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# The Sequences of Fibonacci and Lucas for Real Quadratic Number Fields

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ABSTRACT: We construct the sequences of Fibonacci and Lucas in any quadratic field $\mathbb{Q}(\sqrt{d})$ with $d > 0$ square free, noting that the general properties remain valid as those given by the classical sequences of Fibonacci and Lucas for the case $d = 5$, under the respective variants. For this construction, we use the fundamental unit of $\mathbb{Q}(\sqrt{d})$ and then we observe the generalizations for any unit of $\mathbb{Q}(\sqrt{d})$. Under certain conditions some of these constructions correspond to $k$-Fibonacci sequence for some $k \in \mathbb{N}$. Further, for both sequences, we obtain the generating function, Golden ratio, Binet's formula and some identities that they keep.

Key Words: Algebraic integers, Fibonacci and Lucas numbers, real quadratic fields.

## Contents



## 1. Introduction

The Fibonacci sequence was introduced by Leonardo of Pisa in 1202 in his book Liber Abaci (Book of Calculation) [26]. Many of the properties of the Fibonacci sequence were obtained by F. Édouard Lucas who appoints such sequence by "Fibonacci" [24, Section 3.1.2]. For more information about the history of the Fibonacci numbers, one may refer [23]. Lucas is the one who initiated the generalizations and their variants that have emerged from the Fibonacci sequence (See [2], [3], [4], [5] [7], [10], [16], [19] and [25]). Vera W. de Spinadel introduced the Metallic Means family whose members of such a family have many wonderful and amazing properties and applications to almost every areas of sciences and arts, such as in some areas of the physical, biology, astronomy and music (See [11], [12], [13] and [17]). On the other hand, Sergio Falcón and Ángel Plaza gave the properties of $k$–Fibonacci sequence (See [7], [8], [9] and [10]) and these are particular cases of metallic means families. Also, in [6] M. El-Mikkawy and T. Sogabe have given a new family of $k$–Fibonacci numbers. In [18], one can find hundreds of known identities. In [1], Azarian has presented some known identities as binomial sums for quick numerical calculations.

In this paper, for a real quadratic field $\mathbb{Q}(\sqrt{d})$, with $d > 0$ square free, we associate the sequences of Fibonacci and Lucas (Definition 5), which correspond to certain metallic means families (Theorem 3.2

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and Theorem 3.3). These sequences of Fibonacci and Lucas are determined by their generating functions (Theorem 4.2) satisfying each Binet's formula (Theorem 5.2 and Corollary 5.1). This means that every real quadratic field $\mathbb{Q}(\sqrt{d}\,)$ is also associated with its own Golden ratio (Definition 20), characteristic equation (5.2) and its Golden ratio will be the fundamental unit (Theorem 4.1). Finally, for each $k \in \mathbb{N}$, we establish that the $k$-Fibonacci sequence corresponds to Fibonacci sequence of the real quadratic field $\mathbb{Q}(\sqrt{d}\,)$ for a unique $d > 0$ square free (Theorem 7.2).

This paper is organized as follows. In Section 2, we collect results of quadratic fields necessary for the development of the work. In Section 3, we construct the sequences of Fibonacci and Lucas in any real quadratic field. Also, we prove that the properties remain valid as those given by the classical sequence of Fibonacci and Lucas for $d = 5$. In Section 4, we prove that the Fibonacci and Lucas sequences are determined by the generating functions. In Section 5, we give the Golden ratio associated to $\mathbb{Q}(\sqrt{d}\,)$ and we obtain Binet's formula. In Section 6, we extend our construction over rational integers. Finally, in Section 7, we define the sequences of Fibonacci and Lucas of degree $d$ with respect to an arbitrary unit $\eta$ of $\mathbb{Q}(\sqrt{d}\,)$ and we prove that the results of the previous sections are still met.

## 2. Quadratic number fields

In this section, we collect fundamental results from quadratic fields. Throughout this paper, $d$ denotes a square free integer, $\delta$ denotes the discriminant of the quadratic field $\mathbb{Q}(\sqrt{d}\,)$, $\mathcal{O}$ denotes the ring of integers of $\mathbb{Q}(\sqrt{d}\,)$, and $\mathcal{O}^*$ denotes the multiplicative group of all invertible elements of the ring $\mathcal{O}$. For $d > 0$, $\mathbb{Q}(\sqrt{d}\,)$ is a real quadratic field and for $d < 0$, $\mathbb{Q}(\sqrt{d}\,)$ is an imaginary quadratic field. We recall the following well known result ( [14], [15], [21], [22]):

**Theorem 2.1** *(i) If* $d \equiv 1 \bmod 4$, *then the set* $\left\{ 1\ ,\ \dfrac{1+\sqrt{d}}{2} \right\}$ *is an integral basis of* $\mathbb{Q}(\sqrt{d}\,)$, $\delta = d$,

$$\mathcal{O} = \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] = \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{d}}{2}\right) \text{ and}$$

$$\mathcal{O}^* = \left\{ \frac{a+b\sqrt{d}}{2} \;\middle|\; a, b \in \mathbb{Z},\ a^2 - db^2 = \pm 4 \right\}.$$

*(ii) If* $d \equiv 2 \bmod 4$ *or* $d \equiv 3 \bmod 4$, *then the set* $\left\{1,\ \sqrt{d}\right\}$ *is an integral basis of* $\mathbb{Q}(\sqrt{d}\,)$, $\delta = 4d$, $\mathcal{O} = \mathbb{Z}[\sqrt{d}\,] = \mathbb{Z} + \mathbb{Z}\ \sqrt{d}$ *and*

$$\mathcal{O}^* = \left\{ a + b\sqrt{d} \;\middle|\; a, b \in \mathbb{Z},\ a^2 - db^2 = \pm 1 \right\}.$$

*(iii) If* $d < 0$, *then* $\mathcal{O}^* = \{ -1, 1 \}$ *when* $d \neq -1,\ -3$, $\mathcal{O}^* = \langle i \rangle = \{-1, 1, i, -i\}$ *when* $d = -1$ *and* $\mathcal{O}^* = \langle \zeta_6 \rangle$ *if* $d = -3$, *where* $\zeta_6$ *is a primitive* 6*-th root of unity.*

*(iv) If* $d > 0$, *then*

*(a) There exists a unit* $\varepsilon > 1$ *in* $\mathcal{O}$ *such that* $\mathcal{O}^* = \langle -1 \rangle \times \langle \varepsilon \rangle$.

*(b) If* $u > 1$ *is a unit of* $\mathcal{O}$, *then* $u = a + b\sqrt{d}$ *for some* $a > 0$, $b > 0$ *in* $\mathbb{Q}$.

*(c) If* $N(\varepsilon) = 1$, *then* $N(u) = 1$ *for all* $u \in \mathcal{O}^*$.

The unit $\varepsilon$ of $\mathcal{O}$ in the Theorem 2.1 $(iv)$, is called the fundamental unit of $\mathcal{O}$. Hence the unit $\varepsilon$ of $\mathcal{O}$ completely determines the group $\mathcal{O}^*$.

On the other hand, we denote by $\mathfrak{M}_{2\times 2}(\mathbb{Z})$ the set of all matrices $2 \times 2$ with integer entries. Let $GL_2(\mathbb{Q})$ be the multiplicative group of invertible $2 \times 2$ matrices with rational entries, which is called the general lineal group of degree 2 over $\mathbb{Q}$. The subset of all matrices of $GL_2(\mathbb{Q})$ with determinant 1 is a normal subgroup of $GL_2(\mathbb{Q})$ called the special lineal group of degree 2 over $\mathbb{Q}$ and denoted by $SL_2(\mathbb{Q})$.

For each $\lambda \in \mathbb{Q}$, let

$$G_\lambda = \left\{ A \in GL_2(\mathbb{Q}) \;\middle|\; A = \begin{bmatrix} a & b\lambda \\ b & a \end{bmatrix} \right\} , \qquad L_\lambda = \{A \in G_\lambda \mid, \det(A) = \pm 1\}$$

and

$$T_d = \left\{ A \in \mathfrak{M}_{2\times 2}(\mathbb{Z}) \;\middle|\; A = \begin{bmatrix} a & bd \\ b & a \end{bmatrix} \right\} .$$

**Theorem 2.2** *(i) $T_d$ is a commutative subring with identity of $\mathfrak{M}_{2\times 2}(\mathbb{Z})$.*

*(ii) If $T_d^*$ is the multiplicative group of units of $T_d$, then $T_d^* = L_d \cap \mathfrak{M}_{2\times 2}(\mathbb{Z})$. In particular, $T_d^*$ is a subgroup of $L_d$.*

*(iii) The rings $T_d$ and $\mathbb{Z}[\sqrt{d}\,]$ are isomorphic under the correspondence*

$$\begin{bmatrix} a & bd \\ b & a \end{bmatrix} \longmapsto a + b\sqrt{d}\ .$$

*In particular, $T_d$ is an integral domain.*

*(iv) The isomorphism in (iii) induces an isomorphism between the multiplicative groups $T_d^*$ and $(\mathbb{Z}[\sqrt{d}\,])^*$.*

*(v) $T_d/(T_d \cap SL_2(\mathbb{Q})) \cong \{-1,\ 1\}$.*

**Theorem 2.3** *Let $Q_d$ be the set of all matrices of the form $A = \begin{bmatrix} a & bd \\ b & a \end{bmatrix}$ with $a, b \in \mathbb{Q}$.*

*(i) $Q_d$ is a field isomorphic $\mathbb{Q}(\sqrt{d}\,)$ under the correspondence $\begin{bmatrix} a & bd \\ b & a \end{bmatrix} \longmapsto a + b\sqrt{d}$. Moreover, $Q_d$ is the field of quotients of $T_d$.*

*(ii) There exists a monomorphism of the multiplicative group $\mathbb{Q}(\sqrt{d}\,)^*$ in the group $GL_2(\mathbb{Q})$.*

*(iii) The group $GL_2(\mathbb{Q})$ contains the chain of subgroups $Q_d^* \cap SL_2(\mathbb{Q}) < L_m < G_d = Q_d^* < GL_2(\mathbb{Q})$.*

**Theorem 2.4** *Let $A = \begin{bmatrix} a & bd \\ b & a \end{bmatrix} \in Q_d$, where $a$, $b$ are two rational numbers. Then the powers of $A$, $A^n = \begin{bmatrix} a_n & b_n d \\ b_n & a_n \end{bmatrix}$ with $n \in \mathbb{N}$, are given as follows:*

$$a_n = \begin{cases} \displaystyle\sum_{0 \le t \le \frac{n}{2}} \binom{n}{2t} a^{2t} b^{n-2t} d^{\frac{n}{2}-t}, & \text{if } n \text{ even;} \\[2ex] \displaystyle\sum_{0 \le t \le \frac{n-1}{2}} \binom{n}{2t+1} a^{2t+1} b^{n-2t-1} d^{\frac{n-1}{2}-t}, & \text{if } n \text{ odd} \end{cases} \tag{2.1}$$

*and*

$$b_n = \begin{cases} \displaystyle\sum_{0 \le t \le \frac{n-2}{2}} \binom{n}{2t+1} a^{2t+1} b^{n-2t-1} d^{\frac{n-2}{2}-t}, & \text{if } n \text{ even;} \\[2ex] \displaystyle\sum_{0 \le t \le \frac{n-1}{2}} \binom{n}{2t} a^{2t} b^{n-2t} d^{\frac{n-1}{2}-t}, & \text{if } n \text{ odd.} \end{cases} \tag{2.2}$$

## 3. The sequences of Fibonacci and Lucas in $\mathbb{Q}(\sqrt{d}\,)$

In this section, we construct the sequences of Fibonacci and Lucas in any real quadratic field. We prove that the properties remain valid as those given by the classical sequence of Fibonacci and Lucas for $d=5$. Being $d>0$ a square free integer and $\varepsilon$ the fundamental unit of $\mathbb{Q}(\sqrt{d}\,)$, we will write $\varepsilon = a+b\sqrt{d}$ where $a,b\in\mathbb{Q}$ with its corresponding matrix $A_\varepsilon = \begin{bmatrix} a & bd \\ b & a \end{bmatrix}$ and the $n$-th powers of $A_\varepsilon$ by $A_\varepsilon^n = \begin{bmatrix} a_n & b_n d \\ b_n & a_n \end{bmatrix}$, where $a_n$ and $b_n$ are given as in the equations (2.1) and (2.2) of Theorem 2.4. Also, $\Delta$ will be the determinant of $A_\varepsilon$, that is, $\Delta = a^2 - b^2 d = N(\varepsilon) = \pm 1$, where $N$ is the norm function of the field $\mathbb{Q}(\sqrt{d}\,)$.

**Definition 3.1** *The sequence of Fibonacci (resp. Lucas) of degree d with respect to the fundamental unit $\varepsilon$ is the sequence $\{F_{\varepsilon,n}\}_{n\in\mathbb{N}}$ (resp. $\{L_{\varepsilon,n}\}_{n\in\mathbb{N}}$) of positive numbers given as follows:*

$$F_{\varepsilon,n} := \frac{b_n}{b} \qquad \left(resp.\ \ L_{\varepsilon,n} := \frac{a_n}{a}\right) \qquad (n\in\mathbb{N}) \tag{3.1}$$

*where the sequence $\{b_n\}_{n\in\mathbb{N}}$ (resp. $\{a_n\}_{n\in\mathbb{N}}$) is given as in the equation (2.2) (resp. (2.1)) of Theorem 2.4.*

According to the equation (3.1) of the Definition 3.1, $F_{\varepsilon,n}$ and $L_{\varepsilon,n}$ are given by the follows equations:

$$F_{\varepsilon,n} = \begin{cases} \displaystyle\sum_{0\le t\le \frac{n-2}{2}} \binom{n}{2t+1} a^{2t+1} b^{n-2t-2} d^{\frac{n-2}{2}-t}, & if\ n\ even; \\ \displaystyle\sum_{0\le t\le \frac{n-1}{2}} \binom{n}{2t} a^{2t} b^{n-2t-1} d^{\frac{n-1}{2}-t}, & if\ n\ odd \end{cases} \tag{3.2}$$

and

$$L_{\varepsilon,n} = \begin{cases} \displaystyle\sum_{0\le t\le \frac{n}{2}} \binom{n}{2t} a^{2t-1} b^{n-2t} d^{\frac{n}{2}-t}, & if\ n\ even; \\ \displaystyle\sum_{0\le t\le \frac{n-1}{2}} \binom{n}{2t+1} a^{2t} b^{n-2t-1} d^{\frac{n-1}{2}-t}, & if\ n\ odd. \end{cases} \tag{3.3}$$

**Observation I.** When $d=5$, $\{F_{\varepsilon,n}\}_{n\in\mathbb{N}}$ and $\{L_{\varepsilon,n}\}_{n\in\mathbb{N}}$ are exactly the classical sequences of Fibonacci and Lucas, respectively.

In the rest of the work, by abuse of notation, we write $F_n$ and $L_n$ instead of $F_{\varepsilon,n}$ and $L_{\varepsilon,n}$ if there is no risk of confusion with respect to the classical sequences of Fibonacci and Lucas.

**Theorem 3.1** *For each $m,n\in\mathbb{N}$,*

*(i)* $F_{n+1} = a(L_n + F_n)$.

*(ii)* $L_{n+1} = aL_n + \dfrac{b^2 d}{a} F_n$.

*(iii)* $F_n = \dfrac{a}{\Delta}\left(F_{n+1} - L_{n+1}\right) = \begin{cases} a(L_{n+1} - F_{n+1}), & if \quad \Delta = -1; \\ a(F_{n+1} - L_{n+1}), & if \quad \Delta = 1. \end{cases}$

$$(iv)\ \ L_n = \frac{1}{\Delta}\left(aL_{n+1} - \frac{b^2d}{a}F_{n+1}\right) = \begin{cases} \frac{b^2d}{a}F_{n+1} - aL_{n+1}, & if \quad \Delta = -1; \\ aL_{n+1} - \frac{b^2d}{a}F_{n+1}, & if \quad \Delta = 1. \end{cases}$$

$$(v)\ \ F_{n+1} - a^nF_1 = \sum_{t=0}^{n-1} a^{t+1}L_{n-t}\ .$$

$$(vi)\ \ L_{n+1} - a^nL_1 = b^2d\ \sum_{t=0}^{n-1} a^{t-1}F_{n-t}\ .$$

$$(vii)\ \ F_{m+n} = a(F_mL_n + F_nL_m).$$

$$(viii)\ \ L_{m+n} = \frac{b^2d}{a}\cdot F_mF_n + a\ L_mL_n.$$

$$(ix)\ \ b^2d\ F_n^2 - a^2L_n^2 = -\Delta^n.$$

$$(x)\ \ F_n = \sum_{t=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2t+1}a^{n-2t-1}b^{2t}d^t = \sum_{t=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{n-2t-1}a^{n-2t-1}b^{2t}d^t.$$

$$(xi)\ \ L_n = \sum_{t=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2t}a^{n-2t-1}b^{2t}d^t = \sum_{t=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{n-2t}a^{n-2t-1}b^{2t}d^t.$$

*Here* $\lfloor x\rfloor$ *is the integral part of* $x \in \mathbb{R}$*, i.e., is the greatest integer* $n$ *such that* $n \le x < n+1$*.*

**Proof:** $(i)$ and $(ii)$ are obtained directly from the equations (3.2) and (3.3). $(iii)$ and $(iv)$ are deducted from $(i)$ and $(ii)$. By induction, we obtain $(v)$ and $(vi)$. $(vii)$ and $(viii)$ are obtained from the relationship $A_\varepsilon^{m+n} = A_\varepsilon^m \cdot A_\varepsilon^n$. The relation $\det(A_\varepsilon^n) = \Delta^n$ implies the relation $(ix)$. Finally, $(x)$ and $(xi)$ are obtained from the following relationship:

$$a_n + b_n\sqrt{d} = (a + b\sqrt{d}\,)^n = \sum_{i=0}^{n}\binom{n}{i}a^ib^{n-i}(\sqrt{d}\,)^{n-i} = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i(\sqrt{d}\,)^i.$$

□

**Theorem 3.2** *There exist unique* $r, s \in \mathbb{Q}^*$ *such that* $F_{n+2} = rF_n + sF_{n+1}$ *for all* $n \in \mathbb{N}$*. More precisely,* $F_{n+2} = (-\Delta)F_n + 2aF_{n+1}$ *for all* $n \in \mathbb{N}$*.*

**Proof:** We have for each $n \in \mathbb{N}$,

$$\begin{aligned}(-\Delta)F_n + 2aF_{n+1} &= -(a^2 - b^2d)F_n + 2aF_{n+1} = a\left(\frac{b^2d}{a}\right)F_n - a^2F_n + 2aF_{n+1} \\ &= a(L_{n+1} - aL_n) - a^2F_n + 2aF_{n+1} = a(L_{n+1} + F_{n+1}) \\ &= F_{n+2}\ .\end{aligned}$$

On the other hand, let $r, s \in \mathbb{Q}^*$ be such that

$$F_{n+2} = rF_n + sF_{n+1}, \quad \text{for all } n \in \mathbb{N} \tag{3.4}$$

As $b^2d = a^2 - \Delta$, implies that $F_1 = 1$, $F_2 = 2a$, $F_3 = 4a^2 - \Delta$ and $F_4 = 8a^3 - 4a\Delta$. In particular, by the equation (3.4) for $n = 1$ and $n = 2$, we obtain the system of equations

$$\left.\begin{aligned} r + 2as &= 4a^2 - \Delta \\ 2ar + (4a^2 - \Delta)s &= 8a^3 - 4a\Delta \end{aligned}\right\} \tag{3.5}$$

which has a unique solution, namely $r = -\Delta$ and $s = 2a$. □

**Corollary 3.1** *The Fibonacci sequence* $\{F_n\}_{n\in\mathbb{N}}$ *is a* $k$*-Fibonacci sequence for some* $k \in \mathbb{N}$ *(namely,* $k = 2a$*) if and only if* $\Delta = -1$.

**Proof:** It is immediate by Theorem 3.2. □

**Corollary 3.2** *The following conditions are equivalent:*

*(i)* $F_{n+2} = F_n + F_{n+1}$ *for all* $n \in \mathbb{N}$*;*

*(ii)* $F_3 = F_1 + F_2$*;*

*(iii)* $d = 5$ *and* $\varepsilon = \dfrac{1+\sqrt{5}}{2}$ .

**Proof:** $(i) \Longrightarrow (ii)$: It is immediate.
$(ii) \Longrightarrow (iii)$: We have that $-\Delta+4a^2 = (-\Delta)F_1+2aF_2 = F_3 = F_1+F_2 = 1+2a$, then $4a^2-2a-(\Delta+1) = 0$. If $\Delta = 1$, then $2a^2 - a - 1 = 0$; since $a \neq 1$, necessarily $a = -1/2$. But this implies that $4b^2d = -3$, a contradiction. Therefore $\Delta = -1$, $a = 1/2 = b$ and $d = 5$.
$(iii) \Longrightarrow (i)$: It is clear. □

**Corollary 3.3** *For* $d = 5$, $\{F_n\}_{n\in\mathbb{N}}$ *is the classical Fibonacci sequence, that is,*

$$F_{n+2} = F_n + F_{n+1}, n \in \mathbb{N}.$$

**Proof:** It is immediate. □

We recall, if $d \equiv 2$ or $3 \bmod 4$, then $\varepsilon = a + b\sqrt{d}$ where $a, b \in \mathbb{Z}$. In this case, it is obvious that $F_n \in \mathbb{N}$ for all $n \in \mathbb{N}$. If $d \equiv 1 \bmod 4$, then $\varepsilon = a + b\sqrt{d} = \dfrac{a_0 + b_0\sqrt{d}}{2}$ with $a_0,\ b_0 \in \mathbb{N}$, where either are both even or both odd. When they are both even, we have that $a, b \in \mathbb{N}$ and, hence, $F_n \in \mathbb{N}$. But, in any case, $2a \in \mathbb{N}$. Therefore, we obtain the following result.

**Corollary 3.4** $F_n \in \mathbb{N}$*, for all* $n \in \mathbb{N}$.

**Proof:** By Theorem 3.2, we have $F_{n+2} = (-\Delta)F_n+2aF_{n+1}$ for all $n \in \mathbb{N}$, where $F_1 = 1$ and $F_2 = 2a \in \mathbb{N}$. Then, the proof follows by induction on $n$. □

**Theorem 3.3** *There exist unique* $r, s \in \mathbb{Q}^*$ *such that* $L_{n+2} = rL_n + sL_{n+1}$ *for all* $n \in \mathbb{N}$*. More precisely,* $L_{n+2} = (-\Delta)L_n + 2aL_{n+1}$ *for all* $n \in \mathbb{N}$.

**Proof:** We have that for each $n \in \mathbb{N}$

$$\begin{aligned}(-\Delta)L_n + 2aL_{n+1} &= -\left(aL_{n+1} - \frac{b^2d}{a}F_{n+1}\right) + 2aL_{n+1} = aL_{n+1} + \frac{b^2d}{a}F_{n+1} \\ &= L_{n+2}\ .\end{aligned}$$

Now, we prove the uniqueness. As $b^2d = a^2 - \Delta$, it follows that

$$\begin{aligned}L_1 &= 1 \\ L_2 &= 2a - \frac{\Delta}{a} \\ L_3 &= 4a^2 - 3\Delta \\ L_4 &= 8a^3 - 8a\Delta + \frac{1}{a} \\ &\vdots \qquad \vdots\end{aligned}$$

Let $r, s \in \mathbb{Q}^*$ be such that

$$L_{n+2} = rL_n + sL_{n+1}, \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

In particular, for $n = 1$ and $n = 2$, we have the system of equations

$$\left.\begin{array}{rcl} r + \left(2a - \dfrac{\Delta}{a}\right)s & = & 4a^2 - 3\Delta \\ \left(2a - \dfrac{\Delta}{a}\right)r + (4a^2 - 3\Delta)s & = & 8a^3 - 8a\Delta + \dfrac{1}{a} \end{array}\right\} \tag{3.7}$$

which has a unique solution, namely $r = -\Delta$ and $s = 2a$; so that, this system of equations has the same solution that of the system of equations (3.5) given in the proof of Theorem 3.2. Hence, the theorem. □

Similar to the corollaries of Theorem 3.2 for Fibonacci sequence, we obtain corollaries of the Theorem 3.3 for Lucas sequence.

**Corollary 3.5** *The Lucas sequence $\{L_n\}_{n\in\mathbb{N}}$ is a $k$-Lucas sequence for some $k \in \mathbb{N}$ (namely, $k = 2a$) if and only if $\Delta = -1$.*

**Proof:** It immediate by Theorem 3.3. □

**Corollary 3.6** *The following conditions are equivalent:*

*(i)* $L_{n+2} = L_n + L_{n+1}$ *for all* $n \in \mathbb{N}$;

*(ii)* $L_3 = L_1 + L_2$;

*(iii)* $d = 5$ *and* $\varepsilon = \dfrac{1+\sqrt{5}}{2}$.

**Proof:** $(i) \Longrightarrow (ii)$: It is immediate.
$(ii) \Longrightarrow (iii)$: Since $L_3 = L_1 + L_2$, that is, $4a^2 - 3\Delta = 1 + 2a - \dfrac{\Delta}{a}$, we have that $4a^3 - 2a^2 - a + (1-3a)\Delta = 0$. If $\Delta = 1$, then $4a^3 - 2a^2 - 4a + 1 = 0$ and $a$ can not be a rational number, contradiction. Hence, $\Delta = -1$ and $(2a^2+1)(2a-1) = 0$. This implies that $a = 1/2$ and $4b^2d = 5$. Therefore, $d = 5$ and $a = 1/2 = b$.
$(iii) \Longrightarrow (i)$: It is clear. □

**Corollary 3.7** *If $\{L_n\}_{n\in\mathbb{N}}$ is the Lucas sequence classical, that is $d = 5$, then*

$$L_{n+2} = L_n + L_{n+1},$$

*for each* $n \in \mathbb{N}$.

**Proof:** It is immediate. □

**Corollary 3.8** *For all* $k \in \mathbb{N}$,

*(i)* $L_{2k-1} \in \mathbb{N}$;

*(ii) if* $a \in \mathbb{N}$, *then* $aL_{2k} \in \mathbb{N}$ *and* $(a, aL_{2k}) = 1$;

*(iii) if* $a = \dfrac{a_0}{2}$, *with* $a_0$ *odd, then* $a_0L_{2k} \in \mathbb{N}$ *and* $(a_0, a_0L_{2k}) = 1$.

**Proof:** Applying the Theorem 3.3, the proof follows by induction over all the pairs $(L_{2k-1}, L_{2k})$, $k \in \mathbb{N}$. □

## 4. Generating function

The main goal of this section is to show that the Fibonacci and Lucas sequences given in (4) and (5) are determined by the generating functions.

**Theorem 4.1** *We obtain*

(*i*) $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varepsilon = \lim_{n\to\infty} \frac{L_{n+1}}{L_n}$.

(*ii*) *The series* $\sum_{n=1}^{\infty} F_n x^{n-1}$ *and* $\sum_{n=1}^{\infty} L_n x^{n-1}$ *both have the same radius of convergence, namely* $R = 1/\varepsilon$.

**Proof:** (*i*) By Theorem 3.1, we have

$$\frac{F_{n+1}}{F_n} = \frac{a(L_n + F_n)}{F_n} = a + a \cdot \frac{L_n}{F_n} = a + b \cdot \frac{a_n}{b_n},$$

and

$$\frac{L_{n+1}}{L_n} = \frac{aL_n + \frac{b^2 d}{a} F_n}{L_n} = a + \frac{b^2 d}{a} \cdot \frac{F_n}{L_n} = a + b \cdot \frac{b_n d}{a_n}.$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \sqrt{d} = \lim_{n\to\infty} \frac{b_n d}{a_n}$ (see [20, Theorem 3.1]), It follows that,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varepsilon = \lim_{n\to\infty} \frac{L_{n+1}}{L_n}.$$

(*ii*) For each $x \in \mathbb{R}$, $x \neq 0$, we have

$$\lim_{n\to\infty} \frac{F_{n+1} x^n}{F_n x^{n-1}} = \varepsilon x = \lim_{n\to\infty} \frac{L_{n+1} x^n}{L_n x^{n-1}}.$$

Therefore, $\lim_{n\to\infty} \frac{F_{n+1}|x|^n}{F_n|x|^{n-1}} < 1$ if and only if $|x| < \frac{1}{\varepsilon}$.

Similarly, we can show that, $\lim_{n\to\infty} \frac{L_{n+1}|x|^n}{L_n|x|^{n-1}} < 1$ if and only if $|x| < \frac{1}{\varepsilon}$. Hence, both series have the same radius of convergence $R = 1/\varepsilon$. This complete the proof. □

**Theorem 4.2 (Generating function)** *Let* $x \in \mathbb{R}$ *be such that* $|x| < 1/\varepsilon$.

(*i*) *If* $f(x) = \sum_{n=1}^{\infty} F_n x^{n-1}$, *then* $f(x) = \frac{1}{\Delta x^2 - 2ax + 1}$ .

(*ii*) *If* $g(x) = \sum_{n=1}^{\infty} L_n x^{n-1}$, *then* $g(x) = \left(\frac{a - \Delta x}{a}\right) f(x) = \frac{a - \Delta x}{a(\Delta x^2 - 2ax + 1)}$ .

**Proof:** (*i*): For each $x \in \mathbb{R}$ with $|x| < 1/\varepsilon$, we have that

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} F_n x^{n-1} = 1 + 2ax + \sum_{n=1}^{\infty} F_{n+2} x^{n+1} = 1 + 2ax + \sum_{n=1}^{\infty} (-\Delta F_n + 2aF_{n+1})\, x^{n+1} \\ &= 1 + 2ax - \Delta x^2 f(x) + 2ax(f(x) - 1) = 1 + f(x)(2ax - \Delta x^2). \end{aligned}$$

This implies that $f(x) = \frac{1}{\Delta x^2 - 2ax + 1}$ .

$(ii)$: We observe that, for each $x \in \mathbb{R}$ with $|x| < 1/\varepsilon$,

$$f(x) = 1 + \sum_{n=1}^{\infty} F_{n+1}x^n = 1 + \sum_{n=1}^{\infty} a(L_n + F_n)x^n = 1 + axg(x) + axf(x) .$$

If $x \neq 0$, then

$$g(x) = \frac{(1-ax)f(x)-1}{ax} = \frac{a - \Delta x}{a(\Delta x^2 - 2ax + 1)} = \left(\frac{a-\Delta x}{a}\right) f(x) .$$

□

## 5. Golden ratio and Binet's formula in $\mathbb{Q}(\sqrt{d}\,)$

In this section, we give Golden ratio associated with the quadratic field $\mathbb{Q}(\sqrt{d}\,)$. Also, we obtain Binet's formula in $\mathbb{Q}(\sqrt{d}\,)$. We start with the following definition:

**Definition 5.1** *Let $x, y \in \mathbb{R}$ be such that $0 < y < x$. We say that $x$ and $y$ are in Golden ratio with respect to the quadratic field $\mathbb{Q}(\sqrt{d}\,)$ (or simply that they are in Golden ratio, if there is no risk of confusion with respect to the quadratic field $\mathbb{Q}(\sqrt{d}\,)$), if*

$$\frac{2ax - \Delta y}{x} = \frac{x}{y}. \tag{5.1}$$

If $x$ and $y$ are in Golden ratio and we write $\varphi := \dfrac{x}{y}$, then we have

$$2a - \frac{\Delta}{\varphi} = 2a - \Delta \cdot \frac{y}{x} = \frac{2ax - \Delta y}{x} = \frac{x}{y} = \varphi.$$

Thus, $\varphi$ satisfies the equation

$$\varphi^2 - 2a\varphi + \Delta = 0. \tag{5.2}$$

But $x^2 - 2ax + \Delta$ is the irreducible polynomial of $\varepsilon$ over $\mathbb{Q}$ with $\overline{\varepsilon}$ its other root, where $\overline{\varepsilon}$ is the conjugate of $\varepsilon$. Therefore, $\varphi = \varepsilon$ or $\varphi = \overline{\varepsilon}$. As $x > y > 0$ and $\overline{\varepsilon} = \Delta/\varepsilon$, necessarily $\varphi = \varepsilon$. In consequence, we have the equation

$$\varepsilon^2 = 2a\varepsilon - \Delta. \tag{5.3}$$

**Theorem 5.1** *For each $n \in \mathbb{N}$, with $n \geq 2$,*

$$\varepsilon^n = F_n\ \varepsilon - F_{n-1}\ \Delta. \tag{5.4}$$

**Proof:** We prove the result by induction on $n$. It is clear for $n = 2$, that is, $\varepsilon^2 = 2a\varepsilon - \Delta = F_2\ \varepsilon - F_1\ \Delta$. We assume that that the result holds for $n$. Now,

$$\begin{aligned}
\varepsilon^{n+1} &= \varepsilon(F_n\ \varepsilon - F_{n-1}\ \Delta) \\
&= F_n(2a\varepsilon - \Delta) - F_{n-1}\varepsilon\Delta \\
&= (-\Delta F_{n-1} + 2aF_n)\varepsilon - F_n\Delta \\
&= F_{n+1}\ \varepsilon - F_n\Delta.
\end{aligned}$$

□

Since $\overline{\varepsilon}$ also satisfies the equation (5.3), we have the equation

$$(\overline{\varepsilon})^n = F_n\ \overline{\varepsilon} - F_{n-1}\ \Delta, \tag{5.5}$$

for each $n \geq 2$.

**Theorem 5.2** *For each $n \in \mathbb{N}$,*

$$F_n = \frac{\varepsilon^n - (\overline{\varepsilon})^n}{\varepsilon - \overline{\varepsilon}} \tag{5.6}$$

**Proof:** It follows from the equations (5.4) and (5.5). □

The equation (5.6) is known as the Binet's formula.

**Corollary 5.1** *For each $n \in \mathbb{N}$,*

$$L_n = \frac{\varepsilon^n + (\overline{\varepsilon})^n}{\varepsilon + \overline{\varepsilon}} \tag{5.7}$$

**Proof:** It is immediate from the following:

$$\varepsilon^n + (\overline{\varepsilon})^n \;=\; 2aF_n - 2\Delta F_{n-1} = 2a\left(F_n - \frac{\Delta}{a}\cdot F_{n-1}\right) = 2aL_n = (\varepsilon + \overline{\varepsilon})L_n.$$

□

The following two theorems give the other version of the generating functions of the sequences of Fibonacci and Lucas in $\mathbb{Q}(\sqrt{d}\,)$.

**Theorem 5.3** *Let $f_1(x) = \sum_{n=0}^{\infty} \Delta^n F_{n+1} x^n$ and $g_1(x) = \sum_{n=0}^{\infty} \Delta^n L_{n+1} x^n$. Then, the series $f_1(x)$ and $g_1(x)$ are convergent for $|x| < \min\{|\varepsilon|, |\overline{\varepsilon}|\}$. Furthermore,*

$$f_1(x) \;=\; \frac{\Delta}{x^2 - 2ax + \Delta} \tag{5.8}$$

*and*

$$g_1(x) \;=\; \frac{\Delta(a-x)}{a(x^2 - 2ax + \Delta)} = \left(\frac{a-x}{a}\right) f_1(x) \tag{5.9}$$

**Proof:** For $|x| < \min\{|\varepsilon|, |\overline{\varepsilon}|\}$, we have

$$\begin{aligned}
\frac{2b\sqrt{d}}{(x-\varepsilon)(x-\overline{\varepsilon})} &= \frac{1}{x-\varepsilon} - \frac{1}{x-\overline{\varepsilon}} = \frac{1}{\overline{\varepsilon}\left(1 - \frac{x}{\overline{\varepsilon}}\right)} - \frac{1}{\varepsilon\left(1 - \frac{x}{\varepsilon}\right)} \\
&= \frac{1}{\overline{\varepsilon}} \sum_{n=0}^{\infty} \left(\frac{x}{\overline{\varepsilon}}\right)^n - \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \left(\frac{x}{\varepsilon}\right)^n = \sum_{n=0}^{\infty} \left(\frac{\varepsilon^{n+1} - \overline{\varepsilon}^{\;n+1}}{(\varepsilon\overline{\varepsilon})^{n+1}}\right) x^n \\
&= 2b\sqrt{d}\left(\sum_{n=0}^{\infty} (a^2 - b^2 d)^{n+1} \left(\frac{\varepsilon^{n+1} - \overline{\varepsilon}^{\;n+1}}{\varepsilon - \overline{\varepsilon}}\right) x^n\right) \\
&= 2b\sqrt{d}\left(\sum_{n=0}^{\infty} \Delta^{n+1} F_{n+1} x^n\right).
\end{aligned}$$

This implies that

$$\frac{1}{x^2 - 2ax + \Delta} = \frac{1}{x^2 - 2ax + (a^2 - b^2 d)} = \frac{1}{(x-\varepsilon)(x-\overline{\varepsilon})} = \sum_{n=0}^{\infty} \Delta^{n+1} F_{n+1} x^n,$$

or equivalently,

$$\frac{\Delta}{x^2 - 2ax + \Delta} = \sum_{n=0}^{\infty} \Delta^{n+2} F_{n+1} x^n = \sum_{n=0}^{\infty} \Delta^n F_{n+1} x^n \tag{5.10}$$

Therefore

$$f_1(x) = \frac{\Delta}{x^2 - 2ax + \Delta} .$$

On the other hand, we have

$$\begin{aligned} (a-x)\, f_1(x) &= (a-x)\left(\sum_{n=0}^{\infty} \Delta^{n+2} F_{n+1} x^n\right) \\ &= a\ F_1 + \sum_{n=1}^{\infty} \Delta^{n+1}\left(\frac{a}{\Delta} F_{n+1} - F_n\right) x^n \\ &= a\ L_1 + \sum_{n=1}^{\infty} \Delta^n a L_{n+1} x^n \\ &= a \sum_{n=0}^{\infty} \Delta^n L_{n+1} x^n. \end{aligned}$$

Therefore

$$g_1(x) = \left(\frac{a-x}{a}\right) f_1(x) = \frac{\Delta(a-x)}{a(x^2 - 2ax + \Delta)} .$$

□

## 6. Some Other Properties

Using the equations (5.6) and (5.7), we can extend the definition of the sequences of Fibonacci and Lucas over rational integers. We use the Binet's formula for $n \in \mathbb{Z}$, Theorem 5.2 and Corollary 5.1 and obtain

$$F_{-n} = \begin{cases} 0, & if \quad n = 0; \\ -\Delta^n F_n, & if \quad n \geq 1. \end{cases} \tag{6.1}$$

and

$$L_{-n} = \begin{cases} \frac{1}{a}, & if \quad n = 0; \\ \Delta^n L_n, & if \quad n \geq 1. \end{cases} \tag{6.2}$$

Thus, for all $n \in \mathbb{Z}$,

$$F_{n+2} = (-\Delta) F_n + 2a F_{n+1} \tag{6.3}$$

and

$$L_{n+2} = (-\Delta) L_n + 2a L_{n+1} \tag{6.4}$$

Also, we obtain the identities established by Catalan, Cassini, D' Ocagne and Hosnberger in the the following theorem:

**Theorem 6.1** *For all $m, n \in \mathbb{Z}$, the follows identities hold:*

(i) $F_n^2 - F_{n+m}F_{n-m} = \Delta^{n-m}F_m^2$,

(ii) $F_n^2 - F_{n-1}F_{n+1} = \Delta^{n-1}$,

(iii) $L_n^2 - L_{n+r}L_{n-r} = \dfrac{\Delta^n}{2a^2} - \left(\dfrac{\Delta^{n-r}}{2a}\right) L_{2r}$,

(iv) $F_mF_{n+1} - F_nF_{m+1} = \Delta^n F_{m-n}$,

(v) $F_{m-1}F_n + F_mF_{n+1} = \begin{cases} F_{m+n}, & if \quad \Delta = -1; \\ \dfrac{a}{2b^2d} \cdot \Big(2aL_{m+n} - L_{m-n-1}\Big), & if \quad \Delta = 1. \end{cases}$

(vi) $L_nL_{n+r} = \left(\dfrac{1}{2a}\right) L_{2n+r} + \left(\dfrac{\Delta^n}{2a}\right) L_r$.

**Proof:** All the identities follow from the Binet's Formula. We establish $(iv)$ as follows: We have

$$\begin{aligned} F_mF_{n+1} - F_nF_{m+1} &= \left(\frac{\varepsilon^m - (\bar{\varepsilon})^m}{\varepsilon - \bar{\varepsilon}}\right)\left(\frac{\varepsilon^{n+1} - (\bar{\varepsilon})^{n+1}}{\varepsilon - \bar{\varepsilon}}\right) \\ &\quad - \left(\frac{\varepsilon^n - (\bar{\varepsilon})^n}{\varepsilon - \bar{\varepsilon}}\right)\left(\frac{\varepsilon^{m+1} - (\bar{\varepsilon})^{m+1}}{\varepsilon - \bar{\varepsilon}}\right) \\ &= \frac{\varepsilon^m(\bar{\varepsilon})^n - \varepsilon^n(\bar{\varepsilon})^m}{\varepsilon - \bar{\varepsilon}} = \frac{\varepsilon^{m-n+n}(\bar{\varepsilon})^n - \varepsilon^n(\bar{\varepsilon})^{m-n+n}}{\varepsilon - \bar{\varepsilon}} \\ &= \Delta^n\left(\frac{\varepsilon^{m-n} - (\bar{\varepsilon})^{m-n}}{\varepsilon - \bar{\varepsilon}}\right) = \Delta^n F_{m-n}. \end{aligned}$$

□

## 7. The sequence of Fibonacci and of Lucas of degree $d$ with respect to an arbitrary unit

The unit group of $\mathbb{Q}(\sqrt{d})$, with $d > 0$, is isomorphic to the group $\langle -1 \rangle \times \langle \varepsilon \rangle$ where $\varepsilon$ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$. The cyclic subgroup $\langle \varepsilon \rangle$ is also generated by $1/\varepsilon$, $-\varepsilon$ and $-1/\varepsilon$. Each unit of $\mathbb{Q}(\sqrt{d})$ has the form $\pm\varepsilon^l$ for some $l \in \mathbb{Z}$. Observing the previous development, we can define the sequence of Fibonacci and Lucas of degree $d$ with respect to an arbitrary unit $\eta$ of $\mathbb{Q}(\sqrt{d})$, and the results of the previous sections are still met. Essentially, this is because $N(\eta) = \pm 1$. This allows us to build an infinite number of of sequences in $\mathbb{Q}(\sqrt{d})$ meeting similar properties of the sequences of Fibonacci and Lucas. For example, we consider the unit $\eta = \dfrac{-1+\sqrt{5}}{2}$ of $\mathbb{Q}(\sqrt{5})$. We have that the first terms of the sequence of Fibonacci of degree 5 with respect to the unit $\eta$ are:

$$F_{\eta,1} = 1,\ F_{\eta,2} = -1,\ F_{\eta,3} = 2,\ F_{\eta,4} = -3,\ \ldots,$$

where $N(\eta) = -1$. Comparing the terms of the sequence of Fibonacci with negative index, $F_{-n}$ with $n \geq 1$, we have that $F_{\eta,n} = F_{-n}$ for all $n \in \mathbb{N}$. This is the sequence of Fibonacci with negative index of degree 5 with respect to the unit $\eta = \dfrac{-1+\sqrt{5}}{2}$. This is not a coincidence and this fact is generalized in the following Theorem:

**Theorem 7.1** *The Fibonacci sequence of degree $d$ with respect to the unit $1/\varepsilon$ and $\Delta = -1$ is the Fibonacci sequence with negative index of degree $d$ with respect to the fundamental unit $\varepsilon$.*

**Proof:** We write $\eta = 1/\varepsilon = \Delta\overline{\varepsilon}$. Hence, $\overline{\eta} = \Delta\varepsilon$. Using the Binet's formula, for all $n \in \mathbb{N}$, we have

$$F_{\eta,n} = \frac{\eta^n - (\overline{\eta})^n}{\eta - \overline{\eta}} = \frac{(\Delta\overline{\varepsilon})^n - (\Delta\varepsilon)^n}{\Delta\overline{\varepsilon} - \Delta\varepsilon} = \Delta^{n-1}\,\frac{\varepsilon^n - (\overline{\varepsilon})^n}{\varepsilon - \overline{\varepsilon}} = \Delta^{n-1}F_n = -\Delta^n F_n = F_{-n}\ .$$

□

**Observation II.** If $\Delta = 1$, then the Fibonacci sequence of degree $d$ with respect to the unit $1/\varepsilon$ coincides with the Fibonacci sequence of degree $d$ with respect to the unit $\varepsilon$.

**Theorem 7.2** *For each $k \in \mathbb{N}$, there exist unique $d,\ r \in \mathbb{N}$ such that $d$ is square free and $\frac{k}{2} + \frac{r}{2}\sqrt{d}$ is a unit of the quadratic field $\mathbb{Q}(\sqrt{d}\,)$ with norm $-1$. Therefore, the $k$-Fibonacci sequence is the Fibonacci sequence of degree $d$ with respect to a unit of $\mathbb{Q}(\sqrt{d}\,)$.*

**Proof:** Let $k \in \mathbb{N}$ be arbitrary. We have that $k^2 + 4$ is not a perfect square. Hence, there exist $d, r \in \mathbb{N}$ such that $k^2 + 4 = r^2 d$, where $d$ is positive square free. This implies that

$$\left(\frac{k}{2}\right)^2 - \left(\frac{r}{2}\right)^2 d = -1.$$

Hence, $\frac{k}{2} + \frac{r}{2}\sqrt{d}$ is a unit of $\mathbb{Q}(\sqrt{d}\,)$ with norm $-1$. On the other hand, if $d,\ d_1, r,\ r_1 \in \mathbb{N}$ such that $\frac{k}{2} + \frac{r}{2}\sqrt{d}$ and $\frac{k}{2} + \frac{r_1}{2}\sqrt{d_1}$ are units of the quadratic field $\mathbb{Q}(\sqrt{d}\,)$ both with norm $-1$, then

$$\left(\frac{k}{2}\right)^2 - \left(\frac{r}{2}\right)^2 d = -1 = \left(\frac{k}{2}\right)^2 - \left(\frac{r_1}{2}\right)^2 d_1.$$

Thus, $\left(\frac{r}{2}\right)^2 d = \left(\frac{r_1}{2}\right)^2 d_1$, which implies that $r^2 d = r_1^2 d_1$, where $d$ and $d_1$ are square free. Therefore, $d_1 = d$ and $r_1 = r$. Consequently, the $k$-Fibonacci sequence is the Fibonacci sequence of degree $d$ with respect to a unit of $\mathbb{Q}(\sqrt{d}\,)$. □

**Corollary 7.1** *For each $k \in \mathbb{N}$, the $k$-Fibonacci sequence is a Fibonacci sequence of degree $d$ with respect to a unit of $\mathbb{Q}(\sqrt{d}\,)$ for some square free $d$.*

**Proof:** It is immediate from Theorem 7.2 and Corollary 3.1. □

## 8. Conclusions

In this paper, we have established that every real quadratic number field $\mathbb{Q}(\sqrt{d}\,)$ has its own Fibonacci and Lucas sequences, and variants of these through the fundamental unit. Therefore, the real quadratic field $\mathbb{Q}(\sqrt{d}\,)$ has its own golden ratio. On these lines, further research aimed at obtaining properties, both algebraic and geometric, related to the intrinsic properties of the real quadratic field $\mathbb{Q}(\sqrt{d}\,)$ will be reported in the subsequent paper.

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