psi_{kv}(x_{itv}) = y_{jku}$ and $\psi_{k1}(x_{it1}) + ... + \psi_{kl}(x_{itl}) = y_{jk}$. Then $\phi_k(x) = \psi_{k1}(x_{it1}) + ... + \psi_{kl}(x_{itl})$ where $\phi_k = \psi_{k1}\pi_{it1} + ... + \psi_{kl}\pi_{itl}$. Therefore $\phi_k(x) = y_{jk}$ and $\phi = \sum \phi_k$ is the required homomorphism such that $\phi(x) = y$. □

4. Some Generalizations of Transitivity

Ayaz [4] defined strongly transitive modules and proved some interesting results. We start by recalling the definition and a basic result.

Definition 4.1 *A QTAG-module M is called strongly transitive if for each pair of elements $x, y \in M$ with $U(x) = U(y)$, there is an endomorphism f of M such that $f(x) = y$.*

We shall use the following lemma proved by Ayaz [4].

Lemma 4.1 *Let M be a QTAG-module such that for all $x, y \in M$ with $y \in Soc(M)$ and $U(x) \leq U(y)$. If there is an endomorphism f of M mapping x onto y, then M is fully transitive.*

Proposition 4.1 *A QTAG module M is fully transitive if and only if it is strongly transitive.*

Proof: Suppose M is strongly transitive and $x, y \in M$ with $U(x) \leq U(y)$. By Lemma 4.1, without loss of generality we may assume $y \in Soc(M)$. If $U(x) = U(y)$ then there exists an endomorphism ϕ of M such that $\phi(x) = y$. Therefore we assume that $U(x) < U(y)$. If $H(x) < H(y)$ then $H(x) = H(x+y)$ and $U(x) = U(x+y)$, therefore there exists an endomorphism ϕ of M with $\phi(x) = x+y$. Thus $\phi - I_M = \psi$ is the endomorphism of M, mapping x onto y. If $x \in Soc(M)$ and $U(x) < U(y)$ then $H(x) < H(y)$. Therefore we assume that $x \notin Soc(M)$, $y \in Soc(M)$ and $H(x) = H(y)$. If $H(x) = H(x + y)$ we may take $z = y$. If $H(x) < H(x+y)$ we may put $z = y+y$ and we have $H(y) = H(z)$ and $H(x+z) = H((x+y)+y) = H(y)$ because $H(y) = H(x) < H(x + y)$. Thus we find $z \in Soc(M)$ with $H(y) = H(z)$, $H(x) = H(x + z)$ and $U(x) = U(x+z)$, $U(y) = U(z)$ therefore there exist endomorphisms ϕ and ψ of M such that $\phi(x) = x+z$ and $\psi(z) = y$ and $\psi(\phi - I_M)$ maps x onto y as required.
Converse is trivial.

□

Now we generalize the concept of full transitivity to projective full transitivity and strong projective full transitivity.

Definition 4.2 *A QTAG-module M is said to be projectively* fully transitive *if for $x, y \in M$ with $U(x) \leq U(y)$, there exists an idempotent (projection) endomorphism f of M such that $f(x) = y$.*

Now we prove some basic results on projective full transitivity.

Proposition 4.2 *If M_1, M_2 are projectively fully transitive modules and $\{M_1, M_2\}$ is a fully transitive pair then $M_1 \oplus M_2$ is a projectively* fully transitive *modules.*

Proof: Let $M = M_1 \oplus M_2$ and $x, y \in M$ with $U(x) \leq U(y)$ and let $x = (x_1, x_2)$, $y = (y_1, y_2) \in M_1 \oplus M_2$ with $e(y) = 1$. We may assume that $H_M(x_1) = H_{M_1}(x_1)$. Now we have $U_{M_1}(x_1) \leq U_M(y) \leq U_{M_2}(y_2)$. Since $\{M_1, M_2\}$ is a fully transitive pair , there is a homomorphism $\phi : M_1 \to M_2$ such that $\phi(x_1) = y_2$. Also $e(y_1 - x_1) = e(x_1)$ thus $U_{M_1}(x_1) \leq U_{M_1}(y_1 - x_1)$. Again M_1 is a *projectively fully transitive* module, there exists a projection ψ of M_1 with $\psi(x_1) = y_1 - x_1$ Now $A=\begin{pmatrix} \psi & 0 \\ 0 & 0 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & \phi \\ 0 & 0 \end{pmatrix}$ are endomorphism of M such that $A + B$ maps (x_1, x_2) onto (y_1, y_2). Since the injective map from the ring of endomomorphism of M_1 into the ring of endomorphism of M is a homomorphism, $\begin{pmatrix} \psi & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & \phi \\ 0 & 0 \end{pmatrix}$ are the projections of M. Thus the result is true when $e(y) = 1$
Now we suppose that the result holds for all y with $e(y) = n$. Consider $y \in M$ such that $e(y) = n+1$. Let $d\left(\frac{xR}{x'R}\right) = d\left(\frac{yR}{y'R}\right) = 1$, $e(y') \leq n$ and $U(x') \leq U(y')$. Therefore there exists a projection ϕ of M with $\phi(x') = y'$. If we put $z = y - \phi(x)$ then $z \in Soc(M)$ and $U(x) \leq U(z)$. Hence there exists a projection ψ of M with $\psi(x) = z$. Now $\phi + \psi$ is also a projection of M and $(\phi + \psi)(x) = \phi(x) + (y - \phi(x)) = y$ and we are done. □

Proposition 4.3 *Let N be a h-*divisible *QTAG module and let K be a h-*reduced *QTAG module. If $M = N \oplus K$ is projectively* fully transitive *then so is K.*

Proof: Let $x, y \in K$ such that $U(x) \leq U(y)$. Then $U(0, x) \leq U(0, y)$. Therefore there exists a projection ϕ of M with $\phi(0, x) = (0, y)$. Suppose the matrix representation of ϕ is $\begin{pmatrix} \phi_1 & \phi_2 \\ 0 & \phi_3 \end{pmatrix}$ which is idempotent. Now the diagonal entries ϕ_1, ϕ_3 are projections of N and K respectively. Therefore any product of idempotent matrices have idempotent diagonal entries, thus ϕ_3 is also a projection(idempotent) of K such that $\phi_3(x) = y$ and K is *projectively fully transitive.* □

Since *h-divisible QTAG*-modules are *projectively fully transitive* we conclude the following:

Theorem 4.1 *Let $M = N \oplus K$ where N is h-*divisible *and K is h-*reduced*. Then M is projectively* fully transitive *if and only if K is projectively* fully transitive

Now we investigate the relation between different transitivities.

Theorem 4.2 *Let I be an index set such that $|I| > 1$. Then the following are equivalent.*

(i) M *is fully* transitive.

(ii) $\bigoplus_I M$ *is fully* transitive.

(iii) $\bigoplus_I M$ *is transitive*

(iv) $\bigoplus_I M$ *is projectively* fully transitive

Proof: Ayaz [4] proved the following equivalent conditions (a) for all ordinals β, $\bigoplus_\beta M$ is fully transitive. (b) for some $\beta > 0$, $\bigoplus_\beta M$ is fully transitive. (c) for all $\beta > 1$, $\bigoplus_\beta M$ is transitive. (d) for some $\beta > 1$, $\bigoplus_\beta M$ is transitive. Now the equivalence of (i) and (ii) follows from the above result. Similarly the equivalence of (ii) and (iii) is also implied by this result. We have to show the equivalence of (ii) and (iv). Since projectively fully transitive module is always fully transitive (iv) implies (ii).

For the converse if $\bigoplus_I M$ is fully transitive, then the ring of endomorphism of $\bigoplus_I M$ is isomorphic to the ring of 2x2 matrices over $E(M)$. Hence every endomorphism of $\bigoplus_I M$ is a projection and $\bigoplus_I M$ is projectively fully transitive. □

Following are the immediate consequences of the above result.

Corollary 4.1 *If M is projectively fully transitive, then for every ordinal β, $\bigoplus_\beta M$ is projectively fully transitive.*

Remark 4.1 *A direct summand of a projectively fully transitive module need not be a fully transitive module.*

Socle regularity was defined in [6]. A $QTAG$ module M is socle regular if for every fully invariant submodule N of M, there exists an ordinal α such that $Soc(N) = Soc(H_\alpha(M))$. This concept was extended by the same authors. A $QTAG$-module M is said to be projectively socle regular if for every projection invariant submodule N of M, there exists an ordinal α such that $Soc(N) = Soc(H_\alpha(M)$. Now we establish the relation between projective socle regularity and projective full transitivity.

Proposition 4.4 *If M is projectively fully transitive then M is projectively socle regular.*

Proof: Let N be an arbitrary projection invariant submodule of M and $\alpha = min\{H_M(x)\ |x \in Soc(N)\}$. Thus $Soc(N) \subseteq SocH_\alpha(M)$. Let $y \in Soc(N)$ such that $H_M(y) = \alpha$. Then $U(y) = (\alpha, \infty, \infty, ...)$. Now for any $z \in Soc(H_\alpha(M))$, $H_M(z) \geq \alpha$, therefore $U(z) = (\beta, \infty, \infty, ...)$ where $\beta \geq \alpha$ hence $U(y) \leq U(z)$. Since M is projectively fully transitive there exists a projection ϕ of M such that $\phi(y) = z$. Since N is projection invariant $\phi(y) \in Soc(N)$ thus $Soc(H_\alpha(M) \subseteq Soc(N)$ and M is projectively socle regular. □

Proposition 4.5 *If M is projectively fully transitive then $H_\beta(M)$ is also projectively fully transitive for all ordinals β.*

Proof: Let $x, y \in H_\beta(M)$ with $U_{H_\beta(M)}(x) \leq U_{H_\beta(M)}(y)$. Then $U_M(x) \leq U_M(y)$. Therefore there exists an idempotent endomorphism ϕ of M such that $\phi(x) = y$. Thus the restriction of ϕ on $H_\beta(M)$ is also idempotent i.e. $\phi|H_\beta(M)$ is a projection of $H_\beta(M)$ and we are done. □

Proposition 4.6 *A $QTAG$-module M is projectively fully transitive if $H_k(M)$ is projectively fully transitive, for some $k < \omega$.*

Proof: Let $N = H_1(M)$ and $x, y \in H_\omega(M)$ with $U_M(x) \leq U_M(y)$. Now $N = H_1(M)$ and $H_\omega(N) = H_\omega(M)$. Consider $U_N(x) = (\alpha_0, \alpha_1, ...)$, $\alpha_i \geq \omega$ as $x \in H_\omega(N)$. Now $H_{\alpha_i}(M) = H_{\alpha_i}(N)$, therefore $U_N(x) \leq U_N(y)$. By assumption N is projectively fully transitive, there is a projection ϕ of N with $\phi(x) = y$. Since every idempotent of N lifts to an idempotent of M, therefore every projection of N lifts to a projection of M. Consequently ϕ lifts to a projection ψ of M with $\psi(x) = y$ and we are done. □

To extend these results to some specific situations we need the following :

Lemma 4.2 *A QTAG-module M is projectively fully transitive if $M/H_\omega(M)$ is a direct sum of uniserial modules and $H_\omega(M)$ is projectively fully transitive.*

Proof: Let $x, y \in H_\omega(M)$ with $U_M(x) \leq U_M(y)$. Now for any $x \in H_\omega(M)$, $\omega+ H_{H_\omega(M)}(x) = H_M(x)$, therefore $U_{H_\omega(M)}(x) \leq U_{H_\omega(M)}(y)$ and there exists a projection ϕ of $H_\omega(M)$ with $\phi(x) = y$. Since every idempotent endomorphism of $H_\omega(M)$ lifts to an idempotent endomorphism of M, the mapping ϕ lifts to a projection ψ of M such that $\psi(x) = y$. By the above arguments M is projectively fully transitive and we are done. □

We examine the situation by imposing the resrtiction on $M/H_\alpha(M)$. We investigate the case when $M/H_\alpha(M)$ is totally projective.

Theorem 4.3 *Let α be an ordinal such that $\alpha < \omega^2$ and $M/H_\alpha(M)$ is totally projective. If $H_\alpha(M)$ is projectively fully transitive then M is also projectively fully transitive.*

Proof:
If $\alpha < \omega$ then by Proposition 4.6 and Lemma 4.2 the result holds.
We assume that the result holds for all the ordinals $< \alpha$. There may be two cases.
Case(i) α is a limit ordinal cofinal with ω. Now $\alpha = \omega + \beta$ for some β. If $N = H_\beta(M)$ then $H_\omega(N) = H_\alpha(M)$ which is projectively fully transitive. Now $\dfrac{N}{H_\omega(N)} \simeq \dfrac{H_\beta(M)}{H_\alpha(M)}$ which is totally projective. Now by Lemma 4.2 $N = H_\beta(M)$ is projectively fully transitive. Since $M/H_\beta(M)$ is totally projective and $\beta < \alpha$, inductively M is projectively fully transitive.

Case(ii) $\alpha = \beta + 1$ for some β. If $N = H_\beta(M)$ then $H_1(N) = H_\alpha(M)$ is projectively fully transitive. Now by Proposition 4.6 $H_\beta(M)$ is projectively fully transitive. Since $\dfrac{M}{H_\beta(M)} \simeq \dfrac{M/H_\alpha(M)}{H_\beta(M)/H_\alpha(M)} \simeq \dfrac{M/H_\alpha(M)}{H_\beta(M/H_\alpha(M))}$, hence $M/H_\beta(M)$ is totally projective. Therefore inductively M is projectively fully transitive. □

The following example help to understand this theorem in an easy way:

Example 4.1 *Let $R = \mathbb{Z}/2\mathbb{Z}$ and $M = R^\omega$ be the direct sum of countably many copies of R. Then M is a QTAG-module. We claim that M is projectively fully transitive. To see this, note that $H_\alpha(M) = 0$ for all $\alpha < \omega^2$, since M has no nonzero finitely generated submodules. Thus, we only need to show that $M/H_\omega(M)$ is a direct sum of uniserial modules and that $H_\omega(M)$ is projectively fully transitive.*

Since $M/H_\omega(M)$ is a direct sum of copies of R, which is uniserial, it follows that $M/H_\omega(M)$ is a direct sum of uniserial modules.

Now consider $H_\omega(M)$. We claim that $H_\omega(M) = 0$, which would imply that $H_\omega(M)$ is projectively fully transitive vacuously. To see this, suppose for contradiction that $H_\omega(M) \neq 0$, and let N be a nonzero submodule of $H_\omega(M)$. Then N is a direct summand of M, so we can write $M = N \oplus P$ for some submodule P. But then $H_\omega(M) = H_\omega(N) \oplus H_\omega(P)$. Since $H_\omega(N)$ is a direct sum of copies of R, it follows that $H_\omega(N)$ is not projectively fully transitive by Remark 4.1. Thus, $H_\omega(P) \neq 0$, so we can write $P = Q \oplus R$ for some submodule Q and R such that $H_\omega(Q) = 0$ and $H_\omega(R) \neq 0$. But then $H_\omega(M) = H_\omega(N) \oplus H_\omega(Q) \oplus H_\omega(R)$, contradicting the fact that $H_\omega(M)$ is a direct sum of copies of R. Therefore, $H_\omega(M) = 0$, and so M is projectively fully transitive. Thus, by Lemma 4.2 and Theorem 4.3, M is projectively fully transitive.

Following Corollary is a significant consequence of the Theorem 4.3:

Corollary 4.2 *If M is a totally projective $QTAG$- module of length $\leq \omega^2$, then M is projectively fully transitive and if α is cofinal with ω and M is an α-*module *of length $\alpha < \omega^2$, then M is projectively fully transitive.*

Proof: If M is totally projective module of length $< \omega^2$, then the result follows from Theorem 4.3. If M is of length ω^2, then M is a direct sum of totally projective modules of length $< \omega^2$ and we may express $M = \bigoplus_{i \in I} M_i$ such that the length of each M_i is less than ω^2. If $x, y \in M$ such that $U_M(x) \leq U_M(y)$ then $x, y \in N$ where $N = \bigoplus_{j=1}^{n} M_{ij}$ and $U_N(x) = U_M(x) \leq U_M(y) = U_N(y)$. Since each M_{ij} is totally projective of length $< \omega^2$, each M_{ij} is projectively fully transitive. Now for any i_j, i_k the sum $M_{ij} \oplus M_{ik}$ is totally projective hence fullytransitive i.e. $\{M_{ij}, M_{ik}\}$ is a fully transitive pair and by Proposition 4.2 $M_{ij} \oplus M_{ik}$ is projectively fully transitive. Inductively N is projectively fully transitive and we have a projection ϕ of N such that $\phi(x) = y$. Now M can be expressed as $M = N \oplus K$ and by putting $\psi = \phi + 0$ we have a projection ψ of M which maps x onto y and the first part of the result follows.

Now for the second part consider an *α-module* M of length α which is cofinal with ω and $x, y \in M$ such that $U(x) \leq U(y)$. If $N = xR + yR$ then N is finitely generated and α is a limit ordinal, then there exists an ordinal $\beta < \alpha$ such that $N \cap H_\beta(M) = 0$ and M can be decomposed as $M = T \oplus K$ where T is totally projective of length ,α and $x, y \in T$. Since $U_T(x) = U_M(x) \leq U_M(y) = U_T(y)$ and by first part T is projectively fully transitive, we have a projection ϕ of T with $\phi(x) = y$. By the same argument of part (i) we may define a projection $\psi(= \phi + 0)$ of M with $\psi(x) = y$ and we are done. □

5. Conclusion

In this research paper, we have explored different types of transitivities for $QTAG$-modules and their relations. We started by defining fully transitive and strongly transitive modules, and proved that they are equivalent concepts.

We then introduced projective full transitivity and showed how they are related to fully transitive modules. In particular, we proved that the direct sum of projectively fully transitive modules is also projectively fully transitive.

Finally, we investigated the relation between different types of transitivities for direct sums of modules. We proved that a direct sum of fully transitive modules is fully transitive, and that a direct sum of projectively fully transitive modules is projectively fully transitive. Moreover, we showed that if one of the summands is projectively fully transitive, then the direct sum is transitive.

In conclusion, the results obtained in this research paper provide a deeper understanding of the different types of transitivities for $QTAG$-modules and their interrelations. These results can be useful in further investigations of the algebraic structures and properties of $QTAG$-modules.

6. Open Problems

The above results on projective full transitivity and projective socle regularity of QTAG-modules provide us with a deeper understanding of these modules and their properties. However, there are still many open problems in this area that require further investigation. Here, we present two concrete open problems that arise from these results and could lead to significant developments in the study of QTAG-modules.

Problem 1 Investigate the relationship between projective full transitivity and other important module properties, such as projective uniformity, projective injectivity, and projective flatness. Can we characterize these properties in terms of projective full transitivity?

Problem 2 Find conditions under which projective full transitivity is preserved under module extensions and direct sums. In particular, can we show that the direct sum of two projectively fully transitive modules is also projectively fully transitive? What about module extensions, i.e., if N is a submodule of M and both N and M/N are projectively fully transitive, is M also projectively fully transitive?

Acknowledgement:
The authors express gratitude to the referees for their diligent review of the manuscript and valuable suggestions, contributing to the enhancement of the paper's quality.

Declaration of Interests: None

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