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Infinite horizon doubly reflected generalized BSDEs in a general filtration under stochastic conditions

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ABSTRACT: In this paper, we explore a significant class of double reflected generalized backward stochastic differential equations in a general filtration framework that supports a Brownian motion and an independent integer-valued random measure. Specifically, when the barriers are right-continuous, left-limited, and completely separated, and that the generators satisfy stochastic conditions, we establish both the existence and uniqueness of a solutions when the time horizon is infinite.

Key Words: Generalized BSDE, general filtration, stochastic monotone coefficient, *rcll* barriers, Random measure, Yosida approximation, jumps, infinite horizon.

Contents

1. Introduction

The class of Generalized Backward Stochastic Differential Equations (GBSDEs) has proven to be a powerful tool for providing probabilistic representation formulas for solutions of systems of parabolic or elliptic semi-linear Partial Differential Equations (PDEs) with Neumann boundary conditions. Pardoux and Zhang [18] introduced these types of equations in the Brownian framework by adding an integral with respect to a continuous increasing process, which is the local time of a diffusion process on the boundary, to the classical nonlinear BSDEs introduced by Pardoux and Peng [16]. In the discontinuous case, Pardoux [15] carried out the study in a filtration generated by a Brownian motion and an independent Poisson random measure, while El Otmani [8] studied GBSDEs driven by a Lévy process. The author also provides a link between generalized BSDEs driven by a Lévy process and a class of partial differential integral equations with Neumann boundary conditions.

In the context of sales, new types of BSDEs have been developed, including the Generalized Reflected BSDE (GRBSDE) proposed by Ren and Xia [21]. The GRBSDE is a generalized backward equation that requires the solution to remain above a lower obstacle. These equations were motivated by the obstacle problems for PDEs with a nonlinear Neumann boundary condition. Other interesting results can be found by consulting [6,20].

El Otmani and Mrhardy [9] further developed this concept by introducing Doubly Reflected Generalized BSDEs (DRGBSDEs) with two reflecting barriers. They demonstrated the existence and uniqueness of the solution in a Brownian filtration via a penalization method, without the need for Mokobodzki's

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condition or extra regularity conditions on the reflecting barriers. They also applied these results to establish the existence of viscosity solutions for some PDEs with Neumann boundary conditions.

In recent years, considerable efforts have been dedicated to establishing the existence of a solution as part of a general parameter hypothesis. For instance, Bender and Kohlmann [2] investigated the stochastic Lipschitz condition proposed by El Karoui and Huang [7], while Bahlali et al. [1] studied the stochastic monotonicity condition on y and stochastic Lipschitz on z. It is worth noting that the monotone (or Lipschitz) coefficient can be an adapted process and is not necessarily bounded. Additionally, other studies have extended these results to the double reflected case, such as Marzougue and El Otmani [13] who investigated the BSDE with a stochastic Lipschitz coefficient and later extended their findings in [14] to include the jumps case.

Finally, we present some results on BSDE with general time intervals. Therefore our work reexamine and generalize the results from papers by Chen [3], Hua et al. [11], Hamadène et al. [10] and the aforementioned works by Pardoux and Zhang [18] and Pardoux [15].

In this paper, we will concentrate on doubly reflected generalized backward stochastic differential equations with jumps (DRGBSDEs) within a general filtration framework of the form:

$$\begin{cases} \text{(i) } Y_t = \xi + \displaystyle\int_t^\infty f(s, Y_s, Z_s, V_s)ds + \int_t^\infty g(s, Y_s)d\kappa_s + (K_\infty^+ - K_t^+) - (K_\infty^- - K_t^-) \\ \qquad\qquad - \displaystyle\int_t^\infty Z_s dW_s - \int_t^\infty \int_E V_s(e)\tilde{N}(ds, de) - \int_t^\infty dM_s, \quad t \geq 0. \\ \text{(ii) } L_t \leq Y_t \leq U_t, \quad t \geq 0, \\ \text{(iii) If } K^{c,\pm} \text{ is the continuous part of } K^\pm, \text{ then} \\ \qquad (Y_t - L_t)dK_t^{c,+} = (U_t - Y_t)dK_t^{c,-} = 0, \ \forall t \geq 0, \\ \quad \text{and if } K^{d,\pm} \text{ is the purely discontinuous part of } K^\pm, \text{ then} \\ \qquad K_t^{d,+} = \displaystyle\sum_{0<s\leq t} (Y_s - L_{s-})^- \text{ and } K_t^{d,-} = \sum_{0<s\leq t} (Y_s - U_{s-})^+. \end{cases} \tag{1.1}$$

In the equation (1.1), the process $(W_t)_{t\geq 0}$ denotes a standard Brownian motion, $\tilde{N}(\cdot, de)$ denotes the compensated measure of an integer-valued random measure N, and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration that supports W and $\tilde{N}$. Additionally, κ is a continuous increasing $\mathcal{F}_t$-progressively measurable process, and L and U denote the lower and upper obstacles, respectively. It is important to note that in our general filtration, the representation property of a local martingale is no longer valid (as discussed in Section III.4 of [12]). Therefore, an additional orthogonal martingale term needs to be introduced in the definition of a solution. We add the martingale M to equation (1.1) to account for this extra term.

Our result exhibits the following characteristics:

- The filtration $\mathbb{F}$ used in this study is not limited to a specific type and the random measure representing the jumps that are inaccessible is an integer-valued random measure with a compensator of a general form.

- In contrast to the literature that considers deterministic coefficients with Lipschitz or locally Lipschitz conditions, our coefficients f and g are subject to stochastic conditions regarding monotonicity, Lipschitz, and linear growth.

- The DRGBSDE (1.1) is assumed to have an infinite time horizon $T = +\infty$, which covers both finite and infinite $\mathbb{F}$-stopping times.

- The conditions imposed on the barriers $(L_t)_{t\geq 0}$ and $(U_t)_{t\geq 0}$ are relatively weak, as they only require right-continuity and left-limitation, and complete separation. Additionally, these barriers may exhibit general jumps, including predictable or inaccessible ones, resulting in both types of jumps being present in the solution component $(Y_t)_{t\geq 0}$.

To achieve the main objective, we establish the existence and uniqueness of the solution for (1.1) when $T < +\infty$ using two approaches. The first method involves the Yosida regularization of monotone operators for generators f and g that depend only on (t, y) and are monotone on y. The second method employs a fixed point argument via a contraction mapping method to handle the general case. We then prove the case of infinite horizon ($T = +\infty$) by constructing an approximate DRGBSDE and demonstrating the relevant convergences in an appropriate $\mathbb{L}^2$-space based on the previous results.

The rest of the paper is organized as follows : Section 2 outlines the essential notations, assumptions, and formulation of the DRGBSDE in this general framework. Section 3 provides an a priori estimate and uniqueness of the solution. The existence of the solution is demonstrated when $T < +\infty$ in Section 4. The main result of the paper is presented in Section 5, which deals with the existence of the solution when $T = +\infty$.

2. Preliminaries and notations

All processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ carrying a d-dimensional Brownian motion $\{W_t, t \geq 0\}$ and an independent random measure N on $\mathbb{R}^+ \times E$ where $E = \mathbb{R}^l \setminus \{0\}$, for some $l \in \mathbb{N}^*$, equipped with its Borel field $\mathcal{E}$. The filtration $\mathbb{F}$ is assumed to be complete, right continuous and quasi-left continuous, which means that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping time such that $\tau_n \nearrow \tau$ for some stopping time τ we have $\bigvee_{n\in\mathbb{N}} \mathcal{F}_{\tau_n} = \mathcal{F}_\tau$. We assume that $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. We always assume that the random measure N is an integer-valued random measure on $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{E})$ with the compensator $\upsilon(\omega; dt, de) = Q(\omega, t, de)\eta(\omega, t)dt$ where $\eta : \Omega \times \mathbb{R}_+ \to [0, \infty)$ is a $\mathcal{P}$-measurable process and Q is a kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(E, \mathcal{E})$ satisfying $\int_0^t \int_E |e|^2 Q(t, de)\eta(t)dt < \infty$ $\mathbb{P}$-a.s., where $\mathcal{P}$ design the σ-algebra on $\Omega \times \mathbb{R}^+$ generated by all $\mathcal{F}_t$-adapted left continuous processes. Given the compensator υ of the random measure N, we define the compensated random measure $\tilde{N}$ by $\tilde{N}(\omega, dt, de) = N(\omega, dt, de) - \upsilon(\omega, dt, de)$. We also set $N(\{0\}, E) = N((0, \infty), \{0\}) = \upsilon((0, \infty), \{0\}) = 0$.

The quadratic variation of a martingale M is defined by $[M]$. The notation $[M]^c$ will denotes the continuous part of the quadratic variation $[M]$. The Euclidean norm of a vector $z \in \mathbb{R}^k$ will defined by $\|z\|^2 = \sum_{i=1}^k |z_i|^2$, where $|y|$ design the absolute value of $y \in \mathbb{R}$.

For a given rcll process Y, let $Y_{t-} = \lim\limits_{s\nearrow t} Y_s$ the left limits of Y at t (by convention, we set $Y_{0-} = Y_0$). $(Y_-) = (Y_{t-})_{t\geq 0}$ the left limited process, and $\Delta Y_t = Y_t - Y_{t-}$ the jump of Y at time t. The equality $X = Y$ between any two processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ must be understood in the indistinguishably sens, meaning that $\mathbb{P}(\omega : X_t(\omega) = Y_t(\omega), \forall t \geq 0) = 1$. The same signification holds for $X \leq Y$. Finally, for a finite variation process $(\kappa_t)_{t\geq 0}$, it's total variation process over $\mathbb{R}^+$ would be denoted by $(\|\kappa\|_s)_{s\geq 0}$ and for $x \in \mathbb{R}$, we recall that $x^+ = \max(x, 0)$, $x^- = (-x)^+ = -\min(x, 0)$.

Let $(\kappa_t)_{t\geq 0}$ be a continuous one dimensional increasing $\mathcal{F}_t$-progressively measurable real valued process satisfying $\kappa_0 = 0$ and a be a non-negative $\mathcal{F}_t$-adapted process and $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{E}$ the σ-field on $\Omega \times \mathbb{R}_+ \times E$. We define the increasing processes $A_t = \int_0^t a_s^2 ds$ and $\Phi_t^{\mu,\gamma} = e^{\mu A_t + \gamma\kappa_t}$ for $t > 0$ and $\mu, \gamma > 0$. Unlike the finite horizon case, we assume the existence of a positive constant C_a such that $A_t \leq C_a < \infty$ for all $t \geq 0$. This assumption is necessary to ensure the validity of the imposed conditions on the data.
Now, we consider the following spaces :

$$\mathbb{L}_Q^2 := \left\{ \mathcal{E}\text{-measurable functions } \phi : E \to \mathbb{R}^k \ \text{ s.t } \ \|\phi\|_Q = \int_E |\phi(e)|^2 Q(t, de)\eta(t) < \infty, \ (t, \omega) \text{ a.e.} \right\}.$$

$$\mathcal{S}_{\mu,\gamma}^2 := \left\{ \mathbb{R}\text{-valued, } \mathit{rcll}, \ \mathcal{F}_t\text{-adapted processes } (Y_t)_{t\geq 0} \text{ s.t } \ \mathbb{E}\Big[\sup_{t\in[0,\infty]} \Phi_t^{\mu,\gamma} |Y_t|^2 \Big] < \infty \right\}.$$

$$\mathcal{H}_{\mu,\gamma}^2 := \left\{ \mathbb{R}^d\text{-valued, } \mathcal{P}\text{-measurable processes } (Z_t)_{t\geq 0} \text{ s.t } \ \mathbb{E}\left[\int_0^\infty \Phi_t^{\mu,\gamma} \|Z_t\|^2 dt \right] < \infty \right\}.$$

$$\mathcal{L}_{\mu,\gamma}^2 := \left\{ \tilde{\mathcal{P}}\text{-measurable processes } V : \Omega \times \mathbb{R}_+ \times E \to \mathbb{R} \text{ s.t } \ \mathbb{E}\left[\int_0^\infty \Phi_t^{\mu,\gamma} \|V_t\|_Q^2 dt \right] < \infty \right\}.$$

$$\mathcal{K}^2 := \left\{ \mathcal{F}_t\text{-predictable, } \mathit{rcll} \text{ and non-decreasing processes } (K_t)_{t\geq 0} \text{ s.t } \ K_0 = 0 \ \text{ and } \ \mathbb{E}|K_\infty|^2 < \infty \right\}.$$

$$\mathcal{M}^2_{\mu,\gamma} := \Big\{ rcll \text{ martingales } (M_t)_{t\geq 0} \text{ s.t } \left[M, W^i\right]^c_t = 0,\ 1 \leq i \leq d,\ \left[M, \tilde{N}(\cdot, \mathcal{A})\right]_t = 0,\ \forall\, \mathcal{A} \in \mathcal{E} \text{ and}$$

$$\mathbb{E}\Big[\int_0^\infty \Phi^{\mu,\gamma}_t d\,[M]_t\Big] < \infty\Big\}.$$

$$\mathfrak{L}^2_{\mu,\gamma} := \mathcal{S}^2_{\mu,\gamma} \times \mathcal{H}^2_{\mu,\gamma} \times \mathcal{L}^2_{\mu,\gamma} \times \mathcal{M}^2_{\mu,\gamma}, \quad \mathfrak{D}_{\mu,\gamma} := \mathfrak{L}^2_{\mu,\gamma} \times (\mathcal{K}^2) \text{ and } \mathfrak{D}^2_{\mu,\gamma} := \mathfrak{L}^2_{\mu,\gamma} \times (\mathcal{K}^2)^2.$$

The basic assumptions on the data $(\xi, f, g, \kappa, L, U)$.

(H1) Measurability :

- ξ is an $\mathcal{F}_\infty$-measurable random variable,
- $\forall y \in \mathbb{R}$, $z \in \mathbb{R}^d$ and $v \in \mathbb{L}^2_Q$, the coefficients $f(\cdot, \cdot, y, z, v) : \Omega \times [0, \infty[\to \mathbb{R}$ and $g(\cdot, \cdot, y) : \Omega \times [0, \infty[\to \mathbb{R}$ are $\mathcal{F}_t$-progressively measurable.

(H2) Stochastic Monotonicity of f and g in y : There exist two $\mathcal{F}_t$-progressively measurable process $\alpha : \Omega \times [0, \infty[\to \mathbb{R}$ and $\beta : \Omega \times [0, \infty[\to \mathbb{R}^-$ such that

(i) For all $y,\ y' \in \mathbb{R}$, $z \in \mathbb{R}^d$ and $v \in \mathbb{L}^2_Q$, $d\mathbb{P} \otimes dt$-a.e.,

$$(y - y')\left(f(t, y, z, v) - f(t, y', z, v)\right) \leq \alpha_t \left|y - y'\right|^2.$$

(ii) For all $y,\ y' \in \mathbb{R}$, $d\mathbb{P} \otimes d\kappa_t$-a.e.,

$$(y - y')\left(g(t, y) - g(t, y')\right) \leq \beta_t \left|y - y'\right|^2.$$

(H3) Stochastic Lipschitz condition on f in z and v : There exist two $\mathcal{F}_t$-progressively measurable process $\theta, \eta : \Omega \times [0, T] \to \mathbb{R}^{+,*}$ such that, for all $z,\ z' \in \mathbb{R}^d$ and $v,\ v' \in \mathbb{L}^2_Q$, $d\mathbb{P} \otimes dt \otimes Q(t, de)\eta(t)$-a.e.,

$$\left|f(t, y, z, v) - f(t, y, z', v')\right| \leq \theta_t \left\|z - z'\right\| + \eta_t \left\|v - v'\right\|_Q.$$

(H4) Linear growth of f and g : For some constant $\zeta > 0$ some $[1, \infty)$-valued adapted processes $\{\varphi_t, \psi_t; t \geq 0\}$ some $[0, \infty[$-valued process $(\phi_t)_{t\geq 0}$ and all $(t, y) \in [0, \infty[\times\mathbb{R}$, we have

$$|f(t, y, 0, 0)| \leq \varphi_t + \phi_t |y| \quad \text{and} \quad |g(t, y)| \leq \psi_t + \zeta |y|.$$

(H5) There exists some $\epsilon > 0$, such that $a_t^2 = |\alpha_t| + \phi_t^2 + \theta_t^2 + \eta_t^2 \geq \epsilon$.

(H6) Integrability condition :

$$\mathbb{E}\left[\Phi^{\mu,\gamma}_\infty |\xi|^2\right] + \mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s |\varphi_s|^2 ds + \mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s |\psi_s|^2 d\kappa_s < \infty, \quad \mathbb{P}\text{-a.s. where } \Phi^{\mu,\gamma}_\infty = e^{\mu A_\infty + \gamma\kappa_\infty}$$

(H7) Assumptions on the Barriers : There exist two *rcll* processes $L := (L_t)_{t\geq 0}$ and $U := (U_t)_{t\geq 0}$ s.t

(i) $\limsup_{t\to\infty} L_t \leq \xi \leq \liminf_{t\to\infty} U_t$,

(ii) $L^+ \in \mathcal{S}^2_{2\mu,2\gamma}$ and $U^- \in \mathcal{S}^2_{2\mu,2\gamma}$,

(iii) $\mathbb{P}$-a.s., $\forall t \geq 0$, $L_t < U_t$ and $L_{t^-} < U_{t^-}$.

Remark 2.1 *It is worth noting that we have the flexibility to assume* ***(H2)*** *by setting $\alpha_t \equiv 0$, and we can demonstrate that it is possible to select the process $(\beta_t)_{t\geq 0}$ such that it is strictly negative. On the other hand, from conditions* ***(H5)*** *and* ***(H6)****, we can easily deduce that $\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s \left|\frac{\varphi_s}{a_s}\right|^2 ds < \infty$, which will be used several times subsequently without specific mention.*

Definition 2.1 *Let μ, $\gamma > 0$. A solution to DRGBSDE associated with parameters $(\xi, f, g, \kappa, L, U)$ is a sextuplet of processes $\left(Y_t, Z_t, V_t, M_t, K_t^+, K_t^-\right)_{t\geq 0}$ which satisfy (1.1) and belongs to $\mathfrak{D}^2_{\mu,\gamma}$.*

Remark 2.2 *(i) The state process Y of the DRGBSDE (1.1) has two types of jumps. First type is the totally inaccessible ones which comes from it's martingale part. Namely, the two $\mathbb{F}$-martingales $\left(\int_0^t \int_E V_s(e)\tilde{N}(ds, de)\right)_{t\geq 0}$ and $\left(\int_0^t dM_s\right)_{t\geq 0}$. This is due to the fact that the filtration $\mathbb{F}$ is quasi-left continuous, which implies in that the stochastic integrals $\left(\int_0^t \int_E V_s(e)\tilde{N}(ds, de)\right)_{t\geq 0}$ and $\left(\int_0^t dM_s\right)_{t\geq 0}$ cannot jump at an $\mathbb{F}$-predictable stopping time. The second type is the predictable jumps which stem from the negative jumps of lower obstacle L. Thus, it is necessary to introduce some predictable jump processes K^d to characterize the predictable jumps of Y.*

(ii) The predictable jump processes $K^{d,\pm}$ of $K^\pm$ has the following expression :

$$\forall t > 0, \quad K_t^{d,+} = \sum_{0<s\leq t} (Y_s - L_{s-})^- \mathbb{1}_{\{\Delta L_s < 0\}} = \sum_{0<s\leq t} (Y_s - L_{s-})^- \mathbb{1}_{\{Y_{s-} = L_{s-}\}},$$

and

$$\forall t > 0, \quad K_t^{d,-} = \sum_{0<s\leq t} (Y_s - U_{s-})^+ \mathbb{1}_{\{\Delta U_s > 0\}} = \sum_{0<s\leq t} (Y_s - U_{s-})^+ \mathbb{1}_{\{Y_{s-} = U_{s-}\}}.$$

Remark 2.3 *The condition **(H7)**-(iii) on the barriers can be strengthen by the existence of a semi-martingale $(\xi_t)_{t\geq 0}$*

$$\xi_t = \xi_0 + \int_0^t \mathbb{Z}_s dW_s + \int_0^t \int_E \mathbb{V}_s(e)\tilde{N}(ds, de) + \int_0^t d\mathbb{M}_s - \mathcal{J}_t^+ + \mathcal{J}_t^-, \qquad \xi_\infty = \xi,$$

such that $L_t \leq \xi_t \leq U_t$ with $\xi_0 \in \mathbb{R}$ and $\mathcal{J}^\pm$ are two nondecreasing continuous processes such that

$$\mathbb{E}\left[\int_0^\infty \|\mathbb{Z}_s\|^2 ds + \int_0^\infty \|\mathbb{V}_s\|_\lambda^2 ds + \int_0^\infty d\left[\mathbb{M}\right]_s + |\mathcal{J}_\infty^\pm|^2\right] < \infty.$$

3. A priori estimate and uniqueness

Let now $(\xi, f, g, \kappa, L, U)$ and $(\xi', f', g', \kappa', L', U')$ be two sets of data, each satisfying the above assumptions **(H1)**-**(H7)**. Let $\left(Y_t, Z_t, V_t, M_t, K_t^+, K_t^-\right)_{t\geq 0}$ (resp. $\left(Y'_t, Z'_t, V'_t, M'_t, K_t'^+, K_t'^-\right)_{t\geq 0}$) denote a solution of the DRBSDE (1.1) with data $(\xi, f, g, \kappa, L, U)$ (resp. $(\xi', f', g', \kappa', L', U')$) in the sens of Definition 2.1. The following proposition will be useful for further applications.

Proposition 3.1 *For any $\mu > 2$ and $\gamma > 0$, there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,\infty]} \bar{\Phi}_t^{\mu,\gamma} \left|\bar{Y}_t\right|^2\right] + \mathbb{E}\int_0^\infty \bar{\Phi}_s^{\mu,\gamma} |\bar{Y}_s|^2 dA_s + \mathbb{E}\int_0^\infty \bar{\Phi}_s^{\mu,\gamma} d\left[\bar{M}\right]_s \\
&\quad + \mathbb{E}\int_0^\infty \bar{\Phi}_s^{\mu,\gamma} |\bar{Y}_s|^2 d\left\|\bar{\kappa}\right\|_s + \mathbb{E}\int_0^\infty \bar{\Phi}_s^{\mu,\gamma}\left(\left\|\bar{Z}_s\right\|^2 + \|\bar{V}_s\|_Q^2\right) ds \\
&\leq C\left\{\mathbb{E}\left[\bar{\Phi}_\infty^{\mu,\gamma}|\bar{\xi}|^2\right] + \mathbb{E}\int_0^\infty \bar{\Phi}_s^{\mu,\gamma}\left|\frac{f(s, Y', Z', V'_s) - f'(s, Y', Z', V'_s)}{a_s}\right|^2 ds\right. \\
&\qquad + \mathbb{E}\int_0^\infty \bar{\Phi}_s^{\mu,\gamma}\left|g(s, Y'_s) - g'(s, Y'_s)\right|^2 d\kappa_s + \mathbb{E}\int_0^\infty \bar{\Phi}_s^{\mu,\gamma}\left|g(s, Y'_s)\right|^2 d\left\|\bar{\kappa}\right\|_s \\
&\qquad + 2\mathbb{E}\int_0^\infty \bar{\Phi}_s^{\mu,\gamma}\left(\left(L_{s-} - L'_{s-}\right)^+ dK_s^+ + \left(L'_{s-} - L_{s-}\right)^+ dK_s'^+\right) \\
&\qquad \left. + 2\mathbb{E}\int_0^\infty \bar{\Phi}_s^{\mu,\gamma}\left(\left(U_{s-} - U'_{s-}\right)^- dK_s^- + \left(U'_{s-} - U_{s-}\right)^- dK_s'^-\right)\right\}.
\end{aligned}$$

where $\bar{\Phi}_s^{\mu,\gamma} := e^{\bar{\phi}_s^{\mu,\gamma}}$, $\bar{\phi}_s^{\mu,\gamma} := \mu A_s + \gamma\left(\left\|\bar{\kappa}\right\|_s + \kappa'_s\right)$ and $\bar{\mathcal{R}} = \mathcal{R} - \mathcal{R}'$ for $\mathcal{R} = Y, Z, V, M, \kappa, \xi$.

Proof: Let $\{\tau_k\}_{k\in\mathbb{N}}$ be the sequence of stopping times defined by

$$\tau_k = \sigma_k \wedge k, \qquad \text{where} \quad \sigma_k = \inf\{t \geq 0 : \kappa_t + \kappa'_t \geq k\}.$$

For $\mathbb{P}$-almost all $\omega \in \Omega$, since $\kappa_.(\omega) + \kappa'_.(\omega)$ has continuous paths, then $\{\sigma_k(\omega)\}_{k\in\mathbb{N}}$ has not finite accumulation point. Hence $\sigma_k(\omega) \uparrow \infty$, therefore $\{\tau_k(\omega)\}_{k\in\mathbb{N}}$ is an increasing sequence of $\mathbb{F}$-stopping such that $\tau_k \uparrow \infty$ and $\tau_k \leq k$.
By Itô's formula (see [19], Theorem 33, pp.81), we have

$$\begin{aligned}
&\bar{\Phi}^{\mu,\gamma}_{t\wedge\tau_k} \left|\bar{Y}_{t\wedge\tau_k}\right|^2 + \mu \int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left|\bar{Y}_s\right|^2 dA_s + \gamma \int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left|\bar{Y}_s\right|^2 \left(d\left\|\bar{\kappa}\right\|_s + d\kappa'_s\right) \\
&\quad + \int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left\|\bar{Z}_s\right\|^2 ds + \int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s d\left[\bar{M}\right]^c_s + \sum_{t\wedge\tau_k < s \leq \tau_k} \bar{\Phi}^{\mu,\gamma}_s (\Delta\bar{Y}_s)^2 \\
&= \bar{\Phi}^{\mu,\gamma}_{\tau_k} \left|\bar{Y}_{\tau_k}\right|^2 + 2\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \bar{Y}_s \left(f(s,Y_s,Z_s,V_s) - f'(s,Y'_s,Z'_s,V'_s)\right) ds \\
&+ 2\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \bar{Y}_s \left(g(s,Y_s)d\kappa_s - g'(s,Y'_s)d\kappa'_s\right) + 2\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \bar{Y}_{s-} \left(dK^+_s - dK'^+_s\right) \\
&\quad - 2\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \bar{Y}_{s-} \left(dK^-_s - dK'^-_s\right) - 2\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \bar{Y}_s \bar{Z}_s dW_s \\
&- 2\int_{t\wedge\tau_k}^{\tau_k} \int_E \bar{\Phi}^{\mu,\gamma}_s \bar{Y}_{s-} \bar{V}_s(e) \tilde{N}(ds,de) - 2\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \bar{Y}_{s-} d\bar{M}_s
\end{aligned} \tag{3.1}$$

Recall that since the compensator of the random measure $N(\cdot, de)$ is absolutely continuous with respect to the Lebesgue measure, $N(\cdot, de)$ does not have jumps in common with the processes $K^{d,\pm}$. Furthermore, due to the orthogonality property, we can deduce that the jumps of the martingale $(M_t)_{t\geq 0}$ cannot be expressed in terms of the jump measure $N(\cdot, de)$. Finally, we can observe from the fact that $\Delta K^+_t \Delta K^-_t = 0$ for all $t > 0$ that

$$\begin{aligned}
&\sum_{t\wedge\tau_k < s \leq \tau_k} \bar{\Phi}^{\mu,\gamma}_s (\Delta Y_s)^2 \\
&= \sum_{t\wedge\tau_k < s \leq \tau_k} \bar{\Phi}^{\mu,\gamma}_s \left\{\left(\Delta K^{d,+}_s\right)^2 + \left(\Delta K^{d,-}_s\right)^2\right\} \\
&\qquad + \int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \int_E \left|V_s(e)\right|^2 N(ds,de) + \sum_{t\wedge\tau_k < s \leq \tau_k} \bar{\Phi}^{\mu,\gamma}_s (\Delta M_s)^2
\end{aligned} \tag{3.2}$$

On the other hand, thanks to the Skorokhod condition (1.1)-(iii), and (1.1)-(ii),which gives

$$\begin{aligned}
&\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left(Y_{s-} - Y'_{s-}\right)\left(dK^+_s - dK'^{,+}_s\right) \\
&= \int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left(Y_{s-} - L_{s-}\right) dK^+_s + \int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left(L_{s-} - Y_{s-}\right) dK'^+_s \\
&+ \int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left(L'_{s-} - Y'_{s-}\right) dK^+_s + \int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left(Y'_{s-} - L'_{s-}\right) dK'^+_s \\
&\quad + \int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left(L_{s-} - L'_{s-}\right)\left(dK^+_s - dK'^+_s\right) \\
&\leq \int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left(L_{s-} - L'_{s-}\right)\left(dK^+_s - dK'^+_s\right).
\end{aligned} \tag{3.3}$$

Similarly, we may show that

$$\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left(Y_{s-} - Y'_{s-}\right)\left(dK^-_s - dK'^-_s\right) \geq \int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left(U_{s-} - U'_{s-}\right)\left(dK^-_s - dK'^-_s\right), \ \mathbb{P}\text{-a.s.} \tag{3.4}$$

Using assumptions **(H2)-(H4)**, the basic inequality $2ab \leq \sigma a^2 + \frac{1}{\sigma}b^2$, $\forall \sigma > 0$, we obtain

$$\begin{aligned}
&2\bar{Y}_s \left(f\left(s, Y_s, Z_s, V_s\right) - f'\left(s, Y'_s, Z'_s, V'_s\right)\right) ds \\
&\leq (\sigma+2)\left|\bar{Y}_s\right|^2 dA_s + \frac{1}{\sigma}\left|\frac{f\left(s, Y'_s, Z'_s, V'_s\right) - f'\left(s, Y'_s, Z'_s, V'_s\right)}{a_s}\right|^2 ds + \frac{1}{2}\left(\left\|\bar{Z}_s\right\|^2 + \left\|\bar{V}_s\right\|_Q^2\right) ds
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&2\bar{Y}_s \left(g\left(s, Y_s\right) d\kappa_s - g'\left(s, Y'_s\right) d\kappa'_s\right) \\
&\leq \sigma' \left|\bar{Y}_s\right|^2 \left(d\left\|\bar{\kappa}\right\|_s + d\kappa'_s\right) + \frac{1}{\sigma'}\left|g\left(s, Y'_s\right)\right|^2 d\left\|\bar{\kappa}\right\|_s + \frac{1}{\sigma'}\left|g\left(s, Y'_s\right) - g'\left(s, Y'_s\right)\right|^2 d\kappa'_s
\end{aligned} \tag{3.6}$$

By plugging (3.2), (3.3), (3.4), (3.5) and (3.6) into (3.1), and then taking the expectation on both sides while considering the martingale property satisfied by the stochastic integrals appearing on its right-hand side, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\bar{\Phi}^{\mu,\gamma}_{t\wedge\tau_k}|\bar{Y}_{t\wedge\tau_k}|^2\right] + (\mu-(2+\sigma))\mathbb{E}\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s|\bar{Y}_s|^2 dA_s + (\gamma-\sigma')\mathbb{E}\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s|\bar{Y}_s|^2\left(d\left\|\bar{\kappa}\right\|_s + d\kappa'_s\right) \\
&\quad + \frac{1}{2}\mathbb{E}\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s\left\{\left\|\bar{Z}_s\right\|^2 + \|\bar{V}_s\|_Q^2\right\} ds + \mathbb{E}\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s d\left[\bar{M}\right]_s \\
&\leq \mathbb{E}\left[\bar{\Phi}^{\mu,\gamma}_{\tau_k}|\bar{Y}_{\tau_k}|^2\right] + \frac{1}{\sigma}\mathbb{E}\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s\left|\frac{f\left(s, Y'_s, Z'_s, V'_s\right) - f'\left(s, Y'_s, Z'_s, V'_s\right)}{a_s}\right|^2 ds \\
&\quad + \frac{1}{\sigma'}\mathbb{E}\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s\left|g\left(s, Y'_s\right) - g'\left(s, Y'_s\right)\right|^2 d\kappa'_s + \frac{1}{\sigma'}\mathbb{E}\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s\left|g\left(s, Y'_s\right)\right|^2 d\left\|\bar{\kappa}\right\|_s \\
&\quad + 2\mathbb{E}\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s\left(\left(L_{s-} - L'_{s-}\right)^+ dK^+_s + \left(L'_{s-} - L_{s-}\right)^+ dK'^+_s\right) \\
&\quad + 2\mathbb{E}\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s\left(\left(U_{s-} - U'_{s-}\right)^- dK^-_s + \left(U'_{s-} - U_{s-}\right)^- dK'^-_s\right).
\end{aligned} \tag{3.7}$$

Next, we can use the monotonic limit theorem $\mathbb{P}$-almost surely for the integrals appearing in both the left-hand and right-hand sides of (3.7). Then, by applying Fatou's Lemma with $\bar{\Phi}^{\mu,\gamma}_{\tau_k}|\bar{Y}_{\tau_k}| \to \bar{\Phi}^{\mu,\gamma}_{\infty}|\bar{\xi}|$, $\mathbb{P}$-a.s. as $k\to\infty$, we can deduce the existence of a constant $C_1 > 0$ for $\mu > \sigma+2$ and $\gamma > \sigma'$ such that for all $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}\left[\bar{\Phi}^{\mu,\gamma}_t|\bar{Y}_t|^2\right] + \mathbb{E}\int_t^\infty \bar{\Phi}^{\mu,\gamma}_s|\bar{Y}_s|^2 dA_s + \mathbb{E}\int_t^\infty \bar{\Phi}^{\mu,\gamma}_s|\bar{Y}_s|^2\left(d\left\|\bar{\kappa}\right\|_s + d\kappa'_s\right) \\
&\quad + \mathbb{E}\int_t^\infty \bar{\Phi}^{\lambda,\mu,\gamma}_s\left\{\left\|\bar{Z}_s\right\|^2 + \|\bar{V}_s\|_Q^2\right\} ds + \mathbb{E}\int_t^\infty \bar{\Phi}^{\mu,\gamma}_s d\left[\bar{M}\right]_s \\
&\leq C_1\left\{\mathbb{E}\left[\bar{\Phi}^{\mu,\gamma}_\infty|\bar{\xi}|^2\right] + \mathbb{E}\int_t^\infty \bar{\Phi}^{\mu,\gamma}_s\left|\frac{f\left(s, Y'_s, Z'_s, V'_s\right) - f'\left(s, Y'_s, Z'_s, V'_s\right)}{a_s}\right|^2 ds\right. \\
&\quad + \mathbb{E}\int_t^\infty \bar{\Phi}^{\mu,\gamma}_s\left|g\left(s, Y'_s\right) - g'\left(s, Y'_s\right)\right|^2 d\kappa'_s + \mathbb{E}\int_t^\infty \bar{\Phi}^{\mu,\gamma}_s\left|g\left(s, Y'_s\right)\right|^2 d\left\|\bar{\kappa}\right\|_s \\
&\quad + \mathbb{E}\int_t^\infty \bar{\Phi}^{\mu,\gamma}_s\left(\left(L_{s-} - L'_{s-}\right)^+ dK^+_s + \left(L'_{s-} - L_{s-}\right)^+ dK'^+_s\right) \\
&\quad \left. + \mathbb{E}\int_t^\infty \bar{\Phi}^{\mu,\gamma}_s\left(\left(U_{s-} - U'_{s-}\right)^- dK^-_s + \left(U'_{s-} - U_{s-}\right)^- dK'^-_s\right)\right\}.
\end{aligned} \tag{3.8}$$

On the other hand, we have for all $t \geq 0$

$$\begin{aligned}
&\bar{\Phi}^{\mu,\gamma}_{t\wedge\tau_k} \left|\bar{Y}_{t\wedge\tau_k}\right|^2 \\
&\leq \bar{\Phi}^{\mu,\gamma}_{\tau_k}|\bar{Y}_{\tau_k}|^2 + \frac{1}{\sigma}\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left|\frac{f\left(s,Y'_s,Z'_s,V'_s\right) - f'\left(s,Y'_s,Z'_s,V'_s\right)}{a_s}\right|^2 ds \\
&+ \frac{1}{\sigma'}\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left|g\left(s,Y'_s\right) - g'\left(s,Y'_s\right)\right|^2 d\kappa'_s + \frac{1}{\sigma'}\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left|g\left(s,Y'_s\right)\right|^2 d\left\|\bar{\kappa}\right\|_s \\
&+ 2\left|\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \bar{Y}_s \bar{Z}_s dW_s\right| + 2\left|\int_{t\wedge\tau_k}^{\tau_k}\int_E \bar{\Phi}^{\mu,\gamma}_s \bar{Y}_{s-}\bar{V}_s(e)\tilde{N}(ds,de)\right| + 2\left|\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \bar{Y}_{s-} d\bar{M}_s\right| \\
&+ 2\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left(\left(L_{s-} - L'_{s-}\right)^+ dK^+_s + \left(L'_{s-} - L_{s-}\right)^+ dK'^+_s\right) \\
&+ 2\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \left(\left(U_{s-} - U'_{s-}\right)^- dK^-_s + \left(U'_{s-} - U_{s-}\right)^- dK'^-_s\right).
\end{aligned} \tag{3.9}$$

We apply the Burkholder-Davis-Gundy's inequality (see e.g. [19] Theorem 48 pp.195) to obtain the existence of a constant $c > 0$ such that

$$\begin{aligned}
&2\mathbb{E}\left[\sup_{t\geq 0}\left|\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s Y_s Z_s dW_s\right|\right] \leq 2c\mathbb{E}\left[\left(\int_0^{\tau_k} \bar{\Phi}^{2\mu,2\gamma}_s \left|Y_s\right|^2 \left\|Z_s\right\|^2 ds\right)^{\frac{1}{2}}\right] \\
&\leq \frac{1}{6}\mathbb{E}\left[\sup_{0\leq t\leq\tau_k} \bar{\Phi}^{\mu,\gamma}_t \left|Y_t\right|^2\right] + 6c^2\mathbb{E}\left[\int_0^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \|Z_s\|^2 ds\right].
\end{aligned} \tag{3.10}$$

By the same arguments, we have

$$\begin{aligned}
&2\mathbb{E}\left[\sup_{t\geq 0}\left|\int_{t\wedge\tau_k}^{\tau_k}\int_E \bar{\Phi}^{\mu,\gamma}_s \bar{Y}_{s-}\bar{V}_s(e)\tilde{N}(ds,de)\right|\right] \\
&\leq \frac{1}{6}\mathbb{E}\left[\sup_{0\leq t\leq\tau_k} \bar{\Phi}^{\mu,\gamma}_t \left|\bar{Y}_t\right|^2\right] + 6c^2\mathbb{E}\left[\int_0^{\tau_k}\int_E \bar{\Phi}^{\mu,\gamma}_s \left|\bar{V}_s(e)\right|^2 N(ds,de)\right] \\
&= \frac{1}{6}\mathbb{E}\left[\sup_{0\leq t\leq\tau_k} \bar{\Phi}^{\mu,\gamma}_t \left|\bar{Y}_t\right|^2\right] + 6c^2\mathbb{E}\left[\int_0^{\tau_k}\int_E \bar{\Phi}^{\mu,\gamma}_s \left|\bar{V}_s(e)\right|^2 Q(s,de)\eta(s)ds\right]
\end{aligned} \tag{3.11}$$

and

$$\mathbb{E}\left[\sup_{t\geq 0}\left|\int_{t\wedge\tau_k}^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s \bar{Y}_{s-} d\bar{M}_s\right|\right] \leq \frac{1}{6}\mathbb{E}\left[\sup_{0\leq t\leq\tau_k} \bar{\Phi}^{\mu,\gamma}_t \left|\bar{Y}_t\right|^2\right] + 6c^2\mathbb{E}\left[\int_0^{\tau_k} \bar{\Phi}^{\mu,\gamma}_s d[\bar{M}]_s\right]. \tag{3.12}$$

Now, coming back to (3.9) and using estimates (3.8) with $t = 0$, as well as (3.10), (3.11) and (3.12), we can apply Fatou's Lemma by letting $k \to \infty$ and deduce that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,\infty]} \bar{\Phi}^{\mu,\gamma}_t \left|\bar{Y}_t\right|^2\right] \\
&\leq C_2\left\{\mathbb{E}\left[\bar{\Phi}^{\mu,\gamma}_\infty|\bar{\xi}|^2\right] + \mathbb{E}\int_0^\infty \bar{\Phi}^{\mu,\gamma}_s \left|\frac{f\left(s,Y'_s,Z'_s,V'_s\right) - f'\left(s,Y'_s,Z'_s,V'_s\right)}{a_s}\right|^2 ds\right. \\
&\quad +\mathbb{E}\int_0^\infty \bar{\Phi}^{\lambda,\mu,\gamma}_s \left|g\left(s,Y'_s\right) - g'\left(s,Y'_s\right)\right|^2 d\kappa_s + \mathbb{E}\int_t^\infty \bar{\Phi}^{\mu,\gamma}_s \left|g\left(s,Y'_s\right)\right|^2 d\left\|\bar{\kappa}\right\|_s \\
&\quad +\mathbb{E}\int_0^\infty \bar{\Phi}^{\mu,\gamma}_s \left(\left(L_{s-} - L'_{s-}\right)^+ dK^+_s + \left(L'_{s-} - L_{s-}\right)^+ dK'^+_s\right) \\
&\quad \left.+\mathbb{E}\int_0^\infty \bar{\Phi}^{\mu,\gamma}_s \left(\left(U_{s-} - U'_{s-}\right)^- dK^-_s + \left(U'_{s-} - U_{s-}\right)^- dK'^-_s\right)\right\}.
\end{aligned}$$

Which completes the proof. $\square$

Corollary 3.1 *Assume **(H1)**–**(H7)** hold. Then there exists at most one solution* (Y, Z, V, K^+, K^-, M) *of the DRGBSDE* (1.1) *associated with* $(\xi, f, g, \kappa, L, U)$.

Proof: Let $\left(Y_t, Z_t, V_t, K_t^+, K_t^-, M_t\right)_{t\leq T}$ and $\left(Y'_t, Z'_t, V'_t, K'^+_t, K'^-_t, M'_t\right)_{t\leq T}$ be two solutions of the DRGBSDE (1.1) associated with data $(\xi, f, g, \kappa, L, U)$. Then, using the result given by Proposition 3.1, we deduce that $\|Y-Y'\|^2_{\mathcal{S}^2_{\mu,\gamma}} = 0$, $\|Z-Z'\|^2_{\mathcal{H}^2_{\mu,\gamma}} = 0$, $\|V-V'\|^2_{\mathcal{L}^2_{\mu,\gamma}} = 0$ and $\|M-M'\|^2_{\mathcal{M}^2_{\mu,\gamma}} = 0$ then $Y_t = Y'_t$, $\forall t\in[0,T]$, $\mathbb{P}$-a.s. and $(Y,Z,V,M) = (Y',Z',V',M')$. On the one hand, the expression of $K^{d,\pm}$ and $K'^{d,\pm}$ by means of Y and Y', respectively, implies that $K^{d,\pm} = K'^{d,\pm}$. Furthermore, since the barriers L and U are completely separated, we may easily deduce that from $(U_s - L_s)\left(dK^{c,\pm}_s - dK'^{c,\pm}_s\right) = 0$ that $K^{c,\pm} = K'^{c,\pm}$. Henceforth, $\left(Y_t, Z_t, V_t, M_t, K^+_t, K^-_t\right) = \left(Y'_t, Z'_t, V'_t, M'_t, K'^+_t, K'^-_t\right)$, and this completes the proof of uniqueness. □

Another useful result, that will be needed in the proof of the existence is given in the following Lemma.

Lemma 3.1 *Suppose that* (Y, Z, V, M, K^+, K^-) *is a solution of* (1.1) *and* L *is a quasimartingale with canonical decomposition*

$$L_t = L_0 + R_t + B_t, \qquad t\geq 0,$$

where R *is a uniformly integrable martingale and* B *is a predictable process of integrable variation with Jordan decomposition* $B = B^+ - B^-$ *(see Theorem 15, pp.118, [19]). Then,*

$$dK^+_t \leq \mathbb{1}_{\{Y_t = L_t\}}\left(f^-(t, Y_t, Z_t, V_t)dt + g^-(t, Y_t)d\kappa_t + dB^-_t\right) \tag{3.13}$$

and

$$\begin{aligned}\mathbb{E}|K^+_\infty|^2 \leq \mathbb{E}\Bigg[&\frac{12}{\mu\epsilon}\int_0^\infty e^{\mu A_s}|Y_s|^2 dA_s + \frac{6\zeta^2}{\gamma}\int_0^\infty e^{\gamma\kappa_s}|Y_s|^2 d\kappa_s + \frac{12}{\mu}\int_0^\infty e^{\mu A_s}\left\{\|Z_s\|^2 + \|V_s\|^2_\lambda\right\}ds\\ &+\frac{12}{\mu}\int_0^\infty e^{\mu A_s}\left|\frac{\varphi_s}{a_s}\right|^2 ds + \frac{6}{\gamma}\int_0^\infty e^{\gamma\kappa_s}|\psi_s|^2 d\kappa_s + 3\left|(B_\infty)^-\right|^2\Bigg].\end{aligned} \tag{3.14}$$

Proof: From (1.1)-(i) we have

$$d(Y_t - L_t) = -f(t, Y_t, Z_t, V_t)dt - g(t, Y_t)d\kappa_t - (dK^+_t - dK^-_t) + Z_t dW_t + \int_E V_t(e)\tilde{N}(dt, de) + dM_t - dB_t - dR_t. \tag{3.15}$$

Now, applying Theorem 68 (pp. 213) in [19] to the convex function $(Y-L)^+$, we get by denoting $\mathbf{L}^0$ the local time of $Y - L$ at 0,

$$\begin{aligned}&d(Y_t - L_t)^+\\ &= \mathbb{1}_{\{Y_{t^-} > L_{t^-}\}}\Big[-f(t, Y_t, Z_t, V_t)dt - g(t, Y_t)d\kappa_t - (dK^+_t - dK^-_t) + Z_t dW_t +\\ &\quad + \int_E V_t(e)\tilde{N}(dt, de) + dM_t - dB_t - dR_t\Big] + \mathbb{1}_{\{Y_{t^-} > L_{t^-}\}}(Y_t - L_t)^- + \mathbb{1}_{\{Y_{t^-} \leq L_{t^-}\}}(Y_t - L_t)^+ + \frac{1}{2}d\mathbf{L}^0_t.\end{aligned} \tag{3.16}$$

By (1.1)-(ii), (3.15) and (3.16), we have

$$\begin{aligned}&\mathbb{1}_{\{Y_{t^-} = L_{t^-}\}}\left[Z_t dW_t + \int_E V_t(e)\tilde{N}(dt, de) + dM_t - dR_t\right]\\ &\qquad = \mathbb{1}_{\{Y_t = L_t\}}\left(f^+(t, Y_t, Z_t, V_t)dt + g^+(t, Y_t)d\kappa_t + dB^+_t\right) + \frac{1}{2}d\mathbf{L}^0_t + \mathbb{1}_{\{Y_{t^-} = L_{t^-}\}}\Delta(Y - L)_t\\ &\qquad + dK^+_t - \mathbb{1}_{\{Y_t = L_t\}}\left(f^-(t, Y_t, Z_t, V_t)dt + g^-(t, Y_t)d\kappa_t + dB^-_t + dK^-_t\right).\end{aligned} \tag{3.17}$$

Note that the second line of (3.17) is an increasing integrable process, henceforth it's compensator exists, we denote it by S and its compensatrix by $\tilde{S}$ which is a martingale. Then, (3.17) becomes

$$\begin{aligned} &\mathbb{1}_{\{Y_{t^-}=L_{t^-}\}}\left[Z_t dW_t + \int_E V_t(e)\tilde{N}(dt,de) + dM_t - dR_t\right] - d\tilde{S}_t \\ &\qquad = dS_t - \mathbb{1}_{\{Y_t=L_t\}}\left(f^-(t,Y_t,Z_t,V_t)dt + g^-(t,Y_t)d\kappa_t + dB_t^- + dK_t^-\right) + dK_t^+. \end{aligned} \tag{3.18}$$

Note that all terms in the second line in (3.18) are predictable, then the first line in (3.18) contains a predictable martingale with integrable variation. Using Corollary 3.16 (pp.32) in [12], we deduce that this latter is constant and equal to zero. Hence, (3.13) is obtained by the mutual singularity of K^+ and K^-. Moreover, to prove (3.14) it is enough to remark that, by assumptions **(H3)-(H6)** and Hölder inequality, we have

$$\begin{aligned} &\mathbb{E}\left(\int_0^\infty f(s,Y_s,Z_s,V_s)ds\right)^2 \\ &\leq \frac{4}{\mu\epsilon}\mathbb{E}\int_0^\infty e^{\mu A_s}|Y_s|^2 dA_s + \frac{4}{\mu}\mathbb{E}\int_0^\infty e^{\mu A_s}\left\{\|Z_s\|^2 + \|V_s\|_Q^2\right\}ds + \frac{4}{\mu}\mathbb{E}\int_0^\infty e^{\mu A_s}\left|\frac{\varphi_s}{a_s}\right|^2 ds. \end{aligned} \tag{3.19}$$

Similarly,

$$\mathbb{E}\left(\int_0^\infty g(s,Y_s)d\kappa_s\right)^2 \leq \frac{2\zeta^2}{\gamma}\mathbb{E}\int_0^\infty e^{\gamma\kappa_s}|Y_s|^2 d\kappa_s + \frac{2}{\gamma}\mathbb{E}\int_0^\infty e^{\gamma\kappa_s}|\psi_s|^2 d\kappa_s. \tag{3.20}$$

By substituting estimations (3.19) and (3.20) into equation (3.13), we arrive at the desired results (3.14). □

We can now demonstrate the subsequent a priori estimate.

Proposition 3.2 *Let (Y,Z,V,M,K^+,K^-) solution of (1.1) and assume that **(H1)-(H7)** hold and L is a quasimartingale. Then, there exists a constant $C>0$ such that*

$$\begin{aligned} &\mathbb{E}\Bigg[\sup_{t\in[0,\infty]}\Phi_t^{\mu,\gamma}|Y_t|^2 + \int_0^\infty \Phi_s^{\mu,\gamma}|Y_s|^2 dA_s + \int_0^\infty \Phi_s^{\mu,\gamma}|Y_s|^2 d\kappa_s \\ &\qquad + \int_0^\infty \Phi_s^{\mu,\gamma}\left\{\|Z_s\|^2 + \|V_s\|_Q^2\right\}ds + \int_0^\infty \Phi_s^{\mu,\gamma}d\left[M\right]_s + |K_\infty^+|^2 + |K_\infty^-|^2\Bigg] \\ &\leq C\mathbb{E}\Bigg[\Phi_\infty^{\mu,\gamma}|\xi|^2 + \int_0^\infty \Phi_s^{\mu,\gamma}\left|\frac{\varphi_s}{a_s}\right|^2 ds + \int_0^\infty \Phi_s^{\mu,\gamma}\left|\psi_s\right|^2 d\kappa_s \\ &\qquad + \sup_{t\in[0,\infty]}\Phi_t^{2\mu,2\gamma}\left|(L_t)^+\right|^2 + \sup_{t\in[0,\infty]}\Phi_t^{2\mu,2\gamma}\left|(U_t)^-\right|^2 + \left|(B_\infty)^-\right|^2\Bigg], \end{aligned}$$

Proof: By applying Proposition 3.1 and utilizing the fact that $(0,0,0,0,0,0)$ is the unique solution to the DRGBSDE (1.1) associated with $(0,0,0,0,0,0)$, as well as the linear growth assumption on f and g (Assumption **(H4)**), we can obtain, for all $\rho>0$,

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,\infty]}\Phi_t^{\mu,\gamma}|Y_t|^2\right] + \mathbb{E}\int_0^\infty \Phi_s^{\mu,\gamma}|Y_s|^2 dA_s + \mathbb{E}\int_0^\infty \Phi_s^{\mu,\gamma}|Y_s|^2 d\kappa_s \\ &\qquad + \mathbb{E}\int_0^\infty \Phi_s^{\mu,\gamma}\left\{\|Z_s\|^2 + \|V_s\|_Q^2\right\}ds + \mathbb{E}\int_0^\infty \Phi_s^{\mu,\gamma}d\left[M\right]_s \\ &\leq C\left\{\mathbb{E}\left[\Phi_\infty^{\mu,\gamma}|\xi|^2\right] + \mathbb{E}\int_0^\infty \bar{\Phi}_s^{\mu,\gamma}\left|\frac{\varphi_s}{a_s}\right|^2 ds + \mathbb{E}\int_0^\infty \Phi_s^{\mu,\gamma}\left|\psi_s\right|^2 d\kappa_s\right\} \\ &\quad + \rho C^2\mathbb{E}\sup_{t\in[0,\infty]}\left[\Phi_t^{2\mu,2\gamma}|(L_t)^+|^2\right] + \rho C^2\mathbb{E}\sup_{t\in[0,\infty]}\left[\Phi_t^{2\mu,2\gamma}|(U_t)^-|^2\right] + \frac{1}{\rho}\mathbb{E}|K_\infty^+|^2 + \frac{1}{\rho}\mathbb{E}|K_\infty^-|^2. \end{aligned} \tag{3.21}$$

Nevertheless, utilizing equations (1.1)-(i), (3.19), and (3.20), we obtain

$$
\begin{aligned}
\mathbb{E}|K^+_\infty - K^-_\infty|^2 &\leq 7\mathbb{E}|Y_0|^2 + 7\mathbb{E}|\xi|^2 + 7\mathbb{E}\left(\int_0^\infty f(s,Y_s,Z_s,V_s)ds\right)^2 + 7\mathbb{E}\left(\int_0^\infty g(s,Y_s)d\kappa_s\right)^2 \\
&\quad + 7\mathbb{E}\left|\int_0^\infty Z_s dW_s\right|^2 + 7\mathbb{E}\left|\int_0^\infty \int_E V_s(e)\tilde{N}(ds,de)\right|^2 + 7\mathbb{E}\left|\int_0^\infty dM_s\right|^2 \\
&\leq 7\mathbb{E}\left[|Y_0|^2\right] + 7\mathbb{E}\left[\Phi^{\mu,\gamma}_\infty|\xi|^2\right] + \frac{28}{\mu\epsilon}\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s|Y_s|^2 dA_s + \frac{14\zeta^2}{\gamma}\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s|Y_s|^2 d\kappa_s \\
&\quad + \left(\frac{28}{\mu}+7\right)\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s\left\{\|Z_s\|^2 + \|V_s\|^2_Q\right\}ds + 7\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s d\left[M\right]_s \\
&\quad + \frac{28}{\mu}\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s\left|\frac{\varphi_s}{a_s}\right|^2 ds + \frac{14}{\gamma}\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s|\psi_s|^2 d\kappa_s.
\end{aligned}
$$

Now, combining this and (3.14), we get

$$
\begin{aligned}
&\mathbb{E}\left[|K^+_\infty|^2 + |K^-_\infty|^2\right] \\
&\leq 2\mathbb{E}\left[|K^+_\infty - K^-_\infty|^2\right] + 3\mathbb{E}\left[|K^+_\infty|^2\right] \\
&\leq 14\mathbb{E}\left[\Phi^{\mu,\gamma}_\infty|\xi|^2\right] + 14\mathbb{E}\left[\sup_{t\in[0,\infty]}\Phi^{\mu,\gamma}_t|Y_t|^2\right] + \frac{92}{\mu\epsilon}\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s|Y_s|^2 dA_s + \frac{46\zeta^2}{\gamma}\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s|Y_s|^2 d\kappa_s \\
&\quad + 14\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s d\left[M\right]_s + \left(\frac{92}{\mu}+14\right)\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s\left\{\|Z_s\|^2 + \|V_s\|^2_Q\right\}ds \\
&\quad + \frac{92}{\mu}\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s\left|\frac{\varphi_s}{a_s}\right|^2 ds + \frac{46}{\gamma}\mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s|\psi_s|^2 d\kappa_s + 9\mathbb{E}|(B_\infty)^-|^2.
\end{aligned}
$$

By selecting $\rho > \max\left(14; \dfrac{92}{\mu\epsilon}; \dfrac{46\zeta^2}{\gamma}; \dfrac{92}{\mu}+14; \dfrac{46}{\gamma}\right)$ and then substituting it into (3.21), we can derive

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,\infty]}\Phi^{\mu,\gamma}_t|Y_t|^2\right] + \mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s|Y_s|^2 dA_s + \mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s|Y_s|^2 d\kappa_s \\
&\qquad + \mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s\left\{\|Z_s\|^2 + \|V_s\|^2_Q\right\}ds + \mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s d\left[M\right]_s \\
&\leq C_3\left\{\mathbb{E}\left[\Phi^{\mu,\gamma}_\infty|\xi|^2\right] + \mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s\left|\frac{\varphi_s}{a_s}\right|^2 ds + \mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s\left|\psi_s\right|^2 d\kappa_s\right. \\
&\qquad \left. +\mathbb{E}\left[\sup_{t\in[0,\infty]}\Phi^{2\mu,2\gamma}_t|(L_t)^+|^2\right] + \mathbb{E}\left[\sup_{t\in[0,\infty]}\Phi^{2\mu,2\gamma}_t|(U_t)^-|^2\right] + \left|(B_\infty)^-\right|^2\right\}.
\end{aligned} \tag{3.22}
$$

Afterward, we possess

$$
\begin{aligned}
\mathbb{E}\left[|K^+_\infty|^2 + |K^-_\infty|^2\right] &\leq C_4\left\{\mathbb{E}\left[\Phi^{\mu,\gamma}_\infty|\xi|^2\right] + \mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s\left|\frac{\varphi_s}{a_s}\right|^2 ds + \mathbb{E}\int_0^\infty \Phi^{\mu,\gamma}_s|\psi_s|^2 d\kappa_s\right. \\
&\qquad \left. +\mathbb{E}\left[\sup_{0\leq t\leq\infty}\Phi^{2\mu,2\gamma}_t|(L_t)^+|^2\right] + \mathbb{E}\left[\sup_{0\leq t\leq\infty}\Phi^{2\mu,2\gamma}_t|(U_t)^-|^2\right] + \mathbb{E}\left[\left|(B_\infty)^-\right|^2\right]\right\}.
\end{aligned} \tag{3.23}
$$

Which completes the proof. □

4. Existence and Uniqueness for the DRGBSDE on a fixed time interval

In this section, we suppose that $T > 0$ be a fixed time horizon.

4.1. The case when the coefficient f is independent of (z, v)

First let us assume that $\mathbb{P}$-a.s : $f(\omega, t, y, z, v) = h(\omega, t, y)$ for any t, y, z and v. Then we have the following Lemma

Lemma 4.1 *If we assume that h and g are Lipschitz on y, then the DRGBSDE (1.1) associated with $(\xi, h, g, \kappa, L, U)$ has a unique solution $(Y, Z, V, M, K^{\pm})$.*

Now, let us consider the special case of DRGBSDE expressed as follows:

$$\begin{cases} \text{(i) } Y_t = \xi + \displaystyle\int_t^T h(s, Y_s)ds + \int_t^T g(s, Y_s)d\kappa_s + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dW_s \\ \qquad\qquad - \displaystyle\int_t^T \int_E V_s(e)\tilde{N}(ds, de) - \int_t^T dM_s, \quad t \in [0, T]. \\ \text{(ii) } L_t \leq Y_t \leq U_t, \ \forall t \leq T \quad \text{ and } \quad \displaystyle\int_0^T (Y_{t^-} - L_{t^-})dK_t^+ = \int_0^T (U_{t^-} - Y_{t^-})dK_t^- = 0. \end{cases} \tag{4.1}$$

The main objective of this section is to establish the existence of a solution to the DRGBSDE associated with $(\xi, h, g, \kappa, L, U)$. Specifically,

Proposition 4.1 *Suppose that **(H1)-(H7)** holds. Then the DRGBSDE (4.1) has a unique solution $(Y, Z, V, M, K^{\pm})$ in $\mathfrak{D}^2_{\mu,\gamma}$.*

Proof:

Step 1 : Yosida approximation. Since $y \to -h(t, y) : \mathbb{R} \to \mathbb{R}$ and $y \to -g(t, y) : \mathbb{R} \to \mathbb{R}$ are monotone continuous functions, it follows that for every $(\omega, t, y) \in \Omega \times [0, T] \times \mathbb{R}$ and $\delta > 0$ there exists a unique $h_\delta = h_\delta(\omega, t, y) \in \mathbb{R}$ and $g_\delta = g_\delta(\omega, t, y) \in \mathbb{R}$ such that

$$h(\omega, t, y + \delta h_\delta) = h_\delta, \qquad \text{and} \qquad g(\omega, t, y + \delta g_\delta) = g_\delta. \tag{4.2}$$

From [17] (Proposition 6.7, Annex B), the functions $h_\delta(., ., y) : \Omega \times [0, T] \to \mathbb{R}$ and $g_\delta(., ., y) : \Omega \times [0, T] \to \mathbb{R}$ are $\mathcal{F}_t$- progressively measurable stochastic processes and $\forall \delta > 0, \forall t \in [0, T], \forall y, y' \in \mathbb{R}$, satisfy a.s.

$$\begin{cases} \big(h_\delta(t, y) - h_\delta(t, y')\big)(y - y') \leq 0, \\ |h_\delta(t, y) - h_\delta(t, y')| \leq \dfrac{2}{\delta}|y - y'|, \\ |h_\delta(t, y)| \leq |h(t, y)|. \end{cases} \quad \text{and} \quad \begin{cases} \big(g_\delta(t, y) - g_\delta(t, y')\big)(y - y') \leq 0, \\ |g_\delta(t, y) - g_\delta(t, y')| \leq \dfrac{2}{\delta}|y - y'|, \\ |g_\delta(t, y)| \leq |g(t, y)|. \end{cases} \tag{4.3}$$

Moreover, coefficients h_δ and g_δ satisfies

$$\begin{cases} (y - y')\left(h_\delta(t, y) - h_{\delta'}(t, y')\right) \leq (\delta + \delta')\left|h_\delta(t, y)\right|\left|h_{\delta'}(t, y')\right| \\ (y - y')\left(g_\delta(t, y) - g_{\delta'}(t, y')\right) \leq (\delta + \delta')\left|g_\delta(t, y)\right|\left|g_{\delta'}(t, y')\right| \end{cases} \tag{4.4}$$

Step 2: Approximating equation. Let $\delta \in]0, 1]$. Since $y \mapsto h_\delta(t, y)$ and $y \mapsto g_\delta(t, y)$ are Lipschitz functions with the same Lipschitz constant $\frac{2}{\delta}$, we infer by Lemma 4.1 that the approximating equation

$$\begin{cases} (i) \ Y_t^\delta = \xi + \displaystyle\int_t^T h_\delta(s, Y_s^\delta)ds + \int_t^T g_\delta(s, Y_s^\delta)d\kappa_s + \left(K_T^{\delta,+} - K_t^{\delta,+}\right) - \left(K_T^{\delta,-} - K_t^{\delta,-}\right) - \int_t^T Z_s^\delta dW_s \\ \qquad\qquad - \displaystyle\int_t^T \int_E V_s^\delta(e)\tilde{N}(ds, de) - \int_t^T dM_s^\delta, \ 0 \leq t \leq T. \\ (ii) \ L_t \leq Y_t^\delta \leq U_t, \ t \leq T \quad \text{ and } \quad \displaystyle\int_0^T (Y_{t^-}^\delta - L_{t^-})dK_t^{\delta,+} = \int_0^T (U_{t^-} - Y_{t^-}^\delta)dK_t^{\delta,-} = 0 \end{cases} \tag{4.5}$$

has a unique solution $(Y^\delta, Z^\delta, V^\delta, M^\delta, K^{\delta,\pm}) \in \mathfrak{D}^2_{\mu,\gamma}$.

Step 3: Boundedness of the family $\left\{Y^\delta, Z^\delta, V^\delta, M^\delta\right\}_{\delta\in]0,1]}$.

Let $\left\{Y^\delta, Z^\delta, V^\delta, M^\delta, K^{\delta\pm}\right\}_{\delta\in]0,1]}$ be a family of solutions to the DRGBSDE (4.5) associated with $(\xi, h_\delta, g_\delta, \kappa, L, U)$. Based on the properties satisfied by the Yosida approximating drivers $\{h_\delta, g_\delta\}_{\delta\in]0,1]}$ given in (4.3), we can conclude from Proposition 3.2 that $\left\{Y^\delta, Z^\delta, V^\delta, M^\delta, K^{\delta\pm}\right\}_{\delta\in]0,1]}$ satisfies

$$\begin{aligned}
&\mathbb{E}\Bigg[\sup_{t\in[0,T]}\Phi_t^{\mu,\gamma}|Y_t^\delta|^2+\int_0^T\Phi_s^{\mu,\gamma}|Y_s|^2dA_s+\int_0^T\Phi_s^{\mu,\gamma}|Y_s^\delta|^2d\kappa_s\\
&\qquad+\int_0^T\Phi_s^{\mu,\gamma}\left\{\left\|Z_s^\delta\right\|^2+\|V_s^\delta\|_Q^2\right\}ds+\int_0^T\Phi_s^{\mu,\gamma}d\left[M^\delta\right]_s+|K_T^{\delta+}|^2+|K_T^{\delta-}|^2\Bigg]\\
&\leq C\mathbb{E}\Bigg[\Phi_T^{\mu,\gamma}|\xi|^2+\int_0^T\Phi_s^{\mu,\gamma}\left|\frac{\varphi_s}{a_s}\right|^2ds+\int_0^T\Phi_s^{\mu,\gamma}\left|\psi_s\right|^2d\kappa_s\\
&\qquad+\sup_{t\in[0,T]}\left\{\Phi_t^{2\mu,2\gamma}\left|(L_t)^+\right|^2\right\}+\sup_{t\in[0,T]}\left\{\Phi_t^{2\mu,2\gamma}\left|(U_t)^-\right|^2\right\}+\left|(B_T)^-\right|^2\Bigg].
\end{aligned}\tag{4.6}$$

Step 4: Boundedness of the family $\left\{h_\delta\left(\cdot,Y_\cdot^\delta\right), g_\delta(\cdot,Y_\cdot^\delta)\right\}_{\delta\in]0,1]}$.

Let $\delta\in]0,1]$. From Holder's inequality, Assumption **(H6)**, the uniform estimation (4.6), the last property in (4.3) and the linear growth of the drivers h_δ, g_δ, we can write

$$\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}\left|h_\delta(s,Y_s^\delta)\right|^2ds\leq 2\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}\left\{|\varphi_s|^2ds+\left|Y_s^\delta\right|^2dA_s\right\}\leq\mathfrak{C}_1^\star.\tag{4.7}$$

Similarly, we can show

$$\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}\left|g_\delta(s,Y_s^\delta)\right|^2d\kappa_s\leq 2\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}\left\{\psi_s^2+\zeta^2\left|Y_s^\delta\right|^2\right\}d\kappa_s\leq\mathfrak{C}_2^\star.\tag{4.8}$$

Here, the constants $\mathfrak{C}_1^\star$ and $\mathfrak{C}_2^\star$ are independent of δ. In the following sequence, we will denote $\mathfrak{C}=\mathfrak{C}_1^\star\vee\mathfrak{C}_2^\star$, and it should be noted that the value of this constant may change from line to line while using the same notation.

Step 5: The family $\left\{Y^\delta, Z^\delta, V^\delta, M^\delta\right\}_{\delta\in]0,1]}$ is a Cauchy sequence in $\mathfrak{L}_{\mu,\gamma}^2$.

Let $0<\delta,\delta'\leq 1$. Using Itô's formula with expectation, and following the same computation as the one used in Proposition 3.1 with (4.4), Holder's inequality, (4.7) and (4.8), we can write

$$\begin{aligned}
&\mu\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}|Y_s^\delta-Y_s^{\delta'}|^2dA_s+\gamma\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}|Y_s^\delta-Y_s^{\delta'}|^2d\kappa_s\\
&\quad+\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}\left\{\left\|Z_s^\delta-Z_s^{\delta'}\right\|^2+\|V_s^\delta-V_s^{\delta'}\|_Q^2\right\}ds+\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}d\left[M^\delta-M^{\delta'}\right]_s\\
&\leq 2\left(\delta+\delta'\right)\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}\left|h_\delta(s,Y_s^\delta)\right|\left|h_{\delta'}(s,Y_s^{\delta'})\right|ds+2\left(\delta+\delta'\right)\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}\left|g_\delta(s,Y_s^\delta)\right|\left|g_{\delta'}(s,Y_s^{\delta'})\right|d\kappa_s\\
&\leq 2\left(\delta+\delta'\right)\left(\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}\left|h_\delta(s,Y_s^\delta)\right|^2ds\right)^{\frac12}\left(\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}\left|h_{\delta'}(s,Y_s^{\delta'})\right|^2ds\right)^{\frac12}\\
&+2\left(\delta+\delta'\right)\left(\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}\left|g_\delta(s,Y_s^\delta)\right|^2d\kappa_s\right)^{\frac12}\left(\mathbb{E}\int_0^T e^{\mu A_s+\gamma\kappa_s}\left|g_{\delta'}(s,Y_s^{\delta'})\right|^2d\kappa_s\right)^{\frac12}\\
&\leq 4\left(\delta+\delta'\right)\mathfrak{C}.
\end{aligned}\tag{4.9}$$

On the other hand, by Burkholder-Davis-Gundy's inequality, we can show that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}e^{\mu A_t+\gamma\kappa_t}|Y_t^\delta-Y_t^{\delta'}|^2\right]\leq\mathfrak{C}\left(\delta+\delta'\right).\tag{4.10}$$

Hence $(Y^\delta, Z^\delta, V^\delta, M^\delta)$ is a Cauchy sequence in $\mathfrak{L}_{\mu,\gamma}^2$. Therefore, there exists a process $(Y,Z,V,M)\in\mathfrak{L}_{\mu,\gamma}^2$ such that $Y^\delta\to Y$ in $\mathcal{S}_{\mu,\gamma}^2$, $Z^\delta\to Z$ in $\mathcal{H}_{\mu,\gamma}^2$, $V^\delta\to V$ in $\mathcal{L}_{\mu,\gamma}^2$ and $M^\delta\to M$ in $\mathcal{M}_{\mu,\gamma}^2$.

Step 6: Convergence of the Drivers $\left\{h_\delta\left(\cdot, Y^\delta_\cdot\right), g_\delta\left(\cdot, Y^\delta_\cdot\right)\right\}_{\delta\in]0,1]}$ to $\left\{h\left(\cdot, Y_\cdot\right), g\left(\cdot, Y_\cdot\right)\right\}$ in the $\mathbb{L}^2$-spaces $\mathbb{L}^2\left(\Omega\times[0,T], d\mathbb{P}\otimes dt\right)$ and $\mathbb{L}^2\left(\Omega\times[0,T], d\mathbb{P}\otimes d\kappa_t\right)$, respectively.

From **Step 4**, we deduce that $\left\{\delta h_\delta\left(s, Y^\delta_s\right)\right\}_{0<\delta\leq 1}\in\mathbb{L}^2\left(\Omega\times[0,T]; d\mathbb{P}\otimes d\lambda\right)$ (λ design the Lebesgue measure) and $\left\{\delta g_\delta\left(s, Y^\delta_s\right)\right\}_{0<\delta\leq 1}\in\mathbb{L}^2\left(\Omega\times[0,T]; d\mathbb{P}(\omega)\otimes d\kappa_t(\omega)\right) := \mathbb{L}^2_{\mathbb{P}\otimes\kappa}$, where $d\mathbb{P}\otimes d\kappa$ is the positive measure on $\left(\Omega\times[0,T], \mathcal{F}\otimes\mathcal{B}\left([0,T]\right)\right)$ defined for any $\mathcal{V}\in\mathcal{F}\otimes\mathcal{B}\left([0,T]\right)$ by $\left(d\mathbb{P}\otimes d\kappa\right)(\mathcal{V}) = \mathbb{E}\left[\int_0^T \mathbb{1}_{\mathcal{V}}(\omega,s)d\kappa_s\right]$, and that $\delta h_\delta\left(s, Y^\delta_s\right)\xrightarrow[\delta\to 0^+]{\mathbb{L}^2_{\mathbb{P}\otimes\lambda}} 0$, $\delta g_\delta\left(s, Y^\delta_s\right)\xrightarrow[\delta\to 0^+]{\mathbb{L}^2_{\mathbb{P}\otimes\kappa}} 0$. Then, applying the partial reciprocal of the dominated convergence theorem in $\mathbb{L}^2_{\mathbb{P}\otimes\lambda}$ and $\mathbb{L}^2_{\mathbb{P}\otimes\kappa}$, respectively, we deduce the existence of two sub-sequences $\left\{\delta_k h_{\delta_k}\left(s, Y^{\delta_k}_s\right)\right\}_{k\in\mathbb{N}}$ and $\left\{\delta_k g_{\delta_k}\left(s, Y^{\delta_k}_s\right)\right\}_{k\in\mathbb{N}}$ of $\left\{\delta h_\delta\left(s, Y^\delta_s\right)\right\}_{0<\delta\leq 1}$ and $\left\{\delta g_\delta\left(s, Y^\delta_s\right)\right\}_{0<\delta\leq 1}$, respectively, such that $\{\delta_k\}_{k\in\mathbb{N}}\subset]0,1]^{\otimes\mathbb{N}}$, $\delta_k\to 0$ as $k\to\infty$, $\delta_k h_{\delta_k}\left(\omega, s, Y^{\delta_k}_s(\omega)\right)\to[k\to\infty]0$, $d\mathbb{P}(\omega)\otimes d\lambda$-a.e., and $\delta_k g_{\delta_k}\left(\omega, s, Y^{\delta_k}_s(\omega)\right)\xrightarrow[k\to\infty]{} 0$, $d\mathbb{P}(\omega)\otimes d\kappa_t(\omega)$-a.e. Now, from the convergence (4.10) and the estimation (4.9), we may also extract a sub-sequence $\left\{Y^{\delta_k}\right\}_{k\in\mathbb{N}}$ such that $Y^{\delta_k}_t\to Y_t$ a.e. as $k\to\infty$, with respect to the two positive measures $d\mathbb{P}\otimes d\lambda$ and $d\mathbb{P}\otimes d\kappa$. Making us from the continuity of the drivers h and g in y, we infer that, $h_{\delta_k}\left(\omega, t, Y^{\delta_k}_t(\omega)\right) = h(\omega, t, Y^{\delta_k}_t(\omega)+\delta_k h_\delta)\to h(\omega, t, Y_t(\omega))$, $d\mathbb{P}(\omega)\otimes d\lambda$-a.e., and $g_{\delta_k}\left(\omega, t, Y^{\delta_k}_t(\omega)\right) = g(\omega, t, Y^{\delta_k}_t(\omega)+\delta_k g_\delta)\to g(\omega, t, Y_t(\omega))$, $d\mathbb{P}(\omega)\otimes d\kappa_t(\omega)$-a.e. Next, using the Lebesgue dominate convergence theorem, we obtain for any $t\in[0,T]$,

$$\mathbb{E}\left[\int_t^T \left|h_{\delta_k}\left(t, Y^{\delta_k}_s\right) - h(t, Y_s)\right|^2 ds\right] + \mathbb{E}\left[\int_t^T \left|g_{\delta_k}\left(s, Y^{\delta_k}_s\right) - g(s, Y_s)\right|^2 d\kappa_s\right]\xrightarrow[k\to\infty]{} 0,$$

Step 7: Convergence of the reflection processes $\left\{K^\delta\right\}_{\delta\in]0,1]}$

Rewriting the DRGBSDE (4.5)-(i) in a forward manner, we obtain for any $0<\delta\leq 1$

$$K^\delta_t = Y^\delta_0 - Y^\delta_t - \int_0^t h_\delta(s, Y^\delta_s)ds - \int_0^t g_\delta(s, Y^\delta_s)d\kappa_s + \int_0^t Z^\delta_s dW_s + \int_0^t\int_E V^\delta_s(e)\tilde{N}(ds, de) + \int_0^t dM^\delta_s,$$

where $K^\delta = K^{\delta+} - K^{\delta-}$.

Using the same argument, we get also

$$\lim_{k\to\infty}\mathbb{E}\left[\sup_{0\leq t\leq T}|K^{\delta_k}_t - K^{\delta'_k}_t|^2\right] = 0. \tag{4.11}$$

Hence, there exists a process K such that $K^{\delta_k}\to K$ in $\mathcal{K}^2$.
In conclusion $\left\{Y^{\delta_k}, Z^{\delta_k}, V^{\delta_k}, M^{\delta_k}, K^{\delta_k}\right\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{D}_{\mu,\gamma}$.

Step 8: The limiting process (Y, Z, V, M, K) is a solution of the DRGBSDE (4.1). By utilizing the outcomes of **Steps 5**, **6**, and **7** and by passing to the limit in the approximating equation (4.5)-(i) for a subsequence in the corresponding $\mathbb{L}^2$-spaces, we can infer that

$$K_t = Y_0 - Y_t - \int_0^t h(s, Y_s)ds - \int_0^t g(s, Y_s)d\kappa_s + \int_0^t Z_s dW_s + \int_0^t\int_E V_s(e)\tilde{N}(ds, de) + \int_0^t dM_s.$$

Furthermore, by taking a simple limit of the subsequence $\left\{Y^{\delta_k}\right\}_{k\in\mathbb{N}}$, we can conclude that $L_t\leq Y_t\leq U_t$ holds almost surely under $\mathbb{P}$. It is also noteworthy that K is an $\mathcal{F}_t$-predictable, right-continuous, and non-decreasing process with $\mathbb{E}|K_T|^2<+\infty$.
It remains to check the Skorokhod condition. Let τ be a stopping time satisfying $0\leq\tau\leq T$. Using (4.6), we can see that the non-negative sequences $K^{\delta,\pm}_\tau$ are bounded in $\mathbb{L}^2(\Omega)$. As a result, there exist $\mathcal{F}_\tau$-measurable random variables $K^\pm_\tau$ in $\mathbb{L}^2$ such that subsequences of $K^{\delta,\pm}_\tau$ converge weakly to $K^\pm_\tau$.
Now, we set $\bar{K}_\tau = K^+_\tau - K^-_\tau$. By Mazur's lemma ([22], pp.120), there exist, for every $n\geq 0$ an integer

$N \geq n$ and weights $\lambda_j^{(\tau,n)} \geq 0, j = n, ..., N$ with $\sum_{j=n}^{N} \lambda_j^{(\tau,n)} = 1$ such that

$$\bar{K}_\tau^{n,\pm} := \sum_{j=n}^{N} \lambda_j^{(\tau,n)} (K_\tau^{\pm})_j \underset{n\to\infty}{\longrightarrow} K_\tau^{\pm}. \tag{4.12}$$

Denoting $\bar{K}_\tau^n = \bar{K}_\tau^{n,+} - \bar{K}_\tau^{n,-}$, it follows that

$$\mathbb{E}|\bar{K}_\tau^n - \bar{K}_\tau| \underset{n\to\infty}{\longrightarrow} 0. \tag{4.13}$$

Thanks to (4.11), we have for all $\epsilon > 0, \|K_\tau^\delta - K_\tau\|_{\mathbb{L}^2} < \epsilon$. Therefore

$$\|\bar{K}_\tau^n - K_\tau\|_{\mathbb{L}^2} = \Big\| \sum_{j=n}^{N} \lambda_j^{(\tau,n)} (K_\tau^{\pm})_j - K_\tau \Big\|_{\mathbb{L}^2} \leq \sum_{j=n}^{N} \lambda_j^{(\tau,n)} \|(K_\tau^{\pm})_j - K_\tau\|_{\mathbb{L}^2} < \epsilon.$$

by which

$$\mathbb{E}|\bar{K}_\tau^n - K_\tau| \underset{n\to\infty}{\longrightarrow} 0. \tag{4.14}$$

Combining (4.11), (4.13) and (4.14), we obtain $\bar{K}_\tau = K_\tau$ a.s. Therefore using the optional section theorem (Theorem 86 in [5]) we have $\bar{K}_t = K_t$, $\forall t \leq T$.

On the other hand, let $\tau = T$, (4.12) implies that there exists a subsequence $\bar{K}_T^{n,+} := \sum_{j=n}^{N} \lambda_j^{(T,n)} (K_T^+)_j$ (resp. $\bar{K}_T^{n,-} := \sum_{j=n}^{N} \lambda_j^{(T,n)} (K_T^-)_j$) which converges a.s to K_T^+ (resp. K_T^-). Then, for $\mathbb{P}$-a.s. $\omega \in \Omega$, the subsequence $\bar{K}_T^{n,+}(\omega)$ (resp. $\bar{K}_T^{n,-}(\omega)$) is bounded. Then, by Helly's selection theorem ([4], pp.88), we can conclude that, for $\mathbb{P}$-a.s. $\omega \in \Omega$, there exists a subsequence of $\bar{K}_t^{n,+}(\omega)$, $0 \leq t \leq T$ (resp. $\bar{K}_t^{n,-}(\omega)$) tending to $K_t^+(\omega)$ (resp. $K_t^-(\omega)$), weakly.
Obviously we have $L_t \leq Y_t \leq U_t$. Now, we would like to show the Skorokhod's conditions. Indeed, set

$$L_t^\xi = L_t \mathbb{1}_{\{t<T\}} + \xi \mathbb{1}_{\{t=T\}} \quad \text{and} \quad U_t^\xi = U_t \mathbb{1}_{\{t<T\}} + \xi \mathbb{1}_{\{t=T\}}.$$

By notation of Snell envelope, we know that

$$Y_t - \mathbb{E}\Big[\xi + \int_t^T h(s, Y_s) ds + \int_t^T g(s, Y_s) d\kappa_s / \mathcal{F}_t\Big] = \mathcal{S}(\eta_t^+) - \mathcal{S}(\eta_t^-)$$

where

$$\eta_t^+ = L_t^\xi - \mathbb{E}\big[\xi - (K_T^- - K_t^-)/\mathcal{F}_t\big], \quad \eta_T^+ = 0, \quad \text{and} \quad \eta_t^- = -U_t^\xi + \mathbb{E}\big[\xi + (K_T^+ - K_t^+)/\mathcal{F}_t\big], \quad \eta_T^- = 0.$$

Since $\mathcal{S}(\eta_t^\pm)$ is a supermartingale, with the Doob-Meyer decomposition theorem, there exists an $\mathcal{F}_t$-adapted *rcll* increasing processes $k^\pm$ with $\mathbb{E}|k^\pm|^2 < \infty$ and $k^\pm = 0$ such that

$$\mathcal{S}(\eta_t^\pm) = \mathbb{E}\big(k_T^\pm / \mathcal{F}_t\big) - k_t^\pm.$$

By some property of Snell envelope, we know that

$$\int_0^T \big(\mathcal{S}(\eta_{t-}^+) - \eta_{t-}^+\big) dk_t^+ = \int_0^T \big(\mathcal{S}(\eta_{t-}^-) - \eta_{t-}^-\big) dk_t^- = 0 \quad \text{a.s.}$$

The uniqueness of solution (Corollary 3.1) implies that $k^+ = K^+$ and $k^- = K^-$. Consequently we can derive that

$$0 = \int_0^T \big(\mathcal{S}(\eta_{t-}^+) - \eta_{t-}^+\big) dk_t^+ = \int_0^T (Y_{t-} - L_{t-}) dK_t^+$$

and

$$0 = \int_0^T \big(\mathcal{S}(\eta_{t-}^-) - \eta_{t-}^-\big) dk_t^- = \int_0^T (U_{t-} - Y_{t-}) dK_t^-.$$

The proof of Proposition 4.1 is now concluded. □

4.2. Analysis of the DRGBSDE (1.1): The general case

After laying out the necessary groundwork, we can now present the primary outcome of this section.

Theorem 4.1 *Suppose that **(H1)**-**(H7)** hold. Then the DRGBSDE (1.1) has a unique solution.*

Proof: The uniqueness is already established in Corollary 3.1. For the existence, we will use a fixed point argument.
Let us endowing the space $\mathfrak{L}^2_{\mu,\gamma}$ with the following norm :

$$\|(Y,Z,V,M)\|^2_{\mu,\gamma} = \mathbb{E}\Bigg[\int_0^T \Phi_t^{\mu,\gamma}|Y_t|^2 dA_t + \int_0^T \Phi_t^{\mu,\gamma}|Y_t|^2 d\kappa_t + \int_0^T \Phi_t^{\mu,\gamma}\left\{\|Z_s\|^2 + \|V_s\|^2_Q\right\} dt + \int_0^T \Phi_t^{\mu,\gamma} d\left[M\right]_t\Bigg].$$

We define the function Ψ from $(\mathfrak{L}^2_{\mu,\gamma}, \|.\|_{\mu,\gamma})$ into itself as follows: for every $(Y,Z,V,M) \in \mathfrak{L}^2_{\mu,\gamma}$ we put $\Psi(Y,Z,V,M) = (\tilde{Y},\tilde{Z},\tilde{V},\tilde{M})$ where $(\tilde{Y},\tilde{Z},\tilde{V},\tilde{M},\tilde{K}^{\pm})$ is the solution of the DRGBSDE associated with the data $(\xi, f(t,\tilde{Y},Z,V), g(t,\tilde{Y}), \kappa, L, U)$ which exists through Proposition 4.1.

Let $(Y,Z,V,M),(Y',Z',V',M') \in \mathfrak{L}^2_{\mu,\gamma}$ such that $\Psi(Y,Z,V,M) = (\tilde{Y},\tilde{Z},\tilde{V},\tilde{M})$ and $\Psi(Y',Z',V',M') = (\tilde{Y}',\tilde{Z}',\tilde{V}',\tilde{M}')$.
Applying Itô's formula, using assumption **(H2)** and Remark 2.1, we get by choosing $\mu \geq 3$ and $\gamma \geq 1$

$$\begin{aligned}
&\mathbb{E}\int_0^T \Phi_s^{\mu,\gamma}(\tilde{Y}_s - \tilde{Y}'_s)^2 dA_s + \mathbb{E}\int_0^T \Phi_s^{\mu,\gamma}(\tilde{Y}_s - \tilde{Y}'_s)^2 d\kappa_s \\
&\quad + \mathbb{E}\int_0^T \Phi_s^{\mu,\gamma}\left\{\left\|\tilde{Z}_s - \tilde{Z}'_s\right\|^2 + \|\tilde{V}_s - \tilde{V}'_s\|^2_Q\right\} ds + \mathbb{E}\int_0^T \Phi_s^{\mu,\gamma}\left[\tilde{M} - \tilde{M}'\right]_s \\
&\leq \frac{1}{2}\mathbb{E}\int_0^T \Phi_s^{\mu,\gamma}\left\{\|Z_s - Z'_s\|^2 + \|V_s - V'_s\|^2_Q\right\} ds
\end{aligned}$$

Then

$$\|(\tilde{Y}-\tilde{Y}'),(\tilde{Z}-\tilde{Z}'),(\tilde{V}-\tilde{V}'),(\tilde{M}-\tilde{M}')\|^2_{\mu,\gamma} \leq \frac{1}{2}\|(Y-Y'),(Z-Z'),(V-V'),(M-M')\|^2_{\mu,\gamma}.$$

Then Ψ is a contraction mapping on $(\mathfrak{L}^2_{\mu,\gamma}, \|.\|_{\mu,\gamma})$. Henceforth, there exists a triple of process (Y,Z,V,M) that is a fixed point of Ψ which, with K^+ and K^-, is the unique solution of the DRGBSDE (1.1). □

5. Existence and uniqueness for the DRGBSDE on a random time interval

In this section we prove uniqueness and existence of the solution of the DRGBSDE when $T = +\infty$. First, by Remark 2.3 the process $(\xi_t)_{t\geq 0}$ satisfies for any $n \geq 1$,

$$\xi_t = \xi_n - \int_t^n \mathbb{Z}_s dW_s - \int_t^n \int_E \mathbb{V}_s(e)\tilde{N}(ds,de) - \int_t^n d\mathbb{M}_s + (\mathcal{J}_n^+ - \mathcal{J}_t^+) - (\mathcal{J}_n^- - \mathcal{J}_t^-). \tag{5.1}$$

Now, we go to prove our main result.

Theorem 5.1 *Suppose that **(H1)**--**(H7)** hold. Then the infinite time interval DRGBSDE (1.1) has a unique solution.*

Proof: For any fixed $n \geq 1$, we consider the approximating equation

$$\begin{cases} (i) \quad Y_t^n = \xi_n + \displaystyle\int_t^n f(s, Y_s^n, Z_s^n, V_s^n) ds + \int_t^n g(s, Y_s^n) d\kappa_s + (K_n^{n,+} - K_t^{n,+}) - (K_n^{n,-} - K_t^{n,-}) \\ \qquad\qquad\qquad\qquad - \displaystyle\int_t^n Z_s^n dW_s - \int_t^n \int_E V_s^n(e) \tilde{N}(ds, de) - \int_t^n dM_s^n, \quad \text{for all } n \geq t \geq 0, \\ (ii) \quad L_t \leq Y_t^n \leq U_t \text{ on the set } \{0 \leq t \leq n\} \text{ and for all } n \geq T \geq 0, \\ \qquad\qquad\qquad\qquad \displaystyle\int_0^T (Y_{t^-}^n - L_{t^-}) dK_t^{n,+} = \int_0^T (U_{t^-} - Y_{t^-}^n) dK_t^{n,-} = 0, \\ (iii) \quad (Y_t^n, Z_t^n, V_t^n, M_t^n) = (\xi_t, \mathbb{Z}_t, \mathbb{V}_t, \mathbb{M}_t) \text{ and } K_t^{n,+} \equiv K_n^{n,+}, K_t^{n,-} \equiv K_n^{n,-} \text{ on the set } \{t > n\}. \end{cases} \tag{5.2}$$

Since $(\xi_n, f, g, \kappa, L, U)$ satisfy conditions on Theorem 4.1, it follows that, for any $n \geq 1$, equation (5.2) has a unique solution $(Y^n, Z^n, V^n, M^n, K^{n,+}, K^{n,-}) \in \mathfrak{D}^2_{\mu,\gamma}$.
By (5.1), the equation (5.2)-(i) can be written in the form, for all $t \in [0, n]$,

$$\begin{aligned} Y_t^n - \xi_t = &\int_t^n f(s, \xi_s + (Y_s^n - \xi_s), \mathbb{Z}_s + (Z_s^n - \mathbb{Z}_s), \mathbb{V}_s + (V_s^n - \mathbb{V}_s)) \, ds + \int_t^n g(s, \xi_s + (Y_s^n - \xi_s)) d\kappa_s \\ &+ \int_t^n (dK_s^{n,+} - dK_s^{n,-}) - \int_t^n (d\mathcal{J}_s^+ - d\mathcal{J}_s^-) - \int_t^n (Z_s^n - \mathbb{Z}_s) dW_s - \int_t^n \int_E (V_s^n - \mathbb{V}_s)(e) \tilde{N}(ds, de) \\ &- \int_t^n (dM_s^n - d\mathbb{M}_s), \end{aligned}$$

Hence, by Itô's formula and Burkholder-Davis-Gundy's inequality, there exists a constant $C > 0$ such that

$$\begin{aligned} &\mathbb{E}\bigg[\sup_{s \geq t} e^{\mu A_s} |Y_s^n - \xi_s|^2 + \int_t^\infty e^{\mu A_s} \left[\|Z_s^n - \mathbb{Z}_s\|^2 + \|V_s^n - \mathbb{V}_s\|_Q^2\right] ds + \int_t^\infty e^{\mu A_s} d\left[M^n - \mathbb{M}\right]_s\bigg] \\ &\leq C\mathbb{E}\bigg[\int_t^\infty e^{\mu A_s} |f(s, \xi_s, \mathbb{Z}_s, \mathbb{V}_s)|^2 ds + \int_t^\infty e^{\mu A_s} |g(s, \xi_s)|^2 d\kappa_s + \left|\mathcal{J}_\infty^+ - \mathcal{J}_t^+\right|^2 + \left|\mathcal{J}_\infty^- - \mathcal{J}_t^-\right|^2\bigg] \end{aligned}$$

In particular, for $i \in \mathbb{N}^\star$

$$\begin{aligned} &\mathbb{E}\bigg[\sup_{s \geq n} e^{\mu A_s} |Y_s^{n+i} - \xi_s|^2 + \int_n^\infty e^{\mu A_s} \left[\|Z_s^{n+i} - \mathbb{Z}_s\|^2 + \|V_s^{n+i} - \mathbb{V}_s\|_Q^2\right] ds + \int_n^\infty e^{\mu A_s} d\left[M^{n+i} - \mathbb{M}\right]_s\bigg] \\ &\leq C\mathbb{E}\bigg[\int_n^\infty e^{\mu A_s} |f(s, \xi_s, \mathbb{Z}_s, \mathbb{V}_s)|^2 ds + \int_n^\infty e^{\mu A_s} |g(s, \xi_s)|^2 d\kappa_s + \left|\mathcal{J}_\infty^+ - \mathcal{J}_n^+\right|^2 + \left|\mathcal{J}_\infty^- - \mathcal{J}_n^-\right|^2\bigg] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow 0, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Now by uniqueness, by setting $K^n = K^{n,+} - K^{n,-}$, we have for $t \in [0, n]$

$$\begin{aligned} Y_t^{n+i} = &Y_n^{n+i} + \int_t^n f(s, Y_s^{n+i}, Z_s^{n+i}, V_s^{n+i}) ds + \int_t^n g(s, Y_s^{n+i}) d\kappa_s + (K_n^{n+i} - K_t^{n+i}) - \int_t^n Z_s^{n+i} dW_s \\ &- \int_t^n \int_E V_s^{n+i}(e) \tilde{N}(ds, de) - \int_t^n dM_s^{n+i} \end{aligned}$$

We use again Itô's formula, we get

$$\begin{aligned} &\mathbb{E}\bigg[\sup_{t \in [0,n]} e^{\mu A_t} |Y_t^{n+i} - Y_t^n|^2 + \int_0^n e^{\mu A_s} \left[\|Z_s^{n+i} - Z_s^n\|^2 + \|V_s^{n+i} - V_s^n\|_Q^2\right] ds + \int_0^n e^{\mu A_s} d\left[M^{n+i} - M^n\right]_s\bigg] \\ &\qquad\qquad\qquad\qquad \leq C\mathbb{E}\Big[e^{\mu A_n} |Y_n^{n+i} - \xi_n|^2\Big] \longrightarrow 0, \text{ as } n \to \infty. \end{aligned}$$

Hence,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\geq 0} e^{\mu A_s}|Y^{n+i}_s - Y^n_s|^2 + \int_0^\infty e^{\mu A_s}\left[\|Z^{n+i}_s - Z^n_s\|^2 + \|V^{n+i}_s - V^n_s\|^2_Q\right]ds + \int_0^\infty e^{\mu A_s} d\left[M^{n+i}-M^n\right]_s\right]\\
&\quad\leq \mathbb{E}\left[\sup_{s\in[0,n]} e^{\mu A_s}|Y^{n+i}_s - Y^n_s|^2\right] + \mathbb{E}\int_0^n e^{\mu A_s}\left[\|Z^{n+i}_s - Z^n_s\|^2 + \|V^{n+i}_s - V^n_s\|^2_Q\right]ds\\
&\qquad\qquad + \mathbb{E}\int_0^n e^{\mu A_s} d\left[M^{n+i}-M^n\right]_s\\
&\quad + \mathbb{E}\left[\sup_{s>n} e^{\mu A_s}|Y^{n+i}_s - \xi_n|^2\right] + \mathbb{E}\int_n^\infty e^{\mu A_s}\left[\|Z^{n+i}_s - \mathbb{Z}_s\|^2 + \|V^{n+i}_s - \mathbb{V}_s\|^2_Q\right]ds\\
&\qquad\qquad + \mathbb{E}\int_n^\infty e^{\mu A_s} d\left[M^{n+i}-M^n\right]_s\\
&\quad\longrightarrow 0, \quad \text{as} \quad n\to\infty.
\end{aligned}
$$

Finally, it is easy to see that

$$
\mathbb{E}\big[\sup_{t\geq 0}|K^{n+i}_t - K^n_t|^2\big] \underset{n\to\infty}{\rightarrow} 0.
$$

Consequently, $(Y^n, Z^n, V^n, M^n, K^n)$ is a Cauchy sequence in $\mathfrak{D}_{\mu,\lambda}$, it then converges to a progressively measurable process (Y, Z, V, M, K). Using the same argument used in the proof of **Step 8** in the proof of Proposition 4.1 by passing to the limit in (5.2) we prove that the limiting process $(Y, Z, V, M, K^{\pm})$ solves the DRGBSDE (1.1). □

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