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# A class of fractional differential history-dependent hemivariational inequalities with application to thermo-viscoelastic contact problem

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ABSTRACT: The aim of this work is to study a class of fractional differential history-dependent hemivariational inequalities and to provide an example of application in thermo-viscoelasticity. Using the Rothe method and the subjectivity property of multivalued pseudomonotone operators, we first prove existence of a solution. The proof is based on a fixed point argument and a recent finding from hemivariational inequalities theory. Then, we apply the obtained abstract results to a nonlinear thermo-viscoelastic contact problem with a history-dependent with fractional time Kelvin-Voiget constitution law and adhesion.

Key Words: History-dependent differential hemivariational inequalities, Rothe method, fractional Caputo derivative, fractional Kelvin-Voiget constitutive law.

## Contents



## 1. Introduction

In recent years, a few researchers have used fractional calculus as a means of describing natural phenomena in different fields such as physics, biology, finance, economics and bioengineering, giving an accurate representation of some real wonders of viscoelastic materials, as for instance, materials with partial Kelvin-Voigt constitutive laws. For more details about such models, we refer to [16,17] and, for their mathematical analysis and applied developments, we refer to [3,8] and the reference therein.

The differential hemivariational inequalities are a powerful mathematical tool to represent boundary value problems modelling many physical phenomena, like frictional contact problems, see for exemple [7] and the reference therein. We recall that hemivariational inequalities theory has recently played an important role in the study of nonlinear problems arising in contact mechanics, economics, and engineering, see e.g., [19]. The notion of hemivariational inequalities has been introduced first by Panagiotopouls, as a useful generalization of variational inequalities in 1980, see [20,21]. It is based on Clarke's subdifferential of locally Lipschitz function, see e.g., [22] and the reference therein. Moreover, the current paper aims to apply hemivariational inequalities to contact problems for the particular class of materials whose constitutive law is described by history-dependent operators, see e.g., [6,13,15] and reference therein.

In addition, from a mathematical point of view, models characterising a class of fractional differential hemivariational inequalities is recent, see e.g., [2,5]. Then, the first novelty of the current paper is to extend such models to the class of fractional differential history-dependent hemivariational inequalities. The second novelty consists in the mathematical analysis of a fractional differential history-dependent hemivariational inequalities arising in thermo-viscoelastic contact problem with a conductive foundation, long memory effects, and adhesion.

The paper is organized as follows. In Section 2, we state the fractional differential history-dependent hemivariational problem and we state its unique solvability result. In Section 3, we investigate a static frictional contact problem with unilateral constraints. long memory effects, and adhesion between a thermo-viscoelastic body and a conductive foundation. We derive the variational formulation of this problem which is of the form of differential history-dependent hemivariational inequalities for which we apply the obtained abstract results.

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## 2. Abstract fractional differential history-dependent hemivariational inequalities

In this part, we investigate the Roth technique and provide a theoretical solution for a class of fractional differential history-dependent hemivariational inequalities by using the surjectivity result for multivalued pseudomonotone operators. Everywhere below, let $T > 0$ and let $(V, \|\cdot\|_V)$, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be reflexive Banach spaces. Moreover, let $V^*$ be the dual space of $V$ and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $V^*$ and $V$. The symbols "$\rightharpoonup$" and "$\to$" denote the weak and the strong convergence in all used spaces. Then, with the previous data, we consider the following fractional differential history-dependent hemivariational inequalities problem.

**Problem (P1).** *Find $u \in AC(0,T;V)$ and $\lambda \in W^{1,2}(0,T;Y)$ such that*

$$\begin{cases} \langle A({}_0^cD_t^\alpha u(t)) + Bu(t) + (Ru)(t), v\rangle + j^0(\lambda(t), Mu(t); Mv) \geq \langle f(t), v\rangle, \\ \lambda'(t) = F(t, Mu(t), \lambda(t)), \quad \text{for all } v \in V \text{ a.e. } t \in (0,T), \\ \lambda(0) = \lambda_0, \quad u(0) = u_0 \end{cases} \tag{2.1}$$

where ${}_0^cD_t^\alpha u(t)$ is the Caputo derivative of $u(t)$ at order $\alpha \in (0,1)$ defined by

$${}_0^cD_t^\alpha u(t) = {}_0I_t^{\alpha-1} u'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds$$

for a.e. $t \in (0,T)$, where the operator ${}_0I_t^\alpha w$ is the $\alpha$-ordre time fractional integral of $w(t)$ in the sense of Riemann-Liouville and where $\Gamma$ is the Gamma function, for more details, we refer to [2,5]. Next, let us consider an operator $R : C(0,T;V) \to C(0,T;V^*)$ defined as follows

$$(Ru)(t) = E(a_s + \int_0^t q(t,s)u(s)ds) \quad \text{for} \quad t \in [0,t] \tag{2.2}$$

where $E : V \to V^*$, $q : [0,t] \times [0,t] \to \mathcal{L}(V,V)$ and $a_s \in V$. Moreover, let $j^0(u;v)$ represent the generalized directional derivative of $j$ at the point $u$ in the direction $v$. Consider $u \in AC(0,T;V)$ a solution to Problem $(P1)$ and $w = {}_0^cD_t^\alpha u$ its Caputo derivative, then from [2, Proposition 3(b)], we have

$$u(t) = {}_0I_t^\alpha w(t) + u_0 \quad \text{for a.e.} \quad t \in (0,T).$$

The previous Problem $(P1)$ can be reformulated as follows.

**Problem (P2).** *Find $w \in L^1(0,T;V)$ and $\lambda \in W^{1,2}(0,T;Y)$ such that*

$$\begin{cases} \langle A(w(t)) + B({}_0I_t^\alpha w(t) + u_0) + R({}_0I_t^\alpha w(t) + u_0), v\rangle \\ \quad + j^0(\lambda(t), M({}_0I_t^\alpha w(t) + u_0); Mv) \geq \langle f(t), v\rangle \quad \text{for all } v \in V \text{ a.e. } t \in (0,T), \\ \lambda'(t) = F(t, M({}_0I_t^\alpha w(t) + u_0), \lambda(t)) \quad \text{for all } t \in (0,T), \\ \lambda(0) = \lambda_0. \end{cases} \tag{2.3}$$

Furthermore, we consider the following inclusion problem

**Problem (P3).** *Find $w \in L^1(0,T;V)$ and $\lambda \in W^{1,2}(0,T;Y)$ such that*

$$\begin{cases} A(w(t)) + B({}_0I_t^\alpha w(t) + u_0) + R({}_0I_t^\alpha w(t) + u_0) + M^*\partial j(\lambda(t), M({}_0I_t^\alpha w(t) + u_0)) \ni f(t), \\ \lambda'(t) = F(t, M({}_0I_t^\alpha w(t) + u_0), \lambda(t)) \quad \text{for all } v \in V \text{ a.e. } t \in (0,T), \\ \lambda(0) = \lambda_0 \end{cases} \tag{2.4}$$

where $\partial j(u)$ is the generalized Clarke's gradient of $j$ at $u$. Before stating our main result, we assume

$(\mathcal{H}_1)$ The mapping $F : [0,T] \times X \times Y \to Y$ satisfies

$(i)$ for all $(x,y)\in X\times Y$, the function $t\mapsto F(t,x,y)$ is measurable on $[0,T]$,

$(ii)$ there exists a constant $L_F>0$ such that

$$\|F(t,x_1,y_1)-F(t,x_2,y_2)\|_X\le L_F\,(\|x_1-x_2\|_X+\|y_1-y_2\|_Y), \tag{2.5}$$

for all $t\in[0,T]$ and all $(x_1,y_1)$, $(x_2,y_2)\in X\times Y$,

$(iii)$ the function $t\mapsto F(t,0,0)$ belongs to $L^1([0,T]\,;Y)$.

$(\mathcal{H}_2)$ The operator $A:V\to V^*$ satisfies

$(i)$ $A\in\mathcal{L}(V,V^*)$,

$(ii)$ $\langle Av,v\rangle\ge m_A\|v\|^2$.

$(\mathcal{H}_3)$ The operator $B:V\to V^*$ satisfies

$$B\in\mathcal{L}(V,V^*).$$

$(\mathcal{H}_4)$ The operator $E:V\to V^*$ satisfies

$$E\in\mathcal{L}(V,V^*).$$

$(\mathcal{H}_5)$ The function $q:[0,T]\times[0,T]\to\mathcal{L}(V,V)$ satisfies

$$\|q(t_1,s)-q(t_2,s)\|\le L_q|t_1-t_2|.$$

$(\mathcal{H}_6)$ The function $j:Y\times X\to\mathbb{R}$ satisfies

$(i)$ the function $j:x\to j(y,x)$ is locally Lipschitz, for all $y\in Y$

$(ii)$ there exist positive constants $c_j$, such that, for all $x\in X$, $y\in Y$ we have

$$\|\partial j(y,x)\|\le c_j(1+\|x\|_V). \tag{2.6}$$

$(iii)$ $(y,x)\to j^0(y,x,v)$ is upper-semicontinous on $Y\times X$.

$(\mathcal{H}_7)$ The element $f$ satisfies $f\in L^\infty(0,T;V^*)$.

$(\mathcal{H}_8)$ The operator $M\in\mathcal{L}(V,X)$ is linear continuous and compact.

Now, let $N\in\mathbb{N}^*$ be fixed, and consider $\tau=\frac{T}{N}$, $t_k=k\tau$ and $f_\tau^k$ defined by

$$f_\tau^k=\frac{1}{\tau}\int_{t_{k-1}}^{t_k}f(s)ds,\quad\text{for}\quad k=1,...,N.$$

Consider the corresponding discretized issue to System 2.4 called Rothe problem.

**Problem (P4).** *Find* $\{w_\tau^k\}_{k=1}^N\subset V$ *and* $\lambda_\tau\in W^{1,2}(0,T;Y)$ *such that*

$$Aw_\tau^k+Bu_\tau^k+r_\tau^k+\partial j(\lambda_\tau(t_k),Mu_\tau^k)\ni f_\tau^k,\quad \textit{for}\ \ k=1,...N, \tag{2.7}$$

$$\lambda_\tau'(t)=F(t,M\widehat{u}_\tau,\lambda_\tau(t)),\quad \textit{for all}\ \ t\in(0,T) \tag{2.8}$$

where

$$u_\tau^k=u_0+\frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{j=1}^k w_\tau^j[(k-j+1)^\alpha-(k-j)^\alpha] \tag{2.9}$$

$$\widehat{u}_\tau(t)=\begin{cases}\sum\limits_{i=1}^N\chi_{(t_{i-1},t_i)}(t)u_\tau^{i-1}, & 0<t\le T\\ u_0, \quad t=0\end{cases} \tag{2.10}$$

$$r_\tau^k=E(\sum_{j=1}^k\int_{t_{j-1}}^{t_j}q(t_k,s)u_\tau^j ds+a_R). \tag{2.11}$$

To begin with, we will demonstrate the existence of solution to Problem (P4).

**Lemma 2.1** *Assume assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_8)$ hold. Then, there exist $\tau_0 > 0$ such that for all $\tau \in (0, \tau_0)$, Problem (P4) has a unique solution.*

**Proof:** Let $w_\tau^0, ..., w_\tau^{n-1}$ be given, we will prove that there exists $w_\tau^n \in V$, $\lambda_\tau \in W^{1,2}(0, t_n; Y)$ such that (2.7) is verified. Using similar techniques as in [2, Lemma 14], we prove that there exists a unique solution $\lambda_\tau \in W^{1,2}(0, t_n; Y)$ such that (2.8) hold. It stays to prove that there exist element $w_\tau^n \in V$ such that (2.7) hold. To this end, let us denote

$$v_0 = u_0 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{n-1} w_\tau^j [(n-j+1)^\alpha - (n-j)^\alpha], \quad c_0 = \frac{\tau^\alpha}{\Gamma(\alpha+1)}. \tag{2.12}$$

Then, we will show the surjectivity of the following multivalued operator

$$V \ni v \to Av + B(v_0 + c_0 v) + E(\int_{t_{k-1}}^{t_k} q(t_k, s)(v_0 + c_0 v) ds) + M^* \partial j(\lambda_\tau(t_n), M(v_0 + cv)) \subset V^*. \tag{2.13}$$

From hypotheses $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$, we deduce that the operator

$$V \ni v \to Av + B(v_0 + c_0 v) + E(\int_{t_{k-1}}^{t_k} q(t_k, s)(v_0 + c_0 v) ds) \in V^* \tag{2.14}$$

is bounded, continue and for all $u, v \in V$, we have

$$\begin{aligned}
&\langle Av + B(v_0 + c_0 v) + E(\int_{t_{k-1}}^{t_k} q(t_k, s)(v_0 + c_0 v) ds) - Au - B(v_0 + c_0 u) \\
&- E(\int_{t_{k-1}}^{t_k} q(t_k, s)(v_0 + c_0 u) ds), v - u\rangle \\
&= \langle Av - Au, v - u\rangle + \langle B(v_0 + c_0 v) - B(v_0 + c_0 u), v - u\rangle \\
&+ \langle E(\int_{t_{k-1}}^{t_k} q(t_k, s)(v_0 + c_0 v) ds) - E(\int_{t_{k-1}}^{t_k} q(t_k, s)(v_0 + c_0 u) ds), v - u\rangle.
\end{aligned}$$

Let us consider $c_E = \|E\|$ and $c_q = \max_{t,s \in [0,T]} \|q(t, s)\|$. Then, for all $u, v \in V$, we obtain

$$\begin{aligned}
&\langle Av + B(v_0 + c_0 v) + E(\int_{t_{k-1}}^{t_k} q(t_k, s)(v_0 + c_0 v) ds) - Au - B(v_0 + c_0 u) \\
&- E(\int_{t_{k-1}}^{t_k} q(t_k, s)(v_0 + c_0 u) ds), v - u\rangle \\
&\geq (m_A - c_0\|B\| - c_E c_q c_0)\|v - u\|^2.
\end{aligned}$$

In addition, it comes from hypotheses $(\mathcal{H}_6)$, $(\mathcal{H}_8)$ and [11, Proposition 4] that

$$V \ni v \to M^* \partial j(\lambda_\tau(t_n), M(v_0 + c_0 v) \subset V^*$$

is pseudomonotone operator, such that for all $v \in V$, one has

$$\|M^* \partial j(\lambda_\tau(t_n), M(v_0 + c_0 v)\| \leq c_0 c_j \|M\|^2 \|v\| + \|M\|(1 + \|M\|\|v_0\|).$$

Next, we choose $\tau_0 = (\frac{m_A \Gamma(1+\alpha)}{\|B\| + c_E c_q + c_j \|M\|^2})^{\frac{1}{\alpha}}$ to get

$$L : v \to Av + B(v_0 + c_0 v) + E(\int_{t_{k-1}}^{t_k} q(t_k, s)(v_0 + c_0 v) ds)$$

is strongly monotone operator, and for all $\tau \in (0, \tau_0)$, we have

$$c_0\|B\| + c_E c_q c_0 + c_0 c_j \|M\|^2 < m_A.$$

Finally, using [2, Corallary 7], we find that $L$ is surjective and hence Lemma 2.1 is established.

□

Next, we establish the estimate for a solution of Problem (P4).

**Lemma 2.2** *Assume hypotheses of Lemma 2.1 hold. Then, there exists $\tau_0 > 0$ and $C > 0$ independent of $\tau$, such that for all $\tau \in (0, \tau_0)$, the solution to Problem (P4) satisfies*

$$\max_{k=1,\ldots,N} \|w_\tau^k\| \le C\,, \quad \max_{k=1,\ldots,N} \|u_\tau^k\| \le C\,, \quad \max_{k=1,\ldots,N} \|\Psi_\tau^k\| \le C \tag{2.15}$$

*where $\Psi_\tau^k \in \partial j(\lambda_\tau(t_k), M(u_\tau^k))$ and*

$$Aw_\tau^k + Bu_\tau^k + r_\tau^k + M^*\Psi_\tau^k = f_\tau^k \quad \textit{for} \quad k = 1, ..., N. \tag{2.16}$$

**Proof:** We first take $k = n$ in the relation (2.7) to obtain

$$\langle Aw_\tau^n, w_\tau^n\rangle + \langle Bu_\tau^n, w_\tau^n\rangle + \langle \Psi_\tau^n, Mw_\tau^n\rangle + \langle r_\tau^n, w_\tau^n\rangle = \langle f_\tau^n, w_\tau^n\rangle.$$

We us the same techniques as in [2, Lemma 15] to deduce

$$\begin{aligned}
\langle Bu_\tau^n, w_\tau^n\rangle &= \langle B(u_0 + \frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{j=1}^{n} w_\tau^j[(n-j+1)^\alpha - (n-j)^\alpha]), w_\tau^n\rangle \\
&\ge -\|Bu_0\|\|w_\tau^n\| - \|B\|\frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{j=1}^{n-1}\|w_\tau^j\|[(n-j+1)^\alpha - (n-j)^\alpha]\|w_\tau^n\| - \frac{\tau^\alpha}{\Gamma(\alpha+1)}\|w_\tau^n\|^2.
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
\langle \Psi_\tau^n, Mw_\tau^n\rangle &\ge -(c_j\|M\| + c_j\|M\|^2 u_0)\|w_\tau^n\| - \|M\|^2 c_j \frac{\tau^\alpha}{\Gamma(\alpha+1)}\|w_\tau^n\|^2 \\
&- \|M\|^2 c_j \frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{j=1}^{n-1}\|w_\tau^j\|[(n-j+1)^\alpha - (n-j)^\alpha]\|w_\tau^n\|.
\end{aligned} \tag{2.18}$$

Moreover, keeping in mind assumptions $(\mathcal{H}_4)$, we conclude that

$$\begin{aligned}
&\langle r_\tau^n, w_\tau^n\rangle = \langle E(\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} q(t_k, s)u_\tau^j ds + a_R), w_\tau^n\rangle \\
&\ge -c_E\|a_R\|\|w_\tau^n\| - c_E c_q(\sum_{j=1}^{n}\|u_0 + \frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{j}\|w_\tau^i\|[(j-i+1)^\alpha - (j-i)^\alpha]\|\|w_\tau^n\|) \\
&\ge -c_E\|a_R\|\|w_\tau^n\| - c_E c_q\sum_{j=1}^{n}\|u_0\|\|w_\tau^n\| - \frac{c_E c_q \tau^\alpha}{\Gamma(\alpha+1)}\sum_{j=1}^{n}\sum_{i=1}^{j}\|w_\tau^i\|[(j-i+1)^\alpha - (j-i)^\alpha]\|w_\tau^n\| \\
&\ge -c_E\|a_R\|\|w_\tau^n\| - c_E c_q n\|u_0\|\|w_\tau^n\| - \frac{c_E c_q \tau^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{n-1}\|w_\tau^i\|[(n-i+1)^\alpha]\|w_\tau^n\| - \frac{c_E c_q \tau^\alpha}{\Gamma(\alpha+1)}\|w_\tau^n\|^2.
\end{aligned} \tag{2.19}$$

Therefore, it follows from the coercivity of operator $A$ and inequalities (2.17)-(2.19) that

$$\begin{aligned}
&\langle f_\tau, w_\tau^n\rangle = \langle Aw_\tau^n, w_\tau^n\rangle + \langle Bu_\tau^n, w_\tau^n\rangle + \langle r_\tau^n, w_\tau^n\rangle + \langle \Psi_\tau^n, Mw_\tau^n\rangle\\
&\geq (m_A - \frac{\tau^\alpha}{\Gamma(\alpha+1)}\|B\| - \frac{\tau^\alpha}{\Gamma(\alpha+1)}c_j\|M\|^2 - \frac{\tau^\alpha}{\Gamma(\alpha+1)}c_Ec_q)\|w_\tau^n\|^2\\
&-(\|Bu_0\| - c_E\|a_R\| - c_Ec_qn\|u_0\| + c_j\|M\| + c_j\|M\|^2\|u_0\|)\|w_\tau^n\|\\
&-\|B\|\frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{j=1}^{n-1}\|w_\tau^j\|[(n-j+1)^\alpha - (n-j)^\alpha]\|w_\tau^n\|\\
&-\|M\|^2c_j\frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{j=1}^{n-1}\|w_\tau^j\|[(n-j+1)^\alpha - (n-j)^\alpha]\|w_\tau^n\|\\
&-c_Ec_q\frac{\tau^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{n-1}\|w_\tau^i\|[(n-i+1)^\alpha]\|w_\tau^n\|\\
&\geq (m_A - \frac{\tau^\alpha(\|B\| + c_j\|M\|^2 + c_Ec_q)}{\Gamma(1+\alpha)})\|w_\tau^n\|^2\\
&-(\|Bu_0\| + c_E\|a_R\| + c_Ec_qn\|u_0\| + c_j\|M\| + c_j\|M\|^2\|u_0\|)\|w_\tau^n\|\\
&-\sum_{j=1}^{n-1}\|w_\tau^j\|\|w_\tau^n\|[\frac{\tau^\alpha(\|B\| + c_j\|M\|^2)}{\Gamma(1+\alpha)}[(n-j+1)^\alpha - (n-j)^\alpha] + \frac{\tau^\alpha c_Ec_q}{\Gamma(1+\alpha)}(n-j+1)^\alpha].
\end{aligned} \tag{2.20}$$

Thus, the previous inequality can be written as follows

$$\begin{aligned}
&(m_A - \frac{\tau^\alpha(\|B\| + c_j\|M\|^2 + c_Ec_q)}{\Gamma(1+\alpha)})\|w_\tau^n\|\\
&\leq \|f_\tau^n\|_{V^*} + \|Bu_0\| + c_E\|a_R\| + c_Ec_qn\|u_0\| + c_j\|M\| + c_j\|M\|^2\|u_0\|\\
&+\sum_{j=1}^{n-1}\|w_\tau^j\|(\frac{\tau^\alpha(\|B\| + c_j\|M\|^2)}{\Gamma(1+\alpha)}[(n-j+1)^\alpha - (n-j)^\alpha] + \frac{\tau^\alpha c_Ec_q}{\Gamma(1+\alpha)}(n-j+1)^\alpha).
\end{aligned} \tag{2.21}$$

Then, we take $\tau_0 = \left(\frac{m_A\Gamma(1+\alpha)}{2(\|B\|+c_j\|M\|^2+c_Ec_q)}\right)^{\frac{1}{\alpha}}$ in the previous inequality to find

$$m_A - \frac{\tau^\alpha(\|B\| + c_j\|M\|^2 + c_Ec_q)}{\Gamma(1+\alpha)} \geq \frac{m_A}{2} \quad \text{for all} \quad \tau \in (0, \tau_0).$$

Thus, we conclude that

$$\begin{aligned}
\|w_\tau^n\| &\leq \frac{2\,\|f_\tau^n\|_{V^*}}{m_A} + 2\frac{\|Bu_0\| + c_E\|a_R\| + c_Ec_qn\|u_0\| + c_j\|M\| + c_j\|M\|^2\|u_0\|}{m_A}\\
&+ \frac{2\tau^\alpha}{m_A\Gamma(1+\alpha)}\sum_{j=1}^{n-1}\|w_\tau^j\|[(\|B\| + c_j\|M\|^2)((n-j+1)^\alpha - (n-j)^\alpha) + c_Ec_q(n-j+1)^\alpha]
\end{aligned}$$

Moreover, it comes from the assumption $(\mathcal{H}_7)$, there exist a constant $m_f > 0$ such that

$$\|f_\tau^n\| \leq m_f \quad \text{for} \quad \tau > 0 \quad \text{and} \quad n \in \mathbb{N}.$$

To simplify the notations, let us consider

$$c_0 = \frac{2m_f}{m_A} + 2\frac{\|Bu_0\| + c_E\|a_R\| + c_Ec_qn\|u_0\| + c_j\|M\| + c_j\|M\|^2\|u_0\|}{m_A}.$$

So, we can deduce the following estimation

$$\|w_\tau^n\| \leq c_0 + \frac{2\tau^\alpha}{m_A\Gamma(1+\alpha)} \sum_{j=1}^{n-1} \|w_\tau^j\|[(\|B\| + c_j\|M\|^2)((n-j+1)^\alpha - (n-j)^\alpha) + c_E c_q (n-j+1)^\alpha].$$

Hence, by using Gronwall inequality, we obtain

$$\begin{aligned}
&\|w_\tau^n\| \\
&\leq c_0 \exp(\frac{2\tau^\alpha}{m_A\Gamma(1+\alpha)} \sum_{j=1}^{n-1} [(\|B\| + c_j\|M\|^2)((n-j+1)^\alpha - (n-j)^\alpha) + c_E c_q (n-j+1)^\alpha]) \\
&+ c_0 \exp(\frac{2\tau^\alpha}{m_A\Gamma(1+\alpha)} \sum_{j=1}^{n-1} (\|B\| + c_j\|M\|^2)((n-j+1)^\alpha - (n-j)^\alpha) \\
&+ c_E c_q \frac{2\tau^\alpha}{m_A\Gamma(1+\alpha)} \sum_{j=1}^{n-1} (n-j+1)^\alpha) \\
&\leq c_0 \exp(2\frac{\|B\| + c_j\|M\|^2}{m_A\Gamma(1+\alpha)} t_n^\alpha + \frac{2c_E c_q}{m_A\Gamma(1+\alpha)} \sum_{j=1}^{n-1} t_{n-j+1}^\alpha).
\end{aligned}$$

Thus, we conclude that

$$\|w_\tau^n\| \leq C_1 = c_0 \exp\big(2\,\frac{\|B\| + c_j\|M\|^2 + c_E c_q n}{m_A\Gamma(1+\alpha)} T^\alpha\big). \tag{2.22}$$

Then, we find that the first estimation of (2.15) is verified. For the second estimate of (2.15), we use the same arguments as in [2, Lemma 15], and we exploit (2.9) and (2.22) to get

$$\begin{aligned}
\|u_\tau^n\| &= \|u_0 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{n} [(n-j+1)^\alpha - (n-j)^\alpha] w_\tau^j\| \\
&\leq \|u_0\| + \frac{C_1}{\Gamma(\alpha+1)} \sum_{j=1}^{n} [t_{n-j+1}^\alpha - t_{n-j}^\alpha] \\
&\leq \|u_0\| + \frac{C_1}{\Gamma(\alpha+1)} t_n^\alpha \leq C_2 = \|u_0\| + \frac{C_1}{\Gamma(\alpha+1)} T^\alpha.
\end{aligned} \tag{2.23}$$

Then, the desired estimate holds. It remains now to show the last estimate of (2.15), so from the previous estimations and hypothesis $(\mathcal{H}_6)$, we deduce

$$\|\Psi_\tau^n\| \leq c_j(1 + \|Mu_\tau^n\|) \leq c_j(1 + \|M\|C_2). \tag{2.24}$$

Finally, we can deduce the claimed estimate, and this completes the proof of Lemma 2.2. □

We introduce the following piecewise constant interpolant functions

$$w_\tau^*,\ u_\tau^* : [0,T] \to V,\ f_\tau : [0,T] \to V^*,\ \Psi_\tau : [0,T] \to X^* \ \text{ and } \ r_\tau^* : [0,T] \to V,$$

$$w_\tau^*(t) = w_\tau^k,\ u_\tau^*(t) = u_\tau^k,\ f_\tau(t) = f_\tau^k,\ \Psi_\tau(t) = \Psi_\tau^k \ \text{ for all } \ t \in (t_{k-1}, t_k],$$

$$r_\tau^*(t) = a_R + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} q(t_k, s) u_\tau^j \, ds \ \text{ for all } \ t \in (t_{k-1}, t_k] \ \text{ and for } \ k = 1, ..., N.$$

Then, we have the following Theorem.

**Theorem 2.1** *Assume assumptions of Lemma (2.1) hold, and let $\rho \in (0,\alpha)$ and let $\tau_n$ be a sequence such that $\tau = \tau_n \to 0$ as $n \to \infty$. Then, for a subsequence still denoted by $\tau$, we have*

$$w^*_\tau \rightharpoonup w \quad in \quad L^{\frac{1}{\rho}}(0,T;V) \tag{2.25}$$

$$\Psi_\tau \rightharpoonup \Psi \quad in \quad L^2(0,T;X^*) \tag{2.26}$$

$$\lambda_\tau \rightharpoonup \lambda \quad in \quad C(0,T;Y) \tag{2.27}$$

*as $\tau \to 0$, where $(w,\Psi,\lambda) \in L^{\frac{1}{\rho}}(0,T;V) \times L^2(0,T;X^*) \times W^{1,2}(0,T;Y)$ is solution to Problem (P3).*

**Proof:** We consider the Nemitskii operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ corresponding to $A$, $B$ and $M$, i.e.;

$$(\mathcal{A}v)(t) = Av(t),\ (\mathcal{B}v)(t) = B(u_0 + {}_0I^\alpha_t)v(t),\ (\mathcal{M}v)(t) = M(v(t)) \text{ for all } v \in V,\ \text{a.e } t \in (0,T).$$

Next, we consider the Nemitskii operators $\theta$, $\theta_1 : L^2(0,T;V) \to L^2(0,T;X^*)$ defined by

$$(\theta v)(t) = E(\int_0^t q(t,s)v(s)\,ds),\ (\theta_1 v)(t) = Ev(t) \text{ for all } v \in V,\ \text{a.e } t \in (0,T).$$

In order to prove of the solvability of Problem (P3), we need to establish the two facts below.

$(A)$ For all $v \in L^2(0,T;V)$, and as $\tau \to 0$, one has

$$\langle \mathcal{A}w^*_\tau, v\rangle \to \langle Aw, v\rangle, \tag{2.28}$$

$$\langle \mathcal{B}u^*_\tau, v\rangle \to \langle B(u_0 + {}_0I^\alpha_t w(t), v\rangle, \tag{2.29}$$

$$\langle \Psi_\tau, Mv\rangle \to \langle \Psi, Mv\rangle, \tag{2.30}$$

$$\langle f_\tau, v\rangle \to \langle f, v\rangle. \tag{2.31}$$

$(B)$ For all $v \in L^2(0,T;V)$, one has

$$\langle \theta_1(r^*_\tau), v\rangle \to \langle \theta_1(a_R), v\rangle + \langle \theta({}_0I^0_t w(t) + v_0), v\rangle. \tag{2.32}$$

The condition $(A)$ can be proved using similar techniques as in [2, Theorem 16]. Furthermore, for the condition $(B)$, we remark that for all $t \in (t_{k-1}, t_k]$, the assumption $(\mathcal{H}_5)$ and boundedness of $u^*_\tau$ imply

$$\begin{aligned} &\| \int_0^t q(t,s)u^*_\tau(s)ds - \int_0^{t_k} q(t_k,s)u^*_\tau(s)ds \| \\ &\le \int_t^{t_k} \|q(t,s)u^*_\tau(s)\|ds + \int_0^{t_k} \|(q(t,s) - q(t_k,s))u^*_\tau(s)\|ds \le C_0\tau \end{aligned} \tag{2.33}$$

for some $C_0 > 0$ which independent of $\tau$. Using arguments as for convergence (18) in [2], we get

$$u^*_\tau(t) \rightharpoonup {}_0I^\alpha_t w(t) + u_0 \text{ in } V \text{ as } \tau \to 0 \text{ for all } t \in [0,T]. \tag{2.34}$$

So, the convergence (2.34) implies that

$$\lim_{\tau\to 0} \langle \theta u^*_\tau, v\rangle = \langle \theta({}_0I^\alpha_t w(t) + u_0), v\rangle \text{ for all } v \in L^2(0,T;V).$$

Then, it follows from the hypotheses $(\mathcal{H}_4)$ and $(\mathcal{H}_5)$ that

$$\theta_1(r^*_\tau - a_R) - \theta(u^*_\tau) \to 0 \text{ strongly in } L^2(0,T;V^*) \text{ as } \tau \to 0,$$

which implies that for all $v \in L^2(0,T;V)$, we have

$$\begin{aligned}
&\lim_{\tau\to 0}\langle \theta_1(r_\tau^*), v\rangle_{L^2(0,T;V^*)\times L^2(0,T;V)} \\
&= \lim_{\tau\to 0}(\langle \theta_1(r_\tau^* - a_R) - \theta(u_\tau^*), v\rangle + \langle \theta(u_\tau^*), v\rangle + \langle \theta_1(a_R), v\rangle) \\
&= \langle \theta({}_0I_t^\alpha w(t) + u_0), v\rangle + \langle \theta_1(a_R), v\rangle.
\end{aligned} \tag{2.35}$$

Then, the condition $(B)$ is verified. Keeping now in mind the convergences (2.28)-(2.32), we can get

$$\begin{aligned}
&0 \le \lim_{\tau\to 0}\sup\langle Aw_\tau^*, v\rangle + \lim_{\tau\to 0}\sup\langle Bu_\tau^*, v\rangle + \lim_{\tau\to 0}\sup\langle \theta_1(r_\tau^*), v\rangle + \lim_{\tau\to 0}\sup\langle \Psi_\tau, v\rangle - \lim_{\tau\to 0}\inf\langle f_\tau, v\rangle \\
&\le \langle Aw + B(u_0 + {}_0I_t^\alpha w(t)), v\rangle + \langle \theta_1(a_R), v\rangle + \langle \theta({}_0I_t^\alpha w(t) + u_0), v\rangle + \langle \Psi, Mv\rangle - \langle f, v\rangle.
\end{aligned} \tag{2.36}$$

Thus, we conclude that the following inequality hold

$$\begin{aligned}
&\langle Aw, v\rangle + \langle B(u_0 + {}_0I_t^\alpha w(t)), v\rangle \\
&+ \langle E(\int_0^t q(t,s)(u_0 + {}_0I_t^\alpha w(s))ds, v\rangle + \langle E(a_R), v\rangle + \langle \Psi, Mv\rangle \ge \langle f(t), v\rangle.
\end{aligned} \tag{2.37}$$

Therefore, for all $v \in L^2(0,T;V)$, we have $\Psi(t) \in \partial j(\lambda(t), M(u_0 + {}_0I_t^\alpha w(t))$, and hence

$$\begin{aligned}
&\langle Aw + B(u_0 + {}_0I_t^\alpha w(t)), v\rangle \\
&+ \langle E(a_R + \int_0^t q(t,s)(u_0 + {}_0I_t^\alpha w(s))ds), v\rangle + \langle \Psi(t), Mv\rangle \ge \langle f(t), v\rangle \quad \text{for a.e} \quad t \in (0,T).
\end{aligned} \tag{2.38}$$

Then, we conclude that $(w,\lambda) \in L^{\frac{1}{\rho}}(0,T;V) \times W^{1,2}(0,T;V)$ is a solution to Problem (P3), and this completes the proof of Theorem 2.1.

□

## 3. Application to frictional contact problem in thermo-viscoelasticity with long memory

In this section, we will apply results stated in Section 2 to a representative contact problem for a nonlinear thermo-viscoelastic body with history-dependent time fractional Kelvin-Voiget constitutive law and adhesion. We first describe the physical setting of the problem and we then provide its classical formulation in the form of fractional differential history-dependent hemivariational inequalities. Next, we study the existence and uniqueness of solution for this problem.

We discuss a static contact problem for a nonlinear thermo-viscoelastic body that occupies the domain $\Omega \subset \mathbb{R}^d$, with $d = 2, 3$, which is supposed to be an open, bounded and connected subset of $\mathbb{R}^d$ with a Lipschitz boundary $\Gamma = \partial\Omega$. The body is acted upon by body forces $f_0$, a surface traction of density $f_2$, a volume thermal of density $h_0$, heat source $h_n$, and it is also constrained mechanically on part of $\Gamma$.

To describe all these constraints, we consider three open and measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ such that $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3 = \Gamma$ and $meas(\Gamma_1) > 0$. Throughout this paper $i, j, k$ run from 2 to $d$, the summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the variable.

Let $\mathbb{S}^d$ denote the space of second order symmetric tensors on $\mathbb{R}^d$ while $\cdot$ and $\|\cdot\|$ represent the inner product and the associated Euclidean norm on $\mathbb{R}^d$, defined by

$$u \cdot v = u_i\, v_i\ , \qquad \|v\| = (v\cdot v)^{1/2}, \quad \forall u, v \in \mathbb{R}^d.$$

The inner product on $\mathbb{S}^d$ and its associated norm are given by

$$\sigma \cdot \tau = \sigma_{ij}\, \tau_{ij}\ , \qquad \|\tau\| = (\tau\cdot\tau)^{1/2}, \qquad \forall \sigma, \tau \in \mathbb{S}^d.$$

Also, we consider the following subsets

$$\mathbb{Q} = \Omega \times (0,T)\,,\ \Sigma_1 = \Gamma_1 \times (0,T)\,,\ \Sigma_2 = \Gamma_2 \times (0,T)\,,\ \Sigma_3 = \Gamma_3 \times (0,T).$$

Let $\nu$ be the outer normal to $\Gamma$. Then, the normal and tangential components of a vector $v \in \mathbb{R}^d$ and a tensor $\sigma \in \mathbb{S}^d$ on $\Gamma$ are respectively given by

$$v_\nu = v \cdot \nu\ ,\quad v_\tau = v - v_\nu \nu \ \text{ and } \ \sigma_\nu = (\sigma\nu)\cdot\nu\ ,\quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu\,.$$

Then, the classical formulation of the considered contact problem is as follows.

**Problem** $(\mathcal{P})$**.** *Find a displacement field* $u : \mathbb{Q} \to \mathbb{R}^d$*, a stress field* $\sigma : \mathbb{Q} \to \mathbb{S}^d$*, a temperature* $\theta : \mathbb{Q} \to \mathbb{R}_+$ *and a bonding field* $\lambda : \Sigma_3 \to [0,1]$ *such that*

$$\sigma(t) = \mathcal{A}(\varepsilon({}_0^cD_t^\alpha u(t))) + \mathcal{B}(\varepsilon(u(t))) - \mathcal{M}\theta(t) + \int_0^t \Lambda(t-s)\varepsilon(u(s))ds \quad \text{in } \mathbb{Q}, \tag{3.1}$$

$${}_0^cD_t^\alpha \theta(t) - div\,\mathcal{K}(\nabla\theta(t)) = -\mathcal{N}\varepsilon(u(t)) + h_0(t) \quad \text{in } \mathbb{Q} \tag{3.2}$$

$$Div\,\sigma(t) + f_0(t) = 0 \quad \text{in } \mathbb{Q}, \tag{3.3}$$

$$q(t) + \mathcal{K}\nabla\theta(t) = 0 \quad \text{in } \mathbb{Q} \tag{3.4}$$

$$u(t) = 0 \quad \text{on } \Sigma_1 \tag{3.5}$$

$$\sigma(t)\,\nu = f_2(t) \quad \text{on } \Sigma_2 \tag{3.6}$$

$$\theta(t) = 0 \quad \text{on } \Sigma_1 \tag{3.7}$$

$$q(t) = h_n \quad \text{on } \Sigma_2 \tag{3.8}$$

$$\begin{cases} u_\nu \le g_0\,,\ \sigma_\nu(t) + \gamma(t) \le 0\,,\ (\sigma_\nu(t) + \gamma(t))(u_\nu(t) - g_0) = 0, \\ with\ \ \gamma(t) \in \partial j_\nu(\lambda(t), u_\nu(t) - g_0) \end{cases} \quad \text{on } \Sigma_3 \tag{3.9}$$

$$-\,\sigma_\tau(t) \in \partial j_\tau(\lambda(t), u_\tau(t)) \quad \text{on } \Sigma_3 \tag{3.10}$$

$$-\,\mathcal{K}(\nabla\theta(t))\cdot\nu \in \partial j_\theta(\lambda(t), \theta(t)) \quad \text{on } \Sigma_3 \tag{3.11}$$

$$\lambda^{'}(t) = g(t, u(t), \lambda(t)) \quad \text{on } \Sigma_3 \tag{3.12}$$

$$\lambda(0) = \lambda_0 \quad \text{on } \Gamma_3 \tag{3.13}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0 \quad \text{on } \Omega. \tag{3.14}$$

Here, equations (3.1) and (3.2) represent the fractional Kelvin-Voiget thermo-viscoelastic constitutive law with long memory of Caputo type, see [2,5,11] for more details, where $\mathcal{A} = (a_{ijkl})$ stands for viscosity tensor, $\mathcal{B} = (b_{ijkl}) \in L^\infty(\Omega)$ is a symmetric and coercive elasticity tensor, $\mathcal{M} = (m_{ij})$ is the thermal expansion tensor, $\mathcal{N} = (n_{ij})$ describes the influence of the displacement, $\Lambda$ is the relaxation tensor and $\varepsilon(u) = (\nabla u + (\nabla u)^T)/2$ is the linearised strain tensor. The heat flux field $q = (q_i)$ is defined through the thermal conductivity tensor $\mathcal{K} = (k_{ij})$ by the Fourier law of heat conduction (3.4). In addition, equations (3.3) and (3.4) represent the equilibrium equations for stress and heat flux fields, where $Div$ and $div$ denote the divergence operator for tensors and vector valued functions. Moreover, (3.5) and (3.7) are the mechanical and the thermal boundary conditions, relation (3.9) represents the multivalued normal compliance contact condition with unilateral constraints, more details can be found in [4]. Condition (3.10) represents the friction law in which $\partial j_\tau(\lambda(t),\cdot)$ denotes the Clarke generalized gradient of the function $j_\tau(\lambda(t),\cdot)$. It has been discussed in a number of articles, such as for example [1]. The relation (3.11) represent the heat exchenge between the contact zone $\Gamma_3$ and the foundation. The evolution of the adhesion field is given by a nonlinear ordinary differential equation (3.12) considered on a part $\Sigma_3$, where $g$ is a given function. We note that for explicite examples of a such fonctions, we refer to [10,18].

Finally, conditios (3.13) and (3.14) denote the initial adhesion conditions.

Now, to derive the weak formulation of Problem $(\mathcal{P})$, we introduce the following function spaces

$$H = L^2(\Omega)^d,\ H_1 = H^1(\Omega)^d,\ \mathcal{H} = \{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega)\},\ \mathcal{H}_1 = \{\sigma \in \mathcal{H} : Div\sigma \in \mathcal{H}\}.$$

These espaces are real Hilbert for the following inner products and their associated norms

$$(u, v)_H = \int_\Omega u_i v_i \, dx \ , \qquad (u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \ ,$$

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, dx \ , \qquad (\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_{\mathcal{H}},$$

We also need to introduce the following variational subspaces

$$V = \{v \in H_1(\Omega) : v = 0 \ \text{ on } \ \Gamma_1\}, \ \ W = \{\theta \in H_1(\Omega) : \theta = 0 \ \text{ on } \ \Gamma_1\}.$$

The spaces $V$ and $W$ are Hilbert for the following inner products and their associated norms

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \ , \qquad \|u\|_V = (u, u)_V^{1/2} \ , \tag{3.15}$$

$$(\theta, \eta)_W = (\nabla\theta, \nabla\eta)_{\mathcal{H}} \ , \qquad \|\theta\|_W = (\theta, \theta)_W^{1/2} \ . \tag{3.16}$$

Since $V$ is a closed subspace of $H_1$ and since $meas(\Gamma_1) > 0$, the Korn's inequality holds and then there exists a constant $c_k > 0$ depending only on $\Omega$ and $\Gamma_1$ such that

$$\|v\|_{H_1} \le c_k \, \|\varepsilon(v)\|_{\mathcal{H}}, \quad \forall\, v \in V. \tag{3.17}$$

Hence, the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on $V$, and then $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by Sobolev trace theorem, there exists $c_0 > 0$ depending only on $\Omega$, $\Gamma_3$ and $\Gamma_1$ such that

$$\|v\|_{L^2(\Gamma)^d} \le c_0 \, \|v\|_V, \quad \forall\, v \in V. \tag{3.18}$$

Moreover, sincee $meas(\Gamma_1) > 0$, the Friedrichs-Poincaré inequality holds and thus we have

$$\|\theta\|_{H^1(\Omega)} \le c_R \, \|\nabla\theta\|_{\mathcal{H}}, \quad \forall\, \theta \in W, \tag{3.19}$$

where $c_R > 0$ is a constant which depends only on $\Omega$ and $\Gamma_1$. It follows from (3.16) and (3.19) that the norms $\|\cdot\|_W$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent on $W$, and thus $(W, \|\cdot\|_W)$ is a real Hilbert space. In addition, Sobolev trace theorem implies that there exists $c_2 > 0$ depending on $\Omega$, $\Gamma_1$ and $\Gamma_3$ such that

$$\|\eta\|_{L^2(\Gamma_3)} \le c_2 \, \|\eta\|_W, \quad \forall\, \eta \in W. \tag{3.20}$$

In the study of Problem $(\mathcal{P})$, we need the following hypotheses.

$(\mathcal{B}_1)$ The viscocity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$(i)$ $\mathcal{A}(x, \xi) = a(x)\xi$ for a.e $x \in \Omega$ and all $\xi \in \mathbb{S}^d$,

$(ii)$ $a(x) = (a_{ijkl}(x))$ with $a_{ijkl} \in L^\infty(\Omega)$,

$(3i)$ $a_{ijkl}(x)\xi_{ij}\xi_{kl} \ge m_{\mathcal{A}}\|\xi\|^2_{\mathbb{S}^d}$ for a.e $x \in \Omega$ and all $\xi = (\xi_{ij}) \in \mathbb{S}^d$ with $m_{\mathcal{A}} > 0$.

$(\mathcal{B}_2)$ The elasticity tensor $\mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is partial symmetric and continuous satisfies

$(i)$ $\mathcal{B}(x, \xi) = b(x)\xi$ for a.e $x \in \Omega$ and all $\xi \in \mathbb{S}^d$,

$(ii)$ $b(x) = (b_{ijkl}(x))$ with $b_{ijkl} \in L^\infty(\Omega)$.

$(\mathcal{B}_3)$ The thermal operator $\mathcal{M} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is symmetric, continuous and satisfies

$(i)$ $\mathcal{M}(x, \varepsilon) = m(x)\varepsilon$ for a.e $x \in \Omega$ and all $\varepsilon \in \mathbb{R}^d$,

$(ii)$ $m(x) = (m_{ij}(x))$ with $m_{ij} \in L^\infty(\Omega)$.

$(\mathcal{B}_4)$ The fonction $\mathcal{N} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is symmetric, continuous and satisfies

$(i)$ $\mathcal{N}(x, \varepsilon) = n(x)\varepsilon$ for a.e $x \in \Omega$ and all $\varepsilon \in \mathbb{R}^d$,

$(ii)$ $n(x) = (n_{ij}(x))$ with $n_{ij} \in L^\infty(\Omega)$.

$(\mathcal{B}_5)$ The thermal conductivity $\mathcal{K} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is symmetric, continuous and satisfies

$(i)$ $\mathcal{K}(x, \varepsilon) = k(x)\varepsilon$ for a.e $x \in \Omega$ and all $\varepsilon \in \mathbb{R}^d$,

$(ii)$ $k(x) = (k_{ij}(x))$ with $k_{ij} \in L^\infty(\Omega)$.

$(\mathcal{B}_6)$ The function $j_\nu : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$(i)$ $j_\nu(\cdot, r, s)$ is measurable on $\Gamma_3$ for all $r \in \mathbb{R}$ and

$$j_\nu(., 0, 0) \in L^2(\Gamma_3),$$

$(ii)$ $j_\nu(x, r, \cdot)$ is locally Lipschitz on $\mathbb{R}$, for all $r \in \mathbb{R}$ and $x \in \Gamma_3$,

$(3i)$ there exist a constants $a > 0$, such that

$$|\partial j_\nu(x, r, s)| \leq a(1 + |s|), \tag{3.21}$$

$(4i)$ either $j_\nu(x, r, .)$ or $-j_\nu(x, r, .)$ is regular for a.e $x \in \Gamma_3$, $r \in \mathbb{R}$

$(5i)$ $(r, s) \to j^0_\nu(x, r, s, z)$ is upper semicontinuous.

$(\mathcal{B}_7)$ $j_\tau : \Gamma_3 \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$(i)$ $j_\tau(\cdot, r, \xi)$ is measurable on $\Gamma_3$ for all $(r, \xi) \in \mathbb{R} \times \mathbb{R}^d$ and

$$j_\tau(., 0, 0) \in L^2(\Gamma_3),$$

$(ii)$ $j_\tau(x, r, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$, for all $r \in \mathbb{R}$ and $x \in \Gamma_3$,

$(3i)$ there exist a constants $b > 0$, such that

$$|\partial j_\tau(x, r, \xi)| \leq b(1 + \|\xi\|), \tag{3.22}$$

$(4i)$ either $j_\tau(x, r, .)$ or $-j_\tau(x, r, .)$ is regular for a.e $x \in \Gamma_3$, $r \in \mathbb{R}$,

$(5i)$ $(r, \xi) \to j^0_\tau(x, r, \xi, \eta)$ is upper semicontinuous for $\eta \in \mathbb{R}^d$.

$(\mathcal{B}_8)$ The function $j_\theta : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$(i)$ $j_\theta(\cdot, r, s)$ is measurable on $\Gamma_3$ for all $r \in \mathbb{R}$ and

$$j_\theta(., 0, 0) \in L^2(\Gamma_3),$$

$(ii)$ $j_\theta(x,r,\cdot)$ is locally Lipschitz on $\mathbb{R}$, for all $r\in\mathbb{R}$ and $x\in\Gamma_3$,

$(3i)$ there exist a constants $c>0$, such that

$$|\partial j_\theta(x,r,s)|\leq c(1+|s|), \tag{3.23}$$

$(4i)$ either $j_\theta(x,r,.)$ or $-j_\theta(x,r,.)$ is regular for a.e $x\in\Gamma_3$, $r\in\mathbb{R}$,

$(5i)$ $(r,s)\to j^0_\theta(x,r,s,z)$ is upper semicontinuous.

$(\mathcal{B}_9)$ The function $g:\Gamma_3\times[0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ satisfies

$(i)$ $g(\cdot,\cdot,\xi,r)$ is measurable on $\Gamma_3\times[0,T]$ for all $\xi\in\mathbb{R}^d$, $r\in\mathbb{R}$,

$(ii)$ $g(\cdot,\cdot,\xi,s)$ is measurable on $\Gamma_3\times(0,T)$ for all $r$, $\xi\in\mathbb{R}\times\mathbb{R}^d$ and $x\in\Gamma_3$,

$(3i)$ there exists $L_g>0$ such that, for all $t\in(0,T)$ and $r_1$, $r_2\in\mathbb{R}$, $\xi_1$, $\xi_2\in\mathbb{R}^d$, we have

$$|g(x,t,\xi_1,r_1)-g(x,t,\xi_2,r_2)|\leq L_g(|r_1-r_2|+\|\xi_1-\xi_2\|)\ \text{ a.e. } x\in\Gamma_3. \tag{3.24}$$

$(\mathcal{B}_{10})$ $\Lambda:(0,T)\to\mathbb{Q}_\infty$ is Lipschitiz continus with constant $L_\lambda$.

$(\mathcal{B}_{11})$ Initial condition, volume and surface forces densities, heat source and gap function satisfy

$$(u_0,\theta_0)\in V\times W,\ \lambda_0\in L^2(\Gamma_3),\ f_0,\,h_0\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)),\ f_2,\,h_n\in L^\infty(0,T;L^2(\Gamma_2;\mathbb{R}^d)).$$

Next, we consider two elements $f\in L^2(0,T;V^*)$ and $h\in L^2(0,T;W^*)$ defined by

$$\langle f(t),v\rangle_V=\langle f_0(t),v\rangle_{L^2(\Omega)^d}+\langle f_2(t),v\rangle_{L^2(\Gamma_2)^d}\ \text{ for all }\ v\in V,\ t\in(0,T), \tag{3.25}$$

$$\langle h(t),\xi\rangle_W=\langle h_0(t),\xi\rangle_{L^2(\Omega)}-\langle h_n(t),\xi\rangle_{L^2(\Gamma_2)}\ \text{ for all }\ \xi\in W,,\ t\in(0,T). \tag{3.26}$$

Using now Green's formula, the variational formulation of Problem $(\mathcal{P})$ is as follows.

**Problem** $(\mathcal{PV})$**.** *Find a displacement $u\in W^{1,2}(0,T;V)$, a temperature $\theta\in W^{1,2}(0,T;W)$ and a bonding field $\lambda\in W^{1,2}(0,T;L^2(\Gamma_3))$ such that for all $(v,\xi)\in V\times W$ a.e $t\in(0,T)$, we have*

$$\begin{cases}\langle\mathcal{A}(\varepsilon({}^c_0D^\alpha_t u(t))),\varepsilon(v)\rangle\ +\langle{}^c_0D^\alpha_t\theta(t),\xi\rangle+\langle\mathcal{B}(\varepsilon(u(t)))-\mathcal{M}\theta(t),\varepsilon(v)\rangle\\ +\langle\mathcal{N}\varepsilon(u(t)),\xi\rangle+\langle\mathcal{K}\nabla\theta(t),\nabla\xi\rangle+\langle\displaystyle\int_0^t\Lambda(t-s)\varepsilon(u(s))ds,\varepsilon(v)\rangle\\ +\displaystyle\int_{\Gamma_3}[j^0_\nu(\lambda(t),u_\nu(t)-g_0;v_\nu)+j^0_\tau(\lambda(t),u_\tau(t);v_\tau)]\,da\\ +\displaystyle\int_{\Gamma_3}j^0_\theta(\lambda(t),\theta(t);\xi)\,da\\ \geq\langle f(t),v\rangle_V+\langle h(t),\xi\rangle_W,\\ \lambda^{'}(t)=g(t,u(t),\lambda(t))\ \text{ on }\ \Sigma_3\\ \lambda(0)=\lambda_0\ \text{ in }\ \Gamma_3\\ u(0)=u_0,\ \ \theta(0)=\theta_0\ \text{ in }\ \Omega.\end{cases} \tag{3.27}$$

Then, we consider $Q=V\times W$ which is a real Hilbert space endowed by the following inner product

$$\langle y,z\rangle_Q=\langle u,v\rangle_V+\langle\theta,\xi\rangle_W\ \text{ for all }\ y=(u,\theta),\,z=(v,\xi)\in Q, \tag{3.28}$$

We introduce the operators A, B : $Q \to Q^*$, given for all $y = (u, \theta)$, $z = (v, \xi) \in Q$ by

$$\langle \mathrm{A}y, z\rangle = \langle \mathcal{A}\varepsilon(u), \varepsilon(v)\rangle + \langle \theta, \xi\rangle, \tag{3.29}$$

$$\langle \mathrm{B}y, z\rangle = \langle \mathcal{B}(\varepsilon(u)) - \mathcal{M}\theta, \varepsilon(v)\rangle + \langle \mathcal{N}\varepsilon(u), \xi\rangle + \langle \mathcal{K}\nabla\theta, \nabla\xi\rangle. \tag{3.30}$$

We also introduce the operator R : $L^2(0, T; Q) \to L^2(0, T; Q^*)$ defined for a.e $t \in (0, T)$ by

$$\langle (\mathrm{R}y)(t), z\rangle = \langle \int_0^t \Lambda(t-s)\varepsilon(u(s))\, da, \varepsilon(v)\rangle \text{ for all } (y, z) \in L^2(0, T; Q) \times Q. \tag{3.31}$$

Let $Y = L^2(\Gamma_3)$, $X = L^2(\Gamma_3, \mathbb{R}^d) \times L^2(\Gamma_3, \mathbb{R}^d)$, we then consider the following functional

$$j : Y \times X \longrightarrow \mathbb{R}, \quad j(\lambda, y) = \int_{\Gamma_3} [j_\nu(x, \lambda, u_\nu - g_0) \ + j_\tau(x, \lambda, u_\tau) + j_\theta(x, \lambda, \theta)] da. \tag{3.32}$$

For all $\lambda \in Y$, it follows from $(\mathcal{B}_5)$-$(\mathcal{B}_7)(4i)$ that $j(\lambda, \cdot)$ or $-j(\lambda, \cdot)$ is regular on $X$, and that

$$j^0(\lambda, y) = \int_{\Gamma_3} [j^0_\nu(x, \lambda, u_\nu - g_0) \ + j^0_\tau(x, \lambda, u_\tau) + j^0_\theta(x, \lambda, \theta)]\, da, \ \forall\, y = (u, \theta) \in X,$$

$$\partial j(\lambda, y) = \int_{\Gamma_3} [\partial j_\nu(x, \lambda, u_\nu - g_0) \ + \partial j_\tau(x, \lambda, u_\tau) + \partial j_\theta(x, \lambda, \theta)]\, da, \ \forall\, y = (u, \theta) \in X.$$

Next, we introduce the element $f_h \in L^2(0, T; Q^*)$ given for all $z = (v, \xi) \in Q$, by

$$\langle f_h, z\rangle_Q = \langle f, v\rangle_V + \langle h, \xi\rangle_W. \tag{3.33}$$

Then, let $M = \gamma$ be the trace operator $\gamma : Q \to X$, and consider the following operator

$$F : (0, T) \times X \times Y \to Y, \quad F(t, u, \lambda) = g(x, t, \lambda(x), u(x)) \tag{3.34}$$

for all $t \in (0, T)$, $u \in X$, $\lambda \in Y$ and a.e. $x \in \Gamma_3$. To simplify the notations, let's pose

$$w(t) = {}^c_0D^\alpha_t y(t) \ \text{ for a.e } \ t \in (0, T).$$

Then, for a.e. $t \in (0, T)$, we have

$$y(t) = {}_0I^\alpha_t w(t) + y_0.$$

Using now the previous notations, we consider the following equivalent problem.

**Problem $(\mathcal{QV})$.** *Find $w \in L^1(0, T; Q)$ and $\lambda \in W^{1,2}(0, T; Y)$ such that*

$$\begin{cases} \langle A(w(t)), v\rangle^*_V + \langle B(u_0 + {}_0I^\alpha_t w(t)), v\rangle + \langle R(u_0 + {}_0I^t_0 w(t)))(t), v\rangle \\ \quad + j^0(\lambda(t), M(u_0 + {}_0I^\alpha_t w(t)); Mv) \geq \langle f(t), v\rangle, \quad \forall\, t \in [0, T], \ \forall\, v \in Q. \\ \lambda'(t) = g(t, M(u_0 + {}_0I^\alpha_t w(t)), \lambda(t)), \quad \forall\, t \in (0, T), \\ \lambda(0) = \lambda_0 \end{cases} \tag{3.35}$$

The equivalence of Problem $(\mathcal{PV})$ and Problem $(\mathcal{QV})$ is obtained, i.e., has the same solution.

**Theorem 3.1** *Assume $(\mathcal{B}_1)$-$(\mathcal{B}_{11})$ satisfied. Then, Problem $(\mathcal{PV})$ has at least one solution*

$$(y, \lambda) \in W^{1,2}(0, T; Q) \times W^{1,2}(0, T; L^2(\Gamma_3)).$$

**Proof:** The proof is based on an abstract result on fractional differential history-dependent hemivariational inequalities, stated in Theorem 2.1. Using Definition 3.34 of the operator $F$ and assumption $(\mathcal{B}_9)$, it is clear that the operator $F$ satisfies $(\mathcal{H}_1)$. Moreover, from Definition 3.30 of the operator B and hypotheses $(\mathcal{B}_2)$-$(\mathcal{B}_5)$, we get that $\mathrm{B} \in \mathcal{L}(Q, Q^*)$, i.e., condition $\mathcal{H}_3$ holds. Next, Definition 3.31 of R and hypothesis $(\mathcal{B}_{10})$ imply that conditions $(H_4)$ and $(H_5)$ are satisfied, with $E = I$ and $q = \Lambda$.

**Lemma 3.1** *Under assumption* $(\mathcal{B}_3)$*, the operator* A *defined by* (3.29) *satisfies condition* $(\mathcal{H}_2)$.

**Proof:** We use Definition 3.29 of A and hypotheses $(\mathcal{B}_3)(3i)$ to obtain

$$\begin{aligned}\langle \mathrm{A}y, y\rangle = \langle \mathcal{A}\varepsilon(u), \varepsilon(u)\rangle + \langle \theta, \theta\rangle \geq m_{\mathcal{A}}\|u\|_V + \|\theta\|_W \\ \geq \min(m_{\mathcal{A}}, 1)(\|u\|_V + \|\theta\|_W).\end{aligned} \tag{3.36}$$

Then, the operator A is $m_{\mathrm{A}}$-coercive with $m_{\mathrm{A}} = \min(m_{\mathrm{A}}, 1)$, i.e., condition $(\mathcal{H}_2)$ holds.

□

**Lemma 3.2** *Under assumptions* $(\mathcal{B}_6)$*-*$(\mathcal{B}_8)$*, the operator* $j$ *defined by* (3.32)*, satisfies condition* $(\mathcal{H}_6)$.

**Proof:** Using Definition 3.32 of $j$, hypotheses $(\mathcal{B}_6)$-$(\mathcal{B}_8)$ and Corollary 4.15 in [12] to obtain

$$\begin{aligned}\|\partial j(\lambda, y)\| \leq \sqrt{3meas(\Gamma_3)}(a+b+c) + \sqrt{3}\max\{a+b, c\}(\|u\|_V + \|\theta\|_W) \\ \leq \max\left\{\sqrt{3meas(\Gamma_3)}(a+b+c), \sqrt{3}\max\{a+b, c\}\right\}(1 + \|y\|_Q).\end{aligned} \tag{3.37}$$

Then, conditions $(\mathcal{H}_6)$(i)-(ii) are verified for the following constant

$$c_j = \max\left\{\sqrt{3meas(\Gamma_3)}(a+b+c), \sqrt{3}\max\{a+b, c\}\right\}.$$

Moreover, assumptions $(\mathcal{B}_6)$-$(\mathcal{B}_8)(5i)$ ensure that the condition $(\mathcal{H}_5)$(3i) hold, see [12, Theorem 1.64].

□

Furthermore, condition $(\mathcal{H}_7)$, i.e., $f_h \in L^\infty(0, T, Q^*)$, is a direct consequence of the definitions 3.25, (3.26) and (3.33) of $f$, $h$ and $f_h$ repectively, and of the hypothesis $(\mathcal{B}_{11})$. Thus, condition $(H_8)$ follows effectively by using the properties of the trace operator. Finally, all the conditions of Theorem (2.1) hold and that concludes the proof Theorem 3.1.

□

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