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Two Adopter Dynamics with Cross Adoption: Effect of Adoption Delays [*]

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ABSTRACT: The present study is aimed at analysing consumer behaviour in the market where two products are offered and delay in adoption exists for both products. For this purpose, a three-compartment model is proposed by classifying a population into three categories, namely, the non-adopter class, the adopter class of product-I, and the adopter class of product-II. Boundedness, positivity, and the basic influence number of the mathematical model are analysed. Stability analysis for adopter-free equilibrium and interior equilibrium is carried out for different cases on delay using Ruan's and Lin-Wang's stability theories. Sensitivity analysis is performed for the basic influence number. Numerical simulations of the present model have been carried out with the help of MATLAB, for supporting our analytical findings. As the delay for the first adopter increases, the system switches from an unstable state to a stable state, whereas as the delay for the second adopter decreases, the system becomes stable.

Key Words: Basic influence number, Boundedness, Delay in adoption, Hopf bifurcation, Sensitivity analysis, Numerical simulation.

Contents

1. Introduction

Innovation diffusion is considered one of the most important components of analysing long-term economic growth. In layman's language, there are only two ways in which the output of an economy increases: (1) by increasing the number of inputs going into the productive process, or (2) if one is intelligent, one can think of new ways by which more output can be obtained with the same number of inputs. This

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intelligence leads towards the concept of innovation, and this theory has proved game-changing in addressing the economic, social, and environmental challenges of globalization. Presently, people's lifestyles are changing rapidly. New products or innovations are emerging and competing for survival in a market where cutthroat competition exists. Based on "The Theory of Adoption and Diffusion", Bass [1] developed the diffusion theory and put forward the following model:

$$\frac{dN}{dt} = p[m - N(t)] + \frac{q}{m}N(t)[m - N(t)].$$

Where $N(t)$, p, q, and m describe the cumulative fraction of adopters, the innovation coefficient, the imitation coefficient, and the possible adopter population respectively. Horsky et al. [3] studied the results of advertising on sales growth of newly launched products and tried to show that the diffusion process gained acceleration through advertising. The level of advertising throughout a product's life cycle is also discussed. Maurer et al. [4] developed a dynamical model of website growth for the sake of exploration of the results of competition among websites with the help of Lotka-Volterra expression and observed how the nature of the market is affected. Zhu et al. [5] modelled the dynamics of the population sizes of n different species and showed the upper boundedness of growth rates.

Thinker class and media awareness play important roles in the dynamics [6,13]. An innovation diffusion dynamics model with three parallel innovations having effects on each other during the diffusion process is conceptually framed to discuss associations in detail, which can be complementary, substitutional, independent, or competitive [8]. Singh et al. [9] suggested and analysed an epidemic model integrating maturation delay and latent period of infection, keeping in mind the dynamics of childhood diseases. Mirzae et al. [10] approximated the solution of a second-type integral equation and heat condition with the help of a stable numerical algorithm. Further, linear hyperbolic telegraph equations of second order were solved by converting into their related Volterra integro-differential equations [11]. Singh et al. [12] developed an SIS epidemic model and discussed the latency period of infection and consciousness of media as control strategies. Kumar et al. [14] solved a non-linear delayed mathematical model for innovation diffusion by using center manifold arguments and normal form theory. Sisodiya et al. [15] developed a mathematical model to analyse the diffusion of pathogen-induced cholera disease and the impact of vaccination. The optimal rate of vaccination and fatality rate of the pathogen population for the control of cholera disease are calculated with the help of the Pontryagin Minimum Principle. Tuli et al. [16,17] proposed an innovation diffusion model with delay in adoption and studied the behaviour of people towards the offer of two products. Kumar et al. [22,23] proposed a mathematical model to investigate the dynamical behaviour of the people and discussed the diffusion process of an innovation. Chugh et al. [24] presented a model with the concept of cooperativeness in a competitive market.

Keeping in mind the literature, most of the developed models considered adoption an instantaneous process. But in reality, no adoption is instantaneous. In this model, we consider different levels of delay in adoption. We also include other important factors for product diffusion, such as cross-adoption, word of mouth, media awareness, and frustration from innovation. The present paper is organised as follows: The introduction of the research study is described in part 1. In part 2, ideas for the development of the model are discussed. In part 3, positiveness as well as boundedness of the model have been performed. Part 4 deals with the dynamical behaviour of the system. In part 5, the sensitivity analysis of the basic influence number has been performed. Finally, in part 6, numerical experimentations have been presented to assist the analytical findings of the proposed model.

2. Development of the Proposed Model

Here, we discuss the ideas that inspired us to develop this model. In [16,17], the concept of two different patches is considered. We know that the word 'market' is meaningless without competition. Previous studies bypassed this important concept. In the present study, the concept of cross-adoption is included to meet the competitiveness of the market. Keeping in mind the concept of globalisation, we combined two patches defined in [16,17]. Assumptions for the proposed model are as follows:

(i) There are two products available in the market, and the population is classified into three non-intersecting classes, namely, non-adopters $N(t)$, adopters of product-I $A_1(t)$, and adopters of product-II $A_2(t)$. It is supposed that Λ is the constant recruitment rate in the non-adopter class and μ is the natural fatality rate of N, A_1 and A_2.

(ii) Word of mouth and media play an important role in attracting the non-adopter population to buy a new product from the market where substitution is available. A non-adopter who wants to buy a new product usually interacts with people who have already adopted it. Here β_1 (β_2) be the interaction rate between non-adopter and adopter of product-I (product-II) and m_1 (m_2) be the rate of media effect associated with product-I (product-II).

(iii) An interacted person takes time to switch from non-adopter class to adopter classes. It is supposed that the waiting period of the adoption in which non-user N becomes the user of the product-I (product-II) is τ_1 (τ_2).

(iv) Competition in the market inspires the adopter of any product to change its respective class. It is supposed that δ_1 and δ_2 are the rates of cross-adoption. Usually, there is a time when adopters get fed up with both innovations and rejoin the non-adopter class. Let ξ_1 (ξ_2) be the frustration rate of adopters of product-I (product-II).

Dependent upon the above assumptions, the compartmental schematic flow of our proposed system is presented in Figure 1.

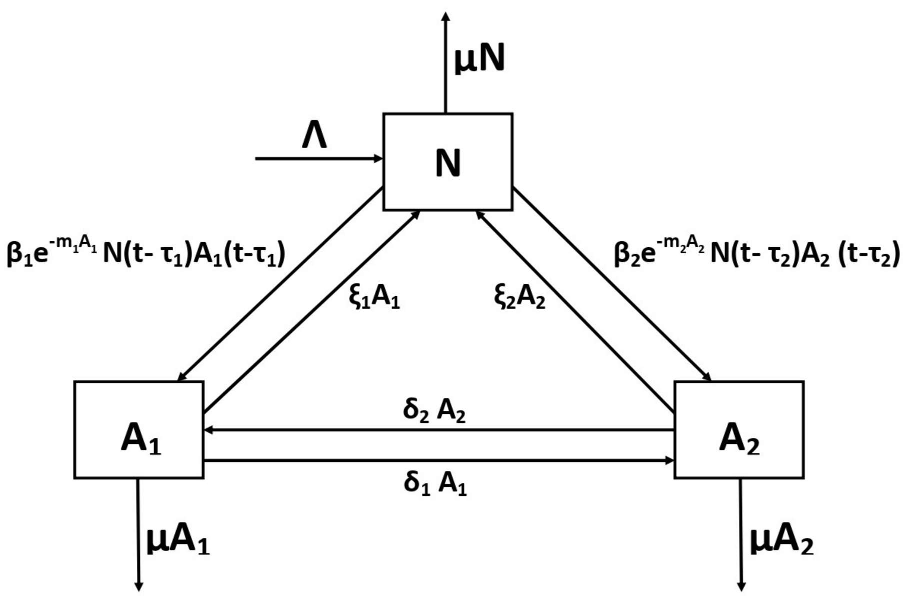

Figure 1: Compartmental schematic flow of proposed system

Keeping in view the schematic flow shown in Figure 1, the mathematical model can be presented by the following system of differential equations:

$$\begin{aligned} \frac{dN}{dt} &= \Lambda - \beta_1 e^{-m_1 A_1} N(t-\tau_1)A_1(t-\tau_1) - \beta_2 e^{-m_2 A_2} N(t-\tau_2)A_2(t-\tau_2) \\ &\quad +\xi_1 A_1 + \xi_2 A_2 - \mu N, \qquad (2.1) \\ \frac{dA_1}{dt} &= \beta_1 e^{-m_1 A_1} N(t-\tau_1)A_1(t-\tau_1) - \xi_1 A_1 - \delta_1 A_1 + \delta_2 A_2 - \mu A_1, \qquad (2.2) \\ \frac{dA_2}{dt} &= \beta_2 e^{-m_2 A_2} N(t-\tau_2)A_2(t-\tau_2) - \xi_2 A_2 + \delta_1 A_1 - \delta_2 A_2 - \mu A_2. \qquad (2.3) \end{aligned}$$

With the initial condition, let's consider the Banach space $C([-\tau, 0], R^3)$ of continuous function ψ: $[-\tau, 0] \to R^3$, with norm $\|\psi\| = \sup_{-\tau \le \theta \le 0}\{| \psi_1(\theta) |, | \psi_2(\theta) |, | \psi_3(\theta) |\}$, where $\tau = \max\{\tau_1, \tau_2\}$ and ψ

$= (\psi_1, \psi_2, \psi_3)$. The initial conditions of the system are given by $N(\theta) = \psi_1(\theta)$, $A_1(\theta) = \psi_2(\theta)$, $A_2(\theta) = \psi_3(\theta)$, $\theta \in [-\tau, 0]$ and the initial function $\psi = (\psi_1, \psi_2, \psi_3)$ belongs to Banach space C. The initial conditions, we choose as $\psi_i(\theta) \geq 0$, $\theta \in [-\tau, 0]$, $i = 1, 2, 3$. By the fundamental theorem of functional differential equations [2], delayed system possesses a unique solution with above the initial conditions.

3. Positivity and Boundedness of the Proposed Model

For boundedness and positivity, the following lemmas have been stated and proved:

Lemma 3.1 *The solution of the system (2.1)-(2.3), with the initial condition is non-negative, when* $t \geq 0$.

Proof: It is assumed that $(N(t), A_1(t), A_2(t))$ is the solution of the system (2.1)-(2.3) with non-negative initial conditions. For $t \in [0, \tau]$, the equation (2.1) becomes

$$\frac{dN}{dt} \geq -\beta_1 e^{-m_1 A_1} N A_1 - \beta_2 e^{-m_2 A_2} N A_2 - \mu N,$$

it implies that

$$N(t) \geq \; N(0)\exp\left\{-\int_0^t \left(\beta_1 e^{-m_1 A_1} A_1 + \beta_2 e^{-m_2 A_2} A_2 + \mu\right) dv\right\} \geq 0.$$

When $t \in [0, \tau]$, the equation (2.2) states that

$$\frac{dA_1}{dt} \geq -\xi_1 A_1 - \delta_1 A_1 - \mu A_1,$$

it implies that

$$A_1(t) \geq \; A_1(0)\exp\left\{-\int_0^t (\xi_1 + \delta_1 + \mu)\, dv\right\} \geq 0.$$

Lastly, for $t \in [0, \tau]$, the equation (2.3) becomes

$$\frac{dA_2}{dt} \geq -\xi_2 A_2 - \delta_2 A_2 - \mu A_2,$$

it implies that

$$A_2(t) \geq \; A_2(0)\exp\left\{-\int_0^t (\xi_2 + \delta_2 + \mu)\, dv\right\} \geq 0.$$

In the same way, for the intervals $[\tau, 2\tau]$, $[2\tau, 3\tau]$, $[3\tau, 4\tau]$,...,$[(n-1)\tau, n\tau]$ where $n \in \mathbb{N}$, the non-negativity of the solution of the system can be proved. Thus, the population densities $N(t), A_1(t)$, and $A_2(t)$ remain positive for $t \geq 0$. □

Lemma 3.2 *The solution for the proposed model (2.1)-(2.3), with the initial condition is bounded uniformly in* ζ, *where*

$$\zeta = \{(N, A_1, A_2) : 0 \leq N_(t) + A_1(t) + A_2(t) \leq \frac{\Lambda}{\mu}\}.$$

Proof: Let $U(t) = N(t) + A_1(t) + A_2(t)$. Differentiate $U(t)$ w.r.t. t, we get

$$\frac{dU(t)}{dt} = \Lambda - \mu(N + A_1 + A_2) = \Lambda - \mu W.$$

After solving, we get $0 \leq U(t) \leq \frac{\delta}{\mu}$ as $t \to \infty$. Hence, $U(t)$ is bounded. Therefore, we can say that the solution is bounded for each group of population. □

4. Dynamical Nature of the System

The dynamical nature of the system has been explored in this section. For this, we will calculate equilibrium points. After that, at adopter-free equilibrium, the basic influence number of the system is evaluated. Finally, stability analysis will be performed at all equilibrium points.

4.1. Equilibrium Points

The proposed system (2.1)-(2.3) has two equilibrium points:

(i) The adopter-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0)$.

(ii) The interior equilibrium $E^*(N^*, A_1^*, A_2^*)$, where N^*, A_1^*, A_2^* is the positive solution of the following system:

$$\begin{aligned} N + A_1 + A_2 &= \frac{\Lambda}{\mu}, \\ \beta_1 e^{-m_1 A_1} N A_1 - \xi_1 A_1 - \delta_1 A_1 + \delta_2 A_2 - \mu A_1 &= 0, \\ \beta_2 e^{-m_2 A_2} N A_2 - \xi_2 A_2 + \delta_1 A_1 - \delta_2 A_2 - \mu A_2 &= 0. \end{aligned}$$

4.2. Basic Influence Number

One of the most important threshold parameters that characterizes the diffusion of an innovation mathematically is the basic influence number. The main advantage of calculating this number, represented by R_A, is to know whether innovation will diffuse or not into the population. f vector contains the term of new adopter after interaction between non-adopter and adopter communities, and v vector contains the remaining term of adopter sections [18,19]:

$$f = \begin{pmatrix} \beta_1 e^{-m_1 A_1} N A_1 \\ \beta_2 e^{-m_2 A_2} N A_2 \end{pmatrix}, v = \begin{pmatrix} \xi_1 A_1 + \delta_1 A_1 - \delta_2 A_2 + \mu A_1 \\ \xi_2 A_2 - \delta_1 A_1 + \delta_2 A_2 + \mu A_2 \end{pmatrix}.$$

Now jacobians of f and v are evaluated at the adopter-free equilibrium E_0 and denoted by F and V respectively, where

$$F = J(f) = \frac{\Lambda}{\mu} \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, V = J(v) = \begin{pmatrix} \xi_1 + \delta_1 + \mu & -\delta_2 \\ -\delta_1 & \xi_2 + \delta_2 + \mu \end{pmatrix}.$$

Also

$$V^{-1} = \frac{1}{(\xi_1 + \delta_1 + \mu)(\xi_2 + \delta_2 + \mu) - \delta_1 \delta_2} \begin{pmatrix} \xi_2 + \delta_2 + \mu & \delta_2 \\ \delta_1 & \xi_1 + \delta_1 + \mu \end{pmatrix}.$$

Now $K = FV^{-1}$, is the next generation matrix.

$$FV^{-1} = \frac{\Lambda}{\mu[(\xi_1 + \delta_1 + \mu)(\xi_2 + \delta_2 + \mu) - \delta_1 \delta_2]} \begin{pmatrix} \beta_1(\xi_2 + \delta_2 + \mu) & \beta_1 \delta_2 \\ \beta_2 \delta_1 & \beta_2(\xi_1 + \delta_1 + \mu) \end{pmatrix}.$$

The basic influence number associated with the model is the largest eigen value, represented by R_A, is given by

$$\begin{aligned} R_A &= \frac{\Lambda}{2\mu[(\xi_1 + \delta_1 + \mu)(\xi_2 + \delta_2 + \mu) - \delta_1 \delta_2]} [\beta_1(\xi_2 + \delta_2 + \mu) + \beta_2(\xi_1 + \delta_1 + \mu) \\ &+ \sqrt{(\beta_1(\xi_2 + \delta_2 + \mu) - \beta_2(\xi_1 + \delta_1 + \mu))^2 + 4\beta_1 \beta_2 \delta_1 \delta_2}]. \end{aligned}$$

4.3. Stability Analysis of the model

Here, we will discuss the behaviour of stability at possible equilibrium points of the system (2.1)-(2.3).

4.3.1. **Local stability of adopter-free equilibrium.** First of all, we will explore the stability dynamics of the system for adopter-free equilibrium. The variational matrix of the system at adopter-free equilibrium is given by

$$J = \begin{pmatrix} -\mu & \xi_1 - \frac{\beta_1\Lambda}{\mu}e^{-\lambda\tau_1} & \xi_2 - \frac{\beta_2\Lambda}{\mu}e^{-\lambda\tau_2} \\ 0 & -\xi_1-\delta_1-\mu+\frac{\beta_1\Lambda}{\mu}e^{-\lambda\tau_1} & \delta_2 \\ 0 & \delta_1 & -\xi_2-\delta_2-\mu+\frac{\beta_2\Lambda}{\mu}e^{-\lambda\tau_2} \end{pmatrix},$$

and the characteristic equation is

$$(\lambda+\mu)F_1(\lambda) = 0, \tag{4.1}$$

with $F_1(\lambda) = y_0(\lambda) + y_1(\lambda) + y_2(\lambda) + y_3(\lambda)$ where $y_0(\lambda) = \lambda^2 + A\lambda + B$, $y_1(\lambda) = (C\lambda + D)e^{-\lambda\tau_1}$, $y_2(\lambda) = (E\lambda+F)e^{-\lambda\tau_2}$, $y_3(\lambda) = Ge^{-\lambda(\tau_1+\tau_2)}$; and $A = \xi_1+\delta_1+\xi_2+\delta_2+2\mu$, $B = (\xi_1+\delta_1+\mu)(\xi_2+\delta_2+\mu)-\delta_1\delta_2$, $C = -\frac{\beta_1\Lambda}{\mu}$, $D = -\frac{\beta_1\Lambda}{\mu}(\xi_2+\delta_2+\mu)$, $E = -\frac{\beta_2\Lambda}{\mu}$, $F = -\frac{\beta_2\Lambda}{\mu}(\xi_1+\delta_1+\mu)$, and $G = \beta_1\beta_2(\frac{\Lambda}{\mu})^2$.

Now, we will discuss all possible cases of stability depending upon the different pairs of values for delay parameters τ_1 and τ_2.

Case(I): $\tau_1 = 0$ and $\tau_2 = 0$, i.e., no delay in adoption for product-I and product-II. Then the equation $F_1(\lambda) = 0$ is reduced to

$$\lambda^2 + (A+E+C)\lambda + (G+D+F+B) = 0, \tag{4.2}$$

which is of the form

$$\lambda^2 + L_1\lambda + L_2 = 0. \tag{4.3}$$

Now, if $(\xi_1+\delta_1+\mu)(\xi_2+\delta_2+\mu)+\beta_1\beta_2(\frac{\lambda}{\mu})^2 > \frac{\lambda}{\mu}\{\beta_1(\xi_2+\delta_2+\mu)+\beta_2(\xi_1+\delta_1+\mu)\}+\delta_1\delta_2$ and $\mu(\xi_1+\xi_2+\delta_1+\delta_2+2\mu) > \Lambda(\beta_1+\beta_2)$ hold, then by the Routh-Hurwitz method, the adopter-free equilibrium E_0 is conditionally asymptotically stable, which is graphically shown in Figure 2. Hence, the adopter-free equilibrium point E_0 is conditionally asymptotically stable.

Case(II): $\tau_1 > 0$ and $\tau_2 = 0$, i.e., delay in adoption to adopt product-I exists but not for product-II. Then the equation $F_1(\lambda) = 0$ is reduced to

$$\lambda^2 + (E+A)\lambda + (F+B) + [C\lambda + (G+D)]e^{-\lambda\tau_1} = 0, \tag{4.4}$$

which is of similar form discussed in [20,21],

$$\lambda^2 + p\lambda + r + (s\lambda+q)e^{-\lambda\tau_1} = 0. \tag{4.5}$$

On comparing the equations (4.4) and (4.5), we get $p = E+A, r = F+B, s = C, q = G+D$. Now, for $\{C^2-(E+A)^2+2(F+B)\}^2 < 4\{(F+B)^2-(D+G)^2\}$, $A+C+E > 0$ and $B+D+F+G > 0$, by Theorem 2.3 discussed in [20], all roots of the system (2.1)-(2.3) possess real parts with negative values for all τ_1, and therefore the system is conditionally asymptotically stable under the conditions given in Appendix A.

Case(III): $\tau_1 = 0$ and $\tau_2 > 0$, i.e., delay in adoption to adopt product-II exists but not for product-I. This case is similar to Case II.

Case(IV): $\tau_1 > 0$ and $\tau_2 > 0$, i.e., delay in adoption to adopt product-I and product-II exists. In this situation, our considered equation $F_1(\lambda) = 0$ is similar to the characteristic equation discussed by Lin Wang [7].

Theorem 4.1 *Let $Z_1^2(w) + Z_2^2(w) > 0$, which implies $z_2\bar{z}_3 \neq z_0\bar{z}_1$, and the stability switching curves for the adopter-free equilibrium E_0 is $\tau = [(\tau^{\pm}_{1,n_1}(w), \tau^{\pm}_{2,n_2}(w)) \in \mathbb{R}^2_+ : w \in \triangle, n_1, n_2 \in \mathbb{Z}]$, where $\triangle = \{w : w > 0\}$.*

Proof: The condition *(ii)* of preliminary [page no. 521, [7]] states that $\lambda \neq 0$, and the zeros of a real function exist in conjugate pairs. Based on this fact, let us assume that $\lambda = iw$ and substitute it in the equation $F_1(\lambda) = 0$, we get

$$(-w^2 + B + Awi) + (D + Cwi)e^{-iw\tau_1} = -(F + Ewi + Ge^{-iw\tau_1})e^{-iw\tau_2}. \tag{4.6}$$

Taking modulus on both sides of the equation (4.6), we get

$$| (-w^2 + B + Awi) + (D + Cwi)e^{-iw\tau_1} |=| (F + Ewi + Ge^{-iw\tau_1}) || e^{-iw\tau_2} | . \tag{4.7}$$

Since $| e^{-iw\tau_2} |= 1$ and $z_0 = -w^2 + B + Awi, z_1 = D + Cwi, z_2 = F + Ewi, z_3 = G$. Therefore, the equation (4.7), can be written as

$$| z_0 + z_1 e^{-iw\tau_1} |=| z_2 + z_3 e^{-iw\tau_1} | . \tag{4.8}$$

On simplifying the equation (4.8), we have

$$| z_0 |^2 + | z_1 |^2 - | z_2 |^2 - | z_3 |^2 = 2Z_1(w)\cos(w\tau_1) - 2Z_2(w)\sin(w\tau_1), \tag{4.9}$$

where

$$Z_1 = Re(z_2\bar{z}_3) - Re(z_0\bar{z}_1), Z_2 = Im(z_2\bar{z}_3) - Im(z_0\bar{z}_1).$$

If there exists w satisfying $Z_1^2(w) + Z_2^2(w) = 0$, then

$$Z_1(w) = Z_2(w) = 0 \Longleftrightarrow z_2\bar{z}_3 = z_0\bar{z}_1. \tag{4.10}$$

The R.H.S. of the equation (4.9) is 0 for any $\tau_1 > 0$, and equation (4.9) can be expressed as

$$| z_0 |^2 + | z_1 |^2 =| z_2 |^2 + | z_3 |^2 . \tag{4.11}$$

Therefore, if there exists w that satisfies equations (4.10) and (4.11), then every $\tau_1 \in \mathbb{R}_+$ is the solution of the equation (4.8).

If $Z_1^2(w) + Z_2^2(w) > 0$, then there exists a continuous function $\phi(w)$ satisfying

$$Z_1 = \sqrt{Z_1^2(w) + Z_2^2(w)}\cos(\phi(w)), Z_2 = \sqrt{Z_1^2(w) + Z_2^2(w)}\sin(\phi(w)),$$

where $\phi(w) = arg(z_2\bar{z}_3 - z_0\bar{z}_1)$. Therefore, the equation (4.9) becomes

$$| z_0 |^2 + | z_1 |^2 - | z_2 |^2 - | z_3 |^2 = 2\sqrt{Z_1^2(w) + Z_2^2(w)}\cos(\phi(w) + w\tau_1). \tag{4.12}$$

As a result, an essential and adequate condition for the existence of $\tau_1 \in \mathbb{R}_+$ satisfying the equation (4.12) is

$$\big| | z_0 |^2 + | z_1 |^2 - | z_2 |^2 - | z_3 |^2 \big| \leq 2\sqrt{Z_1^2(w) + Z_2^2(w)}. \tag{4.13}$$

If $w \in \mathbb{R}_+$ satisfies (4.13), then it represents (4.10) and (4.11), when $Z_1^2(w) + Z_2^2(w) = 0$, Let us suppose

$$\cos(\phi_1(w)) = \frac{| z_0 |^2 + | z_1 |^2 - | z_2 |^2 - | z_3 |^2}{2\sqrt{Z_1^2(w) + Z_2^2(w)}}, \tau_{1,n_1}^{\pm}(w) = \frac{\pm\phi_1(w) - \psi(w) + 2n_1\pi}{w},$$

where $\phi_1(w) \in [0, \pi]$ and $n_1 \in \mathbb{Z}$. If the calculated value of τ_1 is substituted into the equation (4.6), then we obtain the value of $\tau_2(w)$ with each $w \in \triangle$, i.e.,

$$\tau_{2,n_2}^{\pm}(w) = \frac{1}{w}arg\left(-\frac{z_2 + z_3 e^{-iw\tau_1^{\pm}}}{z_0 + z_1 e^{-iw\tau_1^{\pm}}}\right) + 2n_2\pi \quad (n_2 \in \mathbb{Z}).$$

Thus, the stability crossing curves are $\tau = [(\tau_{1,n_1}^{\pm}(w), \tau_{2,n_2}^{\pm}(w)) \in \mathbb{R}_+^2 : w \in \triangle, n_1, n_2 \in \mathbb{Z}]$. □

Remark 4.1 *When $\tau_1 > 0, \tau_2 > 0$, for the proposed system (2.1)-(2.3), the adopter-free equilibrium $E_0(\frac{\Lambda}{\mu}, 0, 0)$ is conditionally asymptotically stable for all $\tau_1 \in [0, \tau_{10^+})$, $\tau_2 \in [0, \tau_{20^+})$, and unstable if $\tau_1 > \tau_{10^+}$, $\tau_2 > \tau_{20^+}$, also the system undergoes the Hopf bifurcation at $\tau_1 = \tau_{10^+}$, $\tau_2 = \tau_{20^+}$. The situation of stability is graphically shown in Figure 3.*

4.3.2. **Local stability of interior equilibrium.** In this section, we will try to learn about the stability dynamics of the system for the interior equilibrium point E^*. The determinantal equation of the jacobian matrix at the interior equilibrium point E^* can be put in the form

$$(\lambda+\mu)F_2(\lambda)=0, \tag{4.14}$$

with $F_2(\lambda) = y_0^*(\lambda) + y_1^*(\lambda) + y_2^*(\lambda) + y_3^*(\lambda)$ where $y_0^*(\lambda) = \lambda^2 + P\lambda + Q$, $y_1^*(\lambda) = (R\lambda + S)e^{-\lambda\tau_1}$, $y_2^*(\lambda) = (T\lambda + U)e^{-\lambda\tau_2}$, $y_3^*(\lambda) = Ve^{-\lambda(\tau_1+\tau_2)}$; and $P = \xi_1 + \delta_1 + \xi_2 + \delta_2 + 2\mu + m_1\beta_1 e^{-m_1A_1^*}N^*A_1^* + m_2\beta_2 e^{-m_2A_2^*}N^*A_2^*, Q = (\xi_1 + \delta_1 + \mu)(\xi_2 + \delta_2 + \mu) + (\xi_1 + \delta_1 + \mu)m_2\beta_2 e^{-m_2A_2^*}N^*A_2^* + (\xi_2 + \delta_2 + \mu)m_1\beta_1 e^{-m_1A_1^*}N^*A_1^* + m_1m_2\beta_1\beta_2 e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_1^*N^*A_2^* - \delta_1\delta_2, R = \beta_1 e^{-m_1A_1^*}(A_1^* - N^*), S = (\xi_2 + \delta_2+\mu)\beta_1 e^{-m_1A_1^*}(A_1^*-N^*)+m_2\beta_1\beta_2 e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_2^*(A_1^*-N^*)+\beta_1\delta_1 e^{-m_1A_1^*}A_1^*, T = \beta_2\ e^{-m_2A_2^*}(A_2^*-N^*), U = (\xi_1 + \delta_1 + \mu)\beta_2 e^{-m_2A_2^*}(A_2^* - N^*) + m_1\beta_1\beta_2 e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_1^*(A_2^* - N^*) + \beta_2\delta_2 e^{-m_2A_2^*}A_2^*$, and $V = \beta_1\beta_2 e^{-m_1A_1^*}e^{-m_2A_2^*}(A_1^* - N^*)(A_2^* - N^*) - \beta_1\beta_2 e^{-m_1A_1^*}e^{-m_2A_2^*}A_1^*A_2^*$.

Now, we will present in detail all possible cases of stability depending upon the different pairs of values for delay parameters τ_1 and τ_2.

Case(I): $\tau_1 = 0$ and $\tau_2 = 0$, i.e., there is non-existence of delay in adoption for both products. Then the equation $F_2(\lambda) = 0$ is reduced to

$$\lambda^2 + (P+R+T)\lambda + (Q+S+U+V) = 0, \tag{4.15}$$

which is of the form

$$\lambda^2 + L_3\lambda + L_4 = 0. \tag{4.16}$$

Now, if $P+R+T > 0$ and $Q+S+U+V > 0$ hold, then by the Routh-Hurwitz method, the interior equilibrium point $E^*(N^*, A_1^*, A_2^*)$ is stable under the conditions given in Appendix B, and it is graphically shown in Figure 4. Hence, the interior equilibrium point $E^*(N^*, A_1^*, A_2^*)$ is conditionally asymptotically stable.

Case(II): $\tau_1 > 0$ and $\tau_2 = 0$, i.e., delay in adoption of product-I exists but not for product-II. Then the equation $F_2(\lambda) = 0$ is reduced to

$$\lambda^2 + (P+T)\lambda + (Q+U) + [R\lambda + (S+V)]e^{-\lambda\tau_1} = 0, \tag{4.17}$$

the form of the above equation is similar as that given in [20,21]:

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau_1} = 0. \tag{4.18}$$

On equating the equations (4.17) and (4.18),we get $p = P+T, r = Q+U, s{=}R, q = S+V$. Now, by Theorem 2.3 discussed in [20], if $P+R+T > 0$, $Q+S+U+V > 0$ and $\{R^2 - (P+T)^2 + 2(Q+U)\}^2 < 4\{(Q+U)^2 - (S+V)^2\}$ hold, then all roots of the system (2.1)-(2.3) possess real parts with negative values for all τ_1, and therefore the system is conditionally asymptotically stable under the conditions given in Appendix C.

Case(III): $\tau_1 = 0$ and $\tau_2 > 0$, i.e., delay in adoption of product-II exists but not for product-I. This case is similar to previous Case II.

Case(IV): $\tau_1 > 0$ and $\tau_2 > 0$, i.e., delay in adoption of both product-I and product-II exists. In this situation, our considered equation $F_2(\lambda) = 0$ is similar to the characteristic equation discussed by Lin Wang [7], and the following theorem will find out the stability switching curves.

Theorem 4.2 *Let $Z_{1*}^2(w) + Z_{2*}^2(w) > 0$, which implies $z_2^*\bar{z_3^*} \neq z_0^*\bar{z_1^*}$, and the stability switching curves for the interior equilibrium E^* is $\tau = [(\tau_{1*,n_1}^{\pm}(w), \tau_{2*,n_2}^{\pm}(w)) \in \mathbb{R}_+^2 : w \in \triangle_1, n_1, n_2 \in \mathbb{Z}]$, where $\triangle_1 = \{w : w > 0\}$.*

Proof: The condition *(ii)* of preliminary [page no. 521, [7]] states that $\lambda \neq 0$, and the zeros of a real function exist in conjugate pairs. Based on this fact, let us assume that $\lambda = iw$ and substitute it in the equation $F_2(\lambda) = 0$, we get

$$(-w^2 + Q + Pwi) + (S + Rwi)e^{-iw\tau_1} = -(U + Twi + Ve^{-iw\tau_1})e^{-iw\tau_2}. \tag{4.19}$$

Taking modulus on both sides of the equation (4.19), we get

$$|\,(-w^2+Q+Pwi)+(S+Rwi)e^{-iw\tau_1}\,|=|\,(U+Twi+Ve^{-iw\tau_1})\,||\,e^{-iw\tau_2}\,|\,. \tag{4.20}$$

Since $|\,e^{-iw\tau_2}\,|=1$ and $z_0^*=-w^2+Q+Pwi,\;\; z_1^*=S+Rwi,\;\; z_2^*=U+Twi, z_3^*=V$. Therefore, the equation (4.20) can be written as

$$|\,z_0^*+z_1^*e^{-iw\tau_1}\,|=|\,z_2^*+z_3^*e^{-iw\tau_1}\,|\,. \tag{4.21}$$

On simplifying the equation (4.21), we have

$$|\,z_0^*\,|^2+|\,z_1^*\,|^2-|\,z_2^*\,|^2-|\,z_3^*\,|^2=2Z_{1*}(w)\cos(w\tau_1)-2Z_{2*}(w)\sin(w\tau_1), \tag{4.22}$$

where

$$Z_{1*}=Re(z_2^*\bar{z_3^*})-Re(z_0^*\bar{z_1^*}), Z_{2*}=Im(z_2^*\bar{z_3^*})-Im(z_0^*\bar{z_1^*}).$$

If there exists w satisfying $Z_{1*}^2(w)+Z_{2*}^2(w)=0$, then

$$Z_{1*}(w)=Z_{2*}(w)=0\Longleftrightarrow z_2^*\bar{z_3^*}=z_0^*\bar{z_1^*}. \tag{4.23}$$

The R.H.S. of the equation (4.22) is 0 for any $\tau_1>0$, and the equation (4.22) can be expressed as

$$|\,z_0^*\,|^2+|\,z_1^*\,|^2=|\,z_2^*\,|^2+|\,z_3^*\,|^2\,. \tag{4.24}$$

Therefore, if existing w satisfies equations (4.23) and (4.24), then every $\tau_1\in\mathbb{R}_+$ is the solution of the equation (4.21). If $Z_{1*}^2(w)+Z_{2*}^2(w)>0$, then there exists a continuous function $\phi^*(w)$ satisfying

$$Z_{1*}=\sqrt{Z_{1*}^2(w)+Z_{2*}^2(w)}\cos(\phi^*(w)), Z_{2*}=\sqrt{Z_{1*}^2(w)+Z_{2*}^2(w)}\sin(\phi^*(w)),$$

where $\phi^*(w)=arg(z_2^*\bar{z_3^*}-z_0^*\bar{z_1^*})$. Therefore, the equation (4.22) becomes

$$|\,z_0^*\,|^2+|\,z_1^*\,|^2-|\,z_2^*\,|^2-|\,z_3^*\,|^2=2\sqrt{Z_{1*}^2(w)+Z_{2*}^2(w)}\cos(\phi^*(w)+w\tau_1). \tag{4.25}$$

As a result, an essential and adequate condition for the existence of $\tau_1\in\mathbb{R}_+$ satisfying the equation (4.25) is

$$||\,z_0^*\,|^2+|\,z_1^*\,|^2-|\,z_2^*\,|^2-|\,z_3^*\,|^2|\leq 2\sqrt{Z_{1*}^2(w)+Z_{2*}^2(w)}. \tag{4.26}$$

If $w\in\mathbb{R}_+$ satisfies (4.26), then the equation (4.26) will also represent (4.23) and (4.24), when $Z_{1*}^2(w)+Z_{2*}^2(w)=0$, Let us assume

$$\cos(\phi_1^*(w))=\frac{|\,z_0^*\,|^2+|\,z_1^*\,|^2-|\,z_2^*\,|^2-|\,z_3^*\,|^2}{2\sqrt{Z_{1*}^2(w)+Z_{2*}^2(w)}}, \tau_{1*,n_1}^{\pm}(w)=\frac{\pm\phi_1^*(w)-\psi^*(w)+2n_1\pi}{w},$$

where $\phi_1^*(w)\in[0,\pi]$ and $n_1\in\mathbb{Z}$. If the calculated value of τ_1 is substituted into the equation (4.19), then we obtain the value of $\tau_2(w)$ with each $w\in\triangle_1$, i.e., $\tau_{2*,n_2}^{\pm}(w)=\frac{1}{w}arg(-\frac{z_{2*}+z_{3*}e^{-iw\tau_{1*}^{\pm}}}{z_{0*}+z_{1*}e^{-iw\tau_{1*}^{\pm}}})+2n_2\pi\;\;(n_2\in\mathbb{Z})$. Thus, the stability crossing curves are $\tau=[(\tau_{1*,n_1}^{\pm}(w),\tau_{2*,n_2}^{\pm}(w))\in\mathbb{R}_+^2:w\in\triangle_1,n_1,n_2\in\mathbb{Z}]$. □

Remark 4.2 *When $\tau_1>0,\tau_2>0$, for the proposed system (2.1)-(2.3), the interior equilibrium point E^* is conditionally asymptotically stable for all $\tau_1\in[0,\tau_{10++})$, $\tau_2\in[0,\tau_{20++})$, and unstable if $\tau_1>\tau_{10++}$, $\tau_2>\tau_{20++}$, also the system undergoes the Hopf bifurcation at $\tau_1=\tau_{10++}$, $\tau_2=\tau_{20++}$. The system undergoes the situation of oscillation for parametric values listed in set 4, and it is graphically shown in Figure 5. The system undergoes the situation of stability for parametric values listed in sets 5 and 6, which is graphically shown in figures 6 and 7.*

Table 1: The normalized sensitivity index for the basic influence number

Parameter (y_g)	$\gamma_{y_g}^{R_A}$
Λ	1
μ	0.217523
ξ_1	0.271048
ξ_2	0.108796
β_1	0.384408
β_2	0.615592
m_1	0
m_2	0
δ_1	0.208813
δ_2	0.193223

5. Sensitivity Analysis

The normalized sensitivity index for the basic influence number R_A is evaluated. The sensitivity index $\gamma_{y_g}^{R_A} = \frac{\partial R_A}{\partial y_g} \times \frac{y_g}{R_A}$ for the basic influence number R_A w.r.t. parameters y_g is shown in Table 1, by taking the interior parametric values: $\Lambda = 0.8; \xi_1 = 0.3; \xi_2 = 0.2; \mu = 0.15; \beta_1 = 0.3; m_1 = 0.5; \beta_2 = 0.31; m_2 = 0.3; \delta_1 = 0.12; \delta_2 = 0.17$. It is observed that each parameter has either a positive or zero impact on R_A.

6. Numerical Simulations

For verifying the analytical conclusions, numerical simulations have been performed with the help of MATLAB. An impression of completeness is provided by numerical simulation. For simulation, we take the parametric values shown in Table 2. Graphical results of the population distribution with respect to time for various data sets are shown in the figures (Figs. 2 to 7).

Table 2: Assumed different sets of parametric values

Set No.	δ_1	δ_2	μ	m_1	m_2	β_1	β_2	ξ_1	ξ_2	Λ	τ_1	τ_2
Set-1	0.12	0.17	0.15	0.35	0.20	0.05	0.02	0.30	0.20	1	0	0
Set-2	0.12	0.17	0.15	0.35	0.20	0.05	0.02	0.30	0.20	1	20	12.25
Set-3	0.12	0.17	0.15	0.35	0.20	0.45	0.42	0.30	0.20	1	0	0
Set-4	0.12	0.17	0.15	0.35	0.20	0.45	0.42	0.30	0.20	1	20	12.25
Set-5	0.12	0.17	0.15	0.35	0.20	0.45	0.42	0.30	0.20	1	20	10.25
Set-6	0.12	0.17	0.15	0.35	0.20	0.45	0.42	0.30	0.20	1	30	12.25

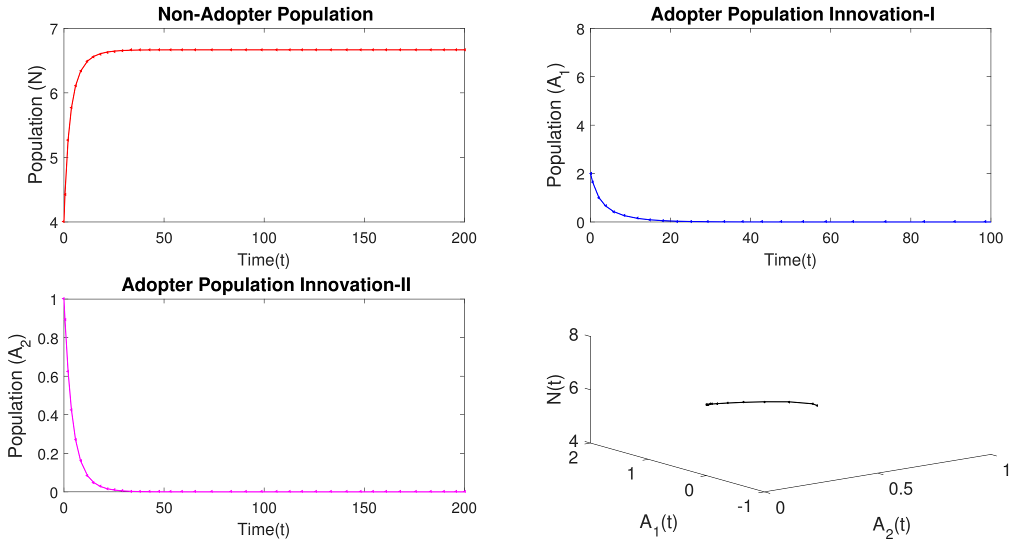

Figure 2: The adopter-free equilibrium $E_0(6.6669, 0, 0)$ is stable for the parametric values defined in Set-1 with τ_1=0 and τ_2=0.

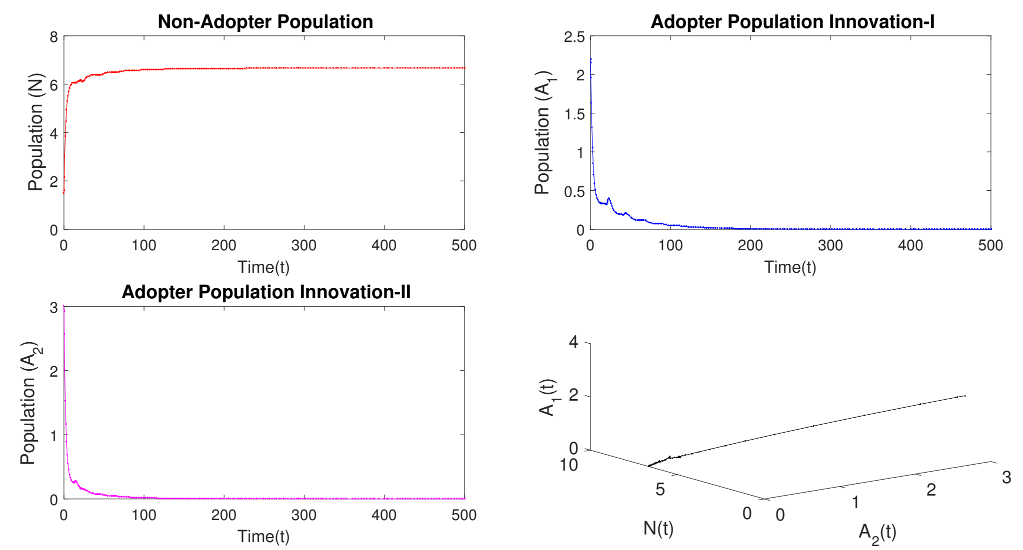

Figure 3: The adopter-free equilibrium $E_0(6.6669, 0, 0)$ is stable for the parametric values defined in Set-2 with τ_1=20 and τ_2=12.25.

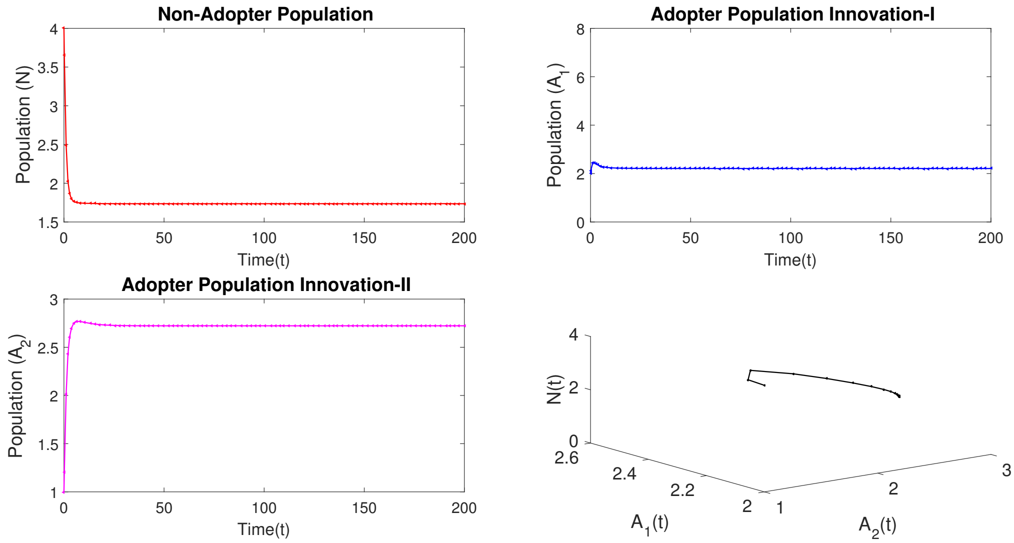

Figure 4: The interior equilibrium $E^*(1.7367, 2.2080, 2.7222)$ is stable for the parametric values defined in Set-3 with τ_1=0 and τ_2=0.

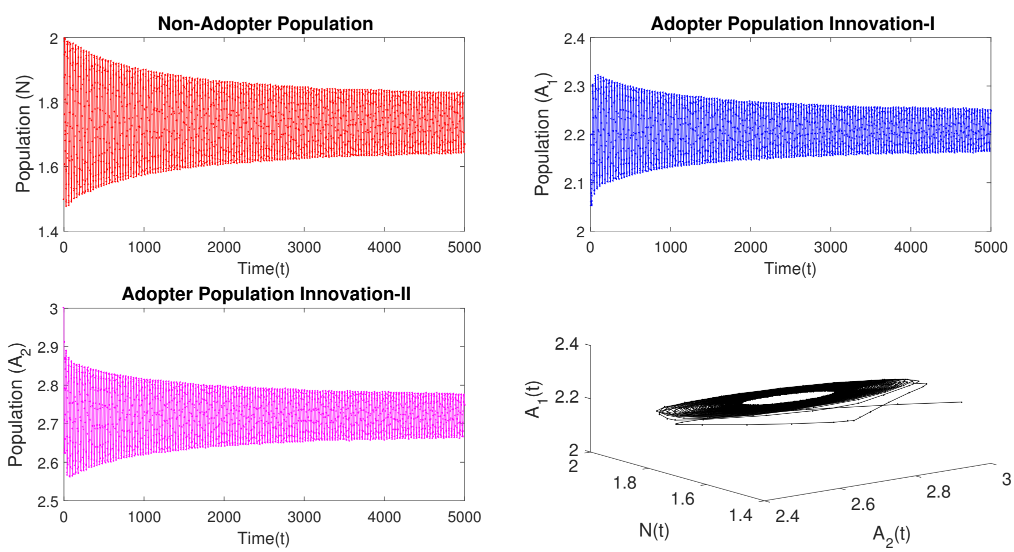

Figure 5: The interior equilibrium $E^*(N^*, A_1^*, A_2^*)$ is unstable, and Hopf bifurcation appears for the parametric values defined in Set-4 with τ_1=20 and τ_2=12.25.

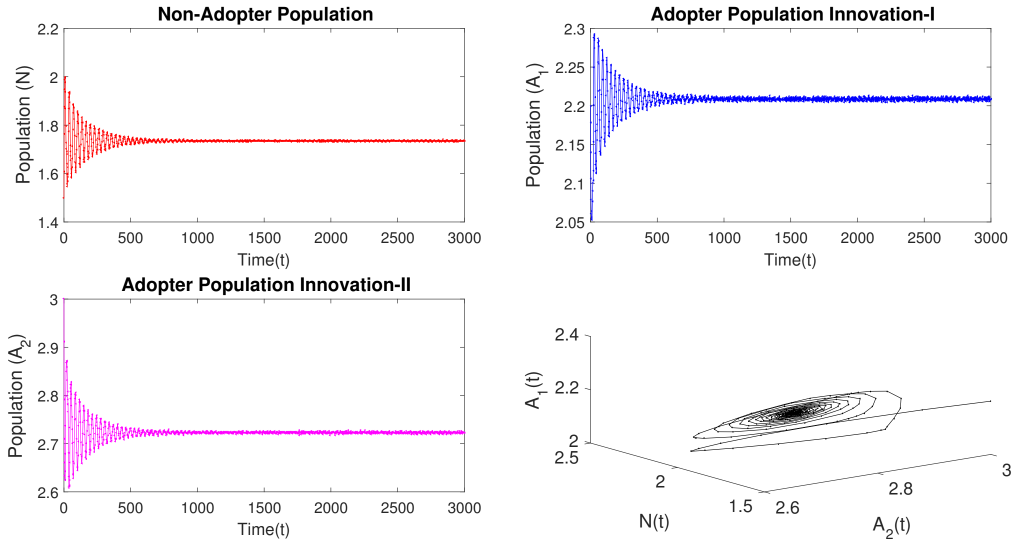

Figure 6: The interior equilibrium $E^*(1.7367, 2.2080, 2.7222)$ is stable for the parametric values defined in Set-5 with τ_1=20 and τ_2=10.25.

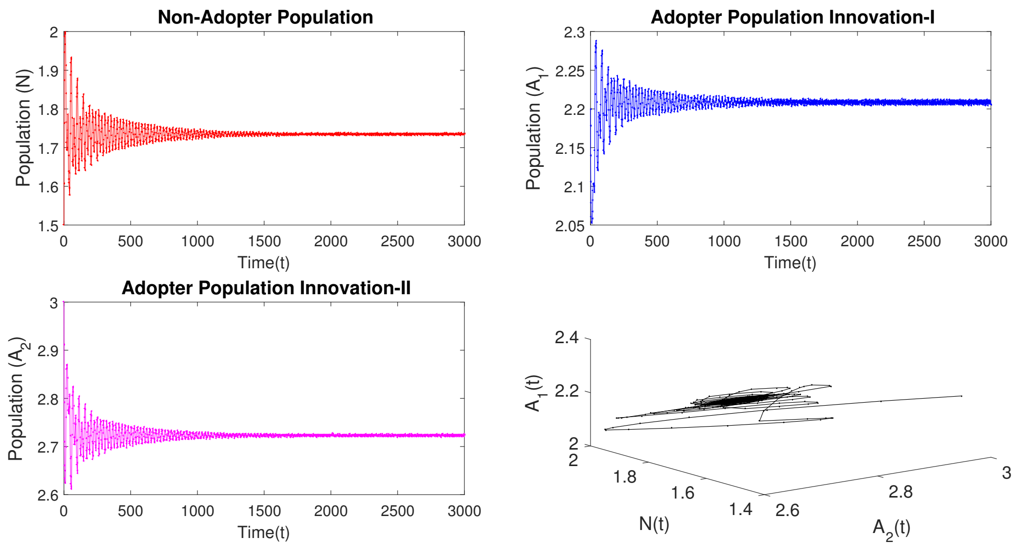

Figure 7: The interior equilibrium $E^*(1.7367, 2.2080, 2.7222)$ is stable for the parametric values defined in Set-6 with τ_1=30 and τ_2=12.25.

7. Conclusions

An innovation diffusion model considering time lags for adoption is proposed and discussed to know the response of the population. We observed two steady states for the model: adopter-free and interior equilibrium. The basic influence number R_A is obtained for the system at the adopter-free equilibrium. The local stability conditions for adopter-free equilibrium as well as interior equilibrium are discussed in detail for all possible cases, depending upon the different pairs of values for delay parameters τ_1 and τ_2. Sensitivity analysis for the basic influence number is performed, and it is found that all the parameters have either no impact or a positive impact. The recruitment rate (Λ) is the most sensitive to R_A, while

R_A is independent of the media effects (m_1 and m_2). Rest parameters μ, ξ_1, ξ_2, β_1, β_2, δ_1, and δ_2 are moderately sensitive for R_A. Finally, numerical simulations are performed to justify the analytical findings. As the delay for the first adopter increases, the system switches from an unstable to a stable state, whereas as the delay for the second adopter decreases, the system becomes stable.

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A. Appendix

(i) $\mu(\xi_1+\xi_2+\delta_1+\delta_2+2\mu) > \Lambda(\beta_1+\beta_2)$.

(ii) $(\xi_1+\delta_1+\mu)(\xi_2+\delta_2+\mu)+\beta_1\beta_2(\frac{\lambda}{\mu})^2 > \frac{\lambda}{\mu}\{\beta_1(\xi_2+\delta_2+\mu)+\beta_2(\xi_1+\delta_1+\mu)\}+\delta_1\delta_2$.

(iii) $[(\frac{\beta_1\Lambda}{\mu})^2-\{\xi_1+\xi_2+\delta_1+\delta_2+2\mu-\frac{\beta_2\Lambda}{\mu}\}^2+2\{(\xi_1+\delta_1+\mu)(\xi_2+\delta_2+\mu)-\delta_1\delta_2-\frac{\beta_2\Lambda}{\mu}(\xi_1+\delta_1+\mu)\}]^2 < 4[\{(\xi_1+\delta_1+\mu)(\xi_2+\delta_2+\mu)-\delta_1\delta_2-\frac{\beta_2\Lambda}{\mu}(\xi_1+\delta_1+\mu)\}^2-\{\beta_1\beta_2(\frac{\Lambda}{\mu})^2-\frac{\beta_1\Lambda}{\mu}(\xi_2+\delta_2+\mu)\}^2]$.

B. Appendix

(i) $\xi_1+\delta_1+\xi_2+\delta_2+2\mu+\beta_1e^{-m_1A_1^*}A_1^*(m_1N^*+1)+\beta_2e^{-m_2A_2^*}A_2^*(m_2N^*+1) > N^*(\beta_1e^{-m_1A_1^*}+\beta_2e^{-m_2A_2^*})$.

(ii) $(\xi_1+\delta_1+\mu)(\xi_2+\delta_2+\mu)+(\xi_1+\delta_1+\mu)m_2\beta_2e^{-m_2A_2^*}N^*A_2^*+(\xi_2+\delta_2+\mu)m_1\beta_1e^{-m_1A_1^*}N^*A_1^*+m_1m_2\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_1^*N^*A_2^*+\{(\xi_2+\delta_2+\mu)\beta_1e^{-m_1A_1^*}+m_2\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_2^*\}(A_1^*-N^*)+\beta_1\delta_1e^{-m_1A_1^*}A_1^*+(\xi_1+\delta_1+\mu)\beta_2e^{-m_2A_2^*}(A_2^*-N^*)+m_1\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_1^*(A_2^*-N^*)+\beta_2\delta_2e^{-m_2A_2^*}A_2^*+\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}(A_1^*-N^*)(A_2^*-N^*) > \beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}A_1^*A_2^*+\delta_1\delta_2$.

C. Appendix

(i) $\xi_1+\delta_1+\xi_2+\delta_2+2\mu+\beta_1e^{-m_1A_1^*}A_1^*(m_1N^*+1)+\beta_2e^{-m_2A_2^*}A_2^*(m_2N^*+1) > N^*(\beta_1e^{-m_1A_1^*}+\beta_2e^{-m_2A_2^*})$.

(ii) $(\xi_1+\delta_1+\mu)(\xi_2+\delta_2+\mu)+(\xi_1+\delta_1+\mu)m_2\beta_2e^{-m_2A_2^*}N^*A_2^*+(\xi_2+\delta_2+\mu)m_1\beta_1e^{-m_1A_1^*}N^*A_1^*+m_1m_2\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_1^*N^*A_2^*+\{(\xi_2+\delta_2+\mu)\beta_1e^{-m_1A_1^*}+m_2\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_2^*\}(A_1^*-N^*)+\beta_1\delta_1e^{-m_1A_1^*}A_1^*+(\xi_1+\delta_1+\mu)\beta_2e^{-m_2A_2^*}(A_2^*-N^*)+m_1\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_1^*(A_2^*-N^*)+\beta_2\delta_2e^{-m_2A_2^*}A_2^*+\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}(A_1^*-N^*)(A_2^*-N^*) > \beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}A_1^*A_2^*+\delta_1\delta_2$.

(iii) $[\{\beta_1e^{-m_1A_1^*}(A_1^*-N^*)\}^2-\{\xi_1+\delta_1+\xi_2+\delta_2+2\mu+m_1\beta_1e^{-m_1A_1^*}N^*A_1^*+m_2\beta_2e^{-m_2A_2^*}N^*A_2^*+\beta_2e^{-m_2A_2^*}(A_2^*-N^*)\}^2+2\{(\xi_1+\delta_1+\mu)(\xi_2+\delta_2+\mu)+(\xi_1+\delta_1+\mu)m_2\beta_2e^{-m_2A_2^*}N^*A_2^*+(\xi_2+\delta_2+\mu)m_1\beta_1e^{-m_1A_1^*}N^*A_1^*+m_1\ m_2\ \beta_1\ \beta_2\ e^{-m_1A_1^*}\ e^{-m_2A_2^*}\ N^*\ A_1^*\ N^*\ A_2^*-\delta_1\ \delta_2+(\xi_1+\delta_1+\mu)\ \beta_2\ e^{-m_2A_2^*}(A_2^*-N^*)+m_1\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_1^*(A_2^*-N^*)+\beta_2\delta_2e^{-m_2A_2^*}A_2^*\}]^2 < 4[\{(\xi_1+\delta_1+\mu)(\xi_2+\delta_2+\mu)+(\xi_1+\delta_1+\mu)m_2\beta_2e^{-m_2A_2^*}N^*A_2^*+(\xi_2+\delta_2+\mu)m_1\beta_1e^{-m_1A_1^*}N^*A_1^*+m_1m_2\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_1^*N^*A_2^*-\delta_1\delta_2+(\xi_1+\delta_1+\mu)\beta_2e^{-m_2A_2^*}(A_2^*-N^*)+m_1\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_1^*(A_2^*-N^*)+\beta_2\delta_2e^{-m_2A_2^*}A_2^*\}^2-\{[(\xi_2+\delta_2+\mu)\beta_1e^{-m_1A_1^*}+m_2\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}N^*A_2^*](A_1^*-N^*)+\beta_1\delta_1e^{-m_1A_1^*}A_1^*+\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}(A_1^*-N^*)(A_2^*-N^*)-\beta_1\beta_2e^{-m_1A_1^*}e^{-m_2A_2^*}A_1^*A_2^*\}^2]$.

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