Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–7.
ISSN-0037-8712
doi:10.5269/bspm.68189

On Omega Topological Groups

Ahmad Al-Omari

ABSTRACT: In this paper by using ω-open sets and ω-continuity, we introduce and investigate the notions of ω-topological groups and obtain several properties of ω-topological groups.

Key Words: ω-open; topological group; ω-topological group; ω-disjoint; ω-discrete.

Contents

1. Introduction

Throughout this paper, (G, τ) and (H, σ) stand for topological spaces with no separation axioms assumed unless otherwise stated. For a subset A of G, the closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively. A point $x \in G$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A set A is said to be ω-closed [10] if it contains all its condensation points. The complement of an ω-closed set is said to be ω-open. It is well known that a subset W of G is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets of G, denoted by τ_ω, forms a topology on G finer than τ. The ω-closure and ω-interior, that can be defined in the same way as $Cl(A)$ and $Int(A)$, respectively, will be denoted by $Cl_\omega(A)$ and $Int_\omega(A)$, respectively.

Recently, Hussain et. al. [13,14] introduced and studied some new notions in topological groups. In this paper, we introduce and study the class of ω-topological groups by using ω-open sets and ω-continuity. Also papers [3-7] have introduced some property related to ω-open sets.

Definition 1.1 [8] A subset A of a G is called an ω-neighbourhood of a point $x \in G$ if there exists an ω-open set B such that $x \in B \subseteq A$.

Definition 1.2 [11] A function $f : (G, \tau) \to (H, \sigma)$ is said to be:

1. ω-continuous if $f^{-1}(V) \in \tau_\omega$ for every $V \in \sigma$.
2. ω^*-continuous if $f^{-1}(V) \in \tau_\omega$ for every $V \in \sigma_\omega$.
3. ω-open if $f(U) \in \sigma_\omega$ for every $U \in \tau$.
4. ω-closed if $f(U) \in (\sigma_\omega)^c$ for every $U \in (\tau)^c$.

Definition 1.3 A function $f : (G \times G, \tau \times \tau) \to (H, \sigma)$ is said to be: ω-continuous if $f^{-1}(V) \in \tau_\omega \times \tau_\omega$ for every $V \in \sigma$.

Lemma 1.1 *[9] Let (G, τ) and (H, σ) be two topological spaces. Then $(\tau \times \sigma)_\omega \subseteq \tau_\omega \times \sigma_\omega$.*

Lemma 1.2 *Let (G, τ) and (H, σ) be a topological spaces. A Map $f : G \times G \to H$ is ω-continuous if and only if for each open neighbourhood C in H there exists $A, B \in \tau_\omega$ such that $f(A \times B) \subseteq C$.*

Proof: Suppose f is ω-continuous then $f^{-1}(C) = S \in \tau_\omega \times \tau_\omega$ for every an open set $C \in H$. And, $S = A \times B$ where $A, B \in \tau_\omega$. Thus $f(S) = f(A \times B) \subseteq C$. Conversely, suppose for each open neighbourhood C in H there exists $A, B \in \tau_\omega$ such that $f(A \times B) \subseteq C$ and, $A \times B \in \tau_\omega \times \tau_\omega$, we have the inverse image of an open set is in $\tau_\omega \times \tau_\omega$. Hence f is ω-continuous. □

Submitted May 10, 2023. Published February 21, 2025
2010 *Mathematics Subject Classification:* 54C05, 54C10. 54C15.

Typeset by $\mathcal{B}^{S}\mathcal{P}_{\mathcal{M}}$ style.
© Soc. Paran. de Mat.

2. ω-Topological Groups

In this section, we introduce and study a new class of topological groups by using ω-open sets and ω-continuity, which is called ω-topological groups.

Definition 2.1 [12] A topological group $(G, \odot, \tau)$ consists of a group $(G, \odot)$ and a topology τ on G for which the multiplication map $\gamma : G \times G \to G$ such that $\gamma(x, y) = x \odot y$ and the inversion map $i : G \to G$ such that $i(x) = x^{-1}$ are continuous.

Now let $i : G \to G$ such that $i(x) = x^{-1}$ be the inverse map which is continuous in a topological group $(G, \odot, \tau)$, and let U be an open neighborhood of h. Then $i^{-1}(U) = U^{-1}$ is open and contains h^{-1}.

Definition 2.2 A 3-tuple $(G, \odot, \tau)$ is called an ω-topological group if

1. for each open neighbourhood W of $x \odot y$ in G there exist an ω-open neighborhoods U of x and V of y such that $U \odot V \subseteq W$.
2. for each open neighbourhood N of x^{-1} there exist an ω-open neighborhoods M of x such that $M^{-1} \subseteq N$.

For a subsets $A, B \subseteq G$, $A^{-1} = \{a^{-1} : a \in A\}$ and $A \odot B = \{a \odot b : a \in A, b \in B\}$.

By Lemma 1.2 and by definition of ω-continues, it is equivalently saying that in an ω-topological group, multiplication and inversion are ω-continuous.

Theorem 2.1 *Let $(G, \odot, \tau)$ be an ω-topological group. Then the functions $i : G \to G$, where $i(x) = x^{-1}$ and $g : G \times G \to G$, where $g(x, y) = x \odot y$ are ω-continuous.*

Proof: (1): Let $x \in G$ and N be the open neighbourhood of x^{-1}, let e be the identity element of G. Then there exist an ω-open neighbourhoods M_e of e and M_x of x such that $M_e \odot M_x^{-1} \subseteq N$. Thus $i(M_x) = M_x^{-1} = e \odot M_x^{-1} \subseteq M_e \odot M_x^{-1} \subseteq N$ implies that i is ω-continuous on G.
(2): Let $(x, y) \in G \times G$, let N be open neighbourhood of $x \odot y$. Then there exist an ω-open neighbourhoods M_x of x and M_y of y^{-1} such that $M_x \odot M_y^{-1} \subseteq N$ and M_y^{-1} is an ω-open neighbourhood of y. Since $M_x \times M_y^{-1} \in \tau_\omega \times \tau_\omega$ is neighbourhood of (x, y). Thus $g(M_x \times M_y^{-1}) = M_x \odot M_y^{-1} \subseteq N$ implies that g is ω-continuous function. □

It follows from the above definition that every topological group is an ω-topological group and if U is open in τ, then U^{-1} is also ω-open.

The following lemma will be used in the sequel.

Lemma 2.1 *If $(G, \odot, \tau)$ is an ω-topological group, then*

1. *If $A \in \tau$, then $A^{-1} \in \tau_\omega$.*
2. *If $A \in \tau$ and $B \subseteq G$, then $A \odot B$ and $B \odot A$ are both in τ_ω.*

Proof: (1): It follows by Theorem 2.1.
(2): Let $x \in B$ and $z \in A \odot x$, $z = y \odot x$ for some $y \in A = (A \odot x) \odot x^{-1}$. Now, $y = z \odot x^{-1}$ and by Definition 2.2, there exists ω-open neighbourhoods M_z of z and $M_{x^{-1}}$ of x^{-1} such that $M_z \odot x^{-1} \subseteq M_z \odot M_{x^{-1}}^{-1} \subseteq A$, we have $z \in M_z \subseteq A \odot x$, thus $A \odot x \in \tau_\omega$. Since arbitrary union of ω-open sets is ω-open, we have $\cup\{A \odot x : x \in B\} = A \odot B \in \tau_\omega$. Also $B \odot A$ in τ_ω. □

Theorem 2.2 *Let $(G, \odot, \tau)$ be an ω-topological group. Then the function $f : G \times G \to G$, where $f(x, y) = x \odot y^{-1}$ is ω-continuous relative to the product topology for $G \times G$.*

Proof: For any $(x, y) \in G \times G$, let N be open neighbourhood of $x \odot y^{-1}$. Then there exist an ω-open neighbourhoods M_x of x and M_y of y such that $M_x \odot M_y^{-1} \subseteq N$ and $M_x \times M_y \in \tau_\omega \times \tau_\omega$ is neighbourhood of (x, y). Thus $f(M_x \times M_y) = M_x \odot M_y^{-1} \subseteq N$ implies that f is ω-continuous on $G \times G$. □

Theorem 2.3 *Let H be any closed subset of an ω-topological group $(G, \odot, \tau)$. Then $g \odot H$ and $H \odot g$ are ω-closed for all $g \in G$.*

Proof: Let $x \in Cl_\omega(g \odot H)$. Let $b = g^{-1} \odot x$ and D be an open neighbourhood of b. Then by Definition of 2.2, there exists $A, B \in \tau_\omega$ of g^{-1} and x in G, respectively such that $A \odot B \subseteq D$. Since $x \in Cl_\omega(g \odot H)$ we have $B \cap (g \odot H) \neq \emptyset$. Let $c \in B \cap (g \odot H)$, then $g^{-1} \odot c \in H \cap (A \odot B) \subseteq H \cap D$ which implies that $H \cap D \neq \emptyset$. Thus b is a limit point of H. Since H is closed we have $b \in H$. Now $x = g \odot b$ and so $x \in (g \odot H)$. Hence $Cl_\omega(g \odot H) \subseteq g \odot H$ and since $g \odot H \subseteq Cl_\omega(g \odot H)$ we have $g \odot H = Cl_\omega(g \odot H)$ and $g \odot H$ is ω-closed for all $g \in G$.

□

Definition 2.3 A bijective function $f : (G, \tau) \to (H, \sigma)$ is said to be ω-homeomorphism if it is ω-continuous and ω-open.

The following simple result is of fundamental importance in what follows.

Theorem 2.4 *Let $(G, \odot, \tau)$ be an ω-topological group. Then each left (right) translation $l_g : G \to G$, $l_g(x) = g \odot x$ ($r_g : G \to G$) and inversion $i : G \to G$, where $i(x) = x^{-1}$ are an ω-homeomorphism*

Proof: Let $a, b \in G$ be arbitrary and D_1 be an open set containing $a \odot b$. By Definition 2.2, there exists ω-open set E_1 containing a and ω-open set F_1 containing b such that $E_1 \odot F_1 \subseteq D_1$ which implies that $a \odot F_1 \subseteq D_1$ and so left translation is ω-continuous. Let A be an open set in G. Then by Lemma 2.1 (2) and Theorem 2.3, the set $l_x(A) = x \odot A = \{x\} \odot A$ is ω-open in G, which means that l_x is an ω-open mapping. Hence each left translation is ω-homeomorphism.

Let H be an open set containing a^{-1}. Since G is ω-topological group, for each an open set H containing a^{-1}, there exists ω-open set K containing a such that $K^{-1} \subseteq H$. Thus, inversion mapping is ω-continuous. Let A be an open set containing a. Since inversion is ω-continuous there exists ω-open set B containing a^{-1} such that $B^{-1} \subseteq A$, which means that the inversion is an ω-open mapping. Hence each inversion is ω-homeomorphism.

□

Recall that a family β of subsets of a topological space G is said to be a neighbourhood base of $x \in G$ if for each open subset U of G containing x there exists $B \in \beta$ such that $x \in B \subseteq U$.

Theorem 2.5 *Let $(G, \odot, \tau)$ be an ω-topological group and let β_e be the base at identity element e of G. Then*

1. *for every $U \in \beta_e$, there exists $V \in \tau_\omega(e)$ such that $V^2 \subseteq U$.*
2. *for every $U \in \beta_e$, there exists $V \in \tau_\omega(e)$ such that $V^{-1} \subseteq U$.*
3. *for every $U \in \beta_e$ and $x \in U$, there exists $V \in \tau_\omega(e)$ such that $V \odot x \subseteq U$.*

Proof: (1). Let $U \in \beta_e$. Then U is an open set containing e. We know that $e = e \odot e$ and by Definition 2.2, there exists two ω-open sets A and B containing e such that $A \odot B \subseteq U$. Let V be the smallest among A and B and so there exists ω-open set V containing e such that $V^2 \subseteq U$.

(2). Let $U \in \beta_e$. Then U is an open set containing e. We know that the inverse of e is itself. Since inversion mapping $a \to a^{-1}$ is ω-continuous, then there exists ω-open set V containing e such that $V^{-1} \subseteq U$.

(3). Let $U \in \beta_e$ and $x \in U$. We know that $x = e \odot x$. Since G is an ω-topological group, by Definition 2.2, there exists ω-open set A containing x and ω-open set V containing e such that $A \odot V \subseteq U$. So for all $x \in U$, there is ω-open set V containing e such that $V \odot x \subseteq U$.

□

Corollary 2.1 *Let $(G, \odot, \tau)$ be an ω-topological group and x be any element of G. Then for any local base β_e at $e \in G$, then each of the families $\beta_x = \{x \odot U : U \in \beta_e\}$ and $\{x \odot U^{-1} : U \in \beta_e\}$ is an ω-open neighbourhood system at x.*

Definition 2.4 An ω-topological space $(G, \odot, \tau)$ is said to be ω- homogeneous if for all $x, y \in X$ there is an ω-homeomorphism f of the space X onto itself such that $f(x) = y$.

Corollary 2.2 *Every ω-topological group $(G, \odot, \tau)$ is an ω-homogeneous space.*

Proof: Take any elements x and y in G and put $z = x^{-1} \odot y$. Then l_z is an ω-homeomorphism of G and $l_z(x) = x \odot z = x \odot (x^{-1} \odot y) = y$. □

Theorem 2.6 *Let $(G, \odot, \tau)$ be an ω-topological group and H a subgroup of G. If H contains a nonempty open set, then H is ω-open in G.*

Proof: Let U be a nonempty open subset of G with $U \subseteq H$. By Theorem 2.4, each translation is ω-homeomorphism, so any $h \in H$ the set $l_h(U) = h \odot U$ is ω-open in G and it is a subset of H. Therefore, the subgroup $H = \cup_{h \in H}(h \odot U)$ is ω-open in G as the union of ω-open sets. □

Theorem 2.7 *Every open subgroup H of an ω-topological group $(G, \odot, \tau)$ is also an ω-topological group (called ω-topological subgroup of G).*

Proof: We have to show that for each $x, y \in H$ and each open set $W \subseteq H$ containing $x \odot y^{-1}$ there exist ω-open set $U \subseteq H$ of x and $V \subseteq H$ of y such that $U \odot V^{-1} \subseteq W$. Since H is open in G, W is an open subset of an ω-topological group G there are an ω-open sets A of x and B of y, respectively such that $A \odot B^{-1} \subseteq W$ The sets $U = A \cap H$ and $V = B \cap H$ are ω-open subsets of H because H is open (also ω-open). Thus, $U \odot V^{-1} \subseteq A \odot B^{-1} \subseteq W$, which means that H is an ω-topological group. □

Theorem 2.8 *Let $(G, \odot, \tau)$ be an ω-topological group. Then every open subgroup of G is ω-closed in G.*

Proof: Let H be an open subgroup of G. Then by Theorem 2.3 every left coset $x \odot H$ of H is ω-open. Thus $Y = \cup_{x \in G \setminus H} x \odot H$ is also ω-open as a union of ω-open sets. Hence $H = G \setminus Y$ is ω-closed. □

A mapping $f : G \to H$ is called a homomorphism if it satisfies $h(x \odot y) = h(x) \odot h(y)$ for all $x, y \in G$. It is easy to see that if e is the identity of G, then $h(e)$ is the identity of H for every homomorphism h of G to H.

Theorem 2.9 *Let $f : G \to H$ be a homomorphism of ω-topological groups. If f is ω^*-continuous at the neutral element e_G of G, then f is ω-continuous on G.*

Proof: Let $x \in G$. Suppose that W is an open set containing $y = f(x)$ in H. Since the left translations in H are ω-continuous mappings, there is an ω-open set V containing of the element e_H of H such that $l_y(V) = y \odot V \subseteq W$. Since f is ω^*-continuous of f at e_G it follows the existence of an ω-open set $U \subseteq G$ containing e_G such that $f(U) \subseteq V$. Since $l_x : G \to G$ is an ω-open mapping, the set $x \odot U$ is an ω-open set containing x, and we have $f(x \odot U) = f(x) \odot f(U) = y \odot f(U) \subseteq y \odot V \subseteq W$. Hence f is ω-continuous at the point x of G, hence on G, because x was an arbitrary element in G.

□

Definition 2.5 [2] An topological space (G, τ) is said to be ω-regular if for each closed set $F \subseteq X$ and each $x \notin F$, there exists two disjoint sets $H, W \in \tau_\omega$ such that $F \subseteq H$ and $x \in W$.

A subset A of a group G is symmetric if $A = A^{-1}$.

Theorem 2.10 *Let $(G, \odot, \tau)$ be an ω-topological group with base $\mathcal{B}_e$ at the identity element e such that for each $U \in \mathcal{B}_e$ there is a symmetric open neighbourhood V of e such that $V \odot V \subseteq U$. Then G satisfies the axiom of ω-regularity at e.*

Proof: Let U be an open set containing the identity e. Then, by assumption, there is a symmetric open neighbourhood V of e satisfying $V \odot V \subseteq U$. We have to show that $Cl_\omega(V) \subseteq U$. Let $x \in Cl_\omega(V)$. The set $x \odot V$ is an ω-open neighbourhood of x, which implies $x \odot V \cap V \neq \emptyset$. Therefore, there are points $a, b \in V$ such that $b = x \odot a$, that is, $x = b \odot a^{-1} \in V \odot V^{-1} = V \odot V \subseteq U$. Hence $Cl_\omega(V) \subseteq U$. □

Theorem 2.11 *Let A and B be subsets of an ω-topological group $(G, \odot, \tau)$. Then:*

1. *$Cl_\omega(A) \odot Cl_\omega(B) \subseteq Cl(A \odot B)$;*
2. *$[Cl_\omega(A)]^{-1} \subseteq Cl(A^{-1})$.*

Proof: (1): Suppose that $x \in Cl_\omega(A)$, $y \in Cl\omega(B)$. Let W be an open neighbourhood of $x \odot y$. Then there are ω-open neighbourhoods U and V of x and y, respectively such that $U \odot V \subseteq W$. Since $x \in Cl_\omega(A)$, $y \in Cl\omega(B)$, there exists $a \in A \cap U$ and $b \in B \cap V$. Then $a \odot b \in (A \odot B) \cap (U \odot V) \subseteq (A \odot B) \cap W$. This means $x \odot y \in Cl(A \odot B)$, i.e. we have $Cl_\omega(A) \odot Cl_\omega(B) \subseteq Cl(A \odot B)$.

(2): Let $x \in [Cl_\omega(A)]^{-1}$ and let U be an open neighbourhood of x. Since the inverse mapping is ω-continuous, the set U^{-1} is ω-open neighbourhood of x^{-1}. Since $x^{-1} \in Cl_\omega(A)$, and $U^{-1} \cap A \neq \emptyset$. Therefore, $U \cap A^{-1} \neq \emptyset$, that is $x \in Cl(A^{-1})$, and so $[Cl_\omega(A)]^{-1} \subseteq Cl(A^{-1})$. □

The inclusions in the previous theorem are not true for ω-topological groups as shown by the following example.

Example 2.1 *The set $G = \{1, 3, 5, 7\}$ is an abelian group under multiplication $m = \odot_8$ the usual multiplication modulo 8. Endow G with the topology $\tau = \{G, \emptyset, \{1\}, \{1, 3, 5\}\}$. We have $\tau_\omega = P(G)$. Take a sets $A = \{1, 3\}$ and $B = \{5, 7\}$. Then $Cl_\omega(A) = A$ and $Cl_\omega(B) = B$. Therefore, $Cl_\omega(A) \odot Cl_\omega(B) = A \odot B = \{5, 7\}$, and $Cl_\omega(A \odot B) = A \odot B = \{5, 7\}$ and $Cl(A \odot B) = \{3, 5, 7\}$. Hence $Cl_\omega(A) \odot Cl_\omega(B) \neq Cl(A \odot B)$.*

Also $B^{-1} = \{5, 7\}$, $Cl(B^{-1}) = \{3, 5, 7\}$, $[Cl_\omega(B)]^{-1} = B^{-1} = \{5, 7\}$, $A^{-1} = \{1, 3\}$, $Cl(A^{-1}) = G$ and $[Cl_\omega(A)]^{-1} = A$. Hence $[Cl_\omega(A)]^{-1} \neq Cl(A^{-1})$.

Remark 2.1 The set $G = \{1, 3, 5, 7\}$ is an abelian group under multiplication $m = \odot_8$ the usual multiplication modulo 8. Endow G with the topology $\tau = \{G, \emptyset, \{1\}, \{1, 3, 5\}\}$. We have $(G, \odot_8, \tau)$ is an ω-topological group since τ_ω is the power set of G and not a topological group since the multiplication map $\gamma : G \times G \to G$ such that $\gamma(x, y) = x \odot y$ is not continuous.

Theorem 2.12 *If V is an open neighbourhood of e in ω-topological group $(G, \odot, \tau)$, then $V \subseteq Cl_\omega(V) \subseteq V^2$.*

Proof: Since $g \odot V$ is an ω-open neighbourhood of g, it must intersects V. Thus there is $t \in V$ of the form $g \odot v^{-1}$, where $v \in V$. But $g = t \odot v \in V \odot V = V^2$ and $Cl_\omega(V) \subseteq V^2$.

□

The previous theorem is not true if V is not an open neighbourhood of e as shown by the following example.

Example 2.2 *The set $G = \{1, 3, 5, 7\}$ is an abelian group under multiplication $m = \odot_8$ the usual multiplication modulo 8. Endow G with the topology $\tau = \{G, \emptyset, \{1\}, \{1, 3, 5\}\}$. Take a set $V = \{5, 7\}$ with $e \notin V$. Then $Cl_\omega(V) = V$ and $V^2 = \{1, 3\}$. Therefore, $V \subseteq Cl_\omega(V) \nsubseteq V^2$.*

Theorem 2.13 *If $(G, \odot, \tau)$ is an ω-topological group, let $A \subseteq G$, then $Cl_\omega(A) \subseteq A \odot U$ every open neighbourhood U of e.*

Proof: Since $(G, \odot, \tau)$ is an ω-topological group, for every open neighbourhood U of e, there exists $V \in \tau_\omega(e)$ such that $V^{-1} \subseteq U$. Let $x \in Cl_\omega(A)$ and V is an ω-open neighbourhood of x. Then there exists $a \in A \cap (x \odot V)$, that is $a \in x \odot V$. This implies that $a = x \odot b$ for some $b \in V$ and $x = a \odot b^{-1} \in a \odot V^{-1} \subseteq A \odot U$. Hence $Cl_\omega(A) \subseteq A \odot U$.

□

The previous theorem is not true if U is not an open neighbourhood of e as shown by the following example.

Example 2.3 *The set $G = \mathbb{Z}_3 = \{0, 1, 2\}$ and $\oplus_3$-the usual addition modulo 3, where $\tau = \{\mathbb{Z}_3, \emptyset, \{0\}, \{0, 1\}\}$. $\tau^c = \{\mathbb{Z}_3, \emptyset, \{1, 2\}, \{2\}\}$. Then $(G, \oplus, \tau)$ is an ω-topological group. Take a sets $A = \{2\}$ and $U = \{1, 2\}$. Then $Cl_\omega(A) = A$ and $A \oplus U = \{0, 1\}$. Therefore, $Cl_\omega(A) \nsubseteq A \oplus U$.*

The previous theorem is not true if $A \subseteq G$, $Cl(A) \nsubseteq A \odot U$ for every open neighbourhood U of e as shown by the following example.

Example 2.4 *The set $G = \mathbb{Z}_3 = \{0, 1, 2\}$ and $\oplus_3$-the usual addition modulo 3, where $\tau = \{\mathbb{Z}_3, \emptyset, \{0\}, \{0, 1\}\}$. $\tau^c = \{\mathbb{Z}_3, \emptyset, \{1, 2\}, \{2\}\}$. Then $(G, \oplus, \tau)$ is an ω-topological group. Take a sets $A = \{0, 2\}$ and $U = \{0\}$ be open neighbourhood of e. Then $Cl(A) = \mathbb{Z}_3$ and $A \oplus U = \{0, 2\}$. Therefore, $Cl(A) \nsubseteq A \oplus U$.*

Theorem 2.14 *If $(G, \odot, \tau)$ is an ω-topological group and $\mathcal{B}_e$ be a base of the space (G, τ) at the neutral element e, then for every subset A of G, we have $Cl_\omega(A) = \{A \odot U : U \in \mathcal{B}_e\}$.*

Proof: We only have to verify that if $x \notin Cl_\omega(A)$, then there exists $U \in \mathcal{B}_e$ such that $x \notin A \odot U$. Since $x \notin Cl_\omega(A)$, then there exists an ω-open neighbourhood W of e such that $x \odot W \cap A = \emptyset$. Take U in $\mathcal{B}_e$ satisfying the condition $U^{-1} \subseteq W$. Then $x \odot U^{-1} \cap A = \emptyset$, that is $\{x\} \cap A \odot U = \emptyset$. This implies that $x \notin A \odot U$. □

Definition 2.6 [1] A topological space (G, τ) is called ω-T_2-space if for every two different points x, y of G, there exist two disjoint ω-open sets U, V of G such that $x \in U$ and $y \in V$.

Theorem 2.15 *Let $(G, \odot, \tau)$ be an ω-topological group, then (G, τ) is ω-regular and ω-T_2-space.*

Proof: Suppose that $F \subseteq G$ is closed and $s \notin F$. Multiplication by s^{-1} allows us to assume that $s = e$. Since F is closed, $W = G \setminus F$ is an open neighbourhood of e. Then there exists $V \in \tau_\omega(e)$ such that $V^2 \subseteq W$. Hence $Cl_\omega(V) \subseteq V \odot W \subseteq W$. Then $U = G \setminus Cl_\omega(V)$ is an ω-neighbourhood containing F which is disjoint from V. This proves that $(G, \odot, \tau)$ is ω-regular. That is, $e \in V \in \tau_\omega$ and $e \neq y \in F \subseteq U \in \tau_\omega$ such that $V \cap U = \emptyset$. Hence G is ω-T_2-space. □

Theorem 2.16 *A nonempty subgroup H of an ω-topological group G is ω-open if its interior is nonempty.*

Proof: Assume that $x \in Int(H)$. Then by definition, there is an open set V such that $x \in V \subseteq H$. For every $y \in H$, we have $y \odot V \subseteq y \odot H = H$. Since V is open, so is $y \odot V$ is ω-open, we conclude that $H = \cup\{y \odot V : y \in H\}$ is an ω-open set. □

Theorem 2.17 *Let $(G, \odot, \tau)$ is an ω-topological group. If U is an open set, then $H = \cup_{n=1}^{\infty} U^n$ is an ω-open set. Also if U be any symmetric open neighbourhood of e. Then the set H is subgroup of G.*

Proof: Since U is open in an ω-topological group $(G, \odot, \tau)$, then by Lemma 2.1, $U \odot U = U^2 \in \tau_\omega$, $U^2 \odot U = U^3 \in \tau_\omega$ and similarly $U^4, U^5, \cdots$ all are ω-open sets in G. Thus the set $H = \cup_{n=1}^{\infty} U^n$ being the union of ω-open sets is an ω-open set.

We prove that $H = \cup_{n=1}^{\infty} U^n$ is a subgroup of G. Let $x, y \in H$. If $x = u^k$, $y = u^l$, $x \odot y = u^k \odot u^l = u^{k+l} \in H$, $x^{-1} = (u^k)^{-1} = (u^{-1})^k = u^k \in H$. This implies that H is a subgroup of G. □

Theorem 2.18 *If A be a subset of an ω-topological group $(G, \odot, \tau)$, then $[Int_\omega(A)]^{-1} = Int_\omega(A^{-1})$.*

Proof: Since the inverse mapping $i : G \to G$ is an ω-homeomorphism, $Int_\omega(i(A)) = Int_\omega(A^{-1}) = i(Int_\omega(A)) = [Int_\omega(A)]^{-1}$. □

Definition 2.7 Let U be an ω-open neighbourhood of the neutral element e of an ω-topological group $(G, \odot, \tau)$. A subset A of G is called ω-disjoint of U if $b \notin a \odot U$ for any disjoint $a, b \in A$.

Example 2.5 *The set $G = \{1, 3, 5, 7\}$ is an abelian group under multiplication $m = \odot_8$ the usual multiplication modulo 8. Endow G with the topology $\tau = \{G, \emptyset, \{1\}, \{1, 3, 5\}\}$. Take a set $A = \{1, 7\}$ and $U = \{1, 3, 5\}$. Then a subset A of G is ω-disjoint of U. Since, $1 \notin 7 \odot U$ and $7 \notin 1 \odot U$.*

Definition 2.8 A collection Ω of subsets of a topological space (G, τ) is ω-discrete, provided each $x \in G$ has an ω-open neighbourhood that intersects at most one member of Ω.

Theorem 2.19 *Let U be an ω-open and V be open neighbourhoods of the neutral element e in an ω-topological group $(G, \odot, \tau)$ such that $V^4 \subseteq U$ and $V^{-1} = V$. If a subset A of G is ω-disjoint of U, then the family of ω-open sets $\{a \odot V : a \in A\}$ is ω-discrete in G.*

Proof: It suffices to verify that, for every $x \in G$, an ω-open neighbourhood $x \odot V$ of x intersects at most one element of the family $\{a \odot V : a \in A\}$. Suppose to the contrary that, for some $x \in G$, there exists distinct elements $a, b \in A$ such that $x \odot V \cap a \odot V \neq \emptyset$ and $x \odot V \cap b \odot V \neq \emptyset$. Then $x^{-1} \odot a \in V^2$ and $b^{-1} \odot x \in V^2$, where $b^{-1} \odot a = (b^{-1} \odot x) \odot (x^{-1} \odot a) \in V^4 \subseteq U$. This implies that $a \in b \odot U$. This contradicts the assumption that A is ω-disjoint of U. □

Acknowledgments

The author wishes to thank the referees for useful comments and suggestions.

References

1. S. Al Ghour and W. Zareer, *Omega open sets in generalized topological spaces*, J. Nonlinear Sci. Appl., **9** (2016), 3010–3017.
2. A. Al-Omari and H. Al-Saadi, *On ω^*-connected spaces*, Songklanakarin J. Sci. Technol., (**42**), No. 2 (2020), 280–283.
3. A. Al-Omari and Mohd Salmi Md Noorani, *Regular Generalized ω-Closed Sets*, Int. J. Math. Math. Sci., vol. 2007, Article ID 016292, 11 pages, 2007. https://doi.org/10.1155/2007/16292
4. A. Al-Omari and Mohd Salmi Md Noorani, *Contra-ω-Continuous and Almost Contra-ω-Continuous*, Int. J. Math. Math. Sci., vol. 2007, Article ID 040469, 13 pages, 2007. https://doi.org/10.1155/2007/40469
5. A. Al-Omari, T. Noiri and Mohd. Salmi Md. Noorani, *Characterizations of Strongly Compact Spaces*, Int. J. Math. Math. Sci., vol. 2009, Article ID 573038, 9 pages, 2009. https://doi.org/10.1155/2009/573038
6. A. Al-Omari, T. Noiri and Mohd. Salmi Md. Noorani, *Weak and Strong Forms of ω-Continuous Functions*, Int. J. Math. Math. Sci., vol. 2009, Article ID 174042, 12 pages, 2009. https://doi.org/10.1155/2009/174042
7. A. Al-Omari and Mohd. Salmi Md. Noorani, *Decomposition of continuity via b-open aet*, Bol. Soc. Paran. Mat., **26**(1-2)(2008), 53-64.
8. k. Al-Zoubi, and B. Al-Nashef, *The topology of ω-open subsets*, Al-Manareh **9** (2) (2003), 169–179.
9. K. Al-Zoubi, *On generalized ω-closed sets*, Intern. J. Math. Math. Sci., **13** (2005), 2011-2021.
10. H. Z. Hdeib, *ω-closed mappings*, Rev. Colombiana Mat., **16** (1982), 65-78.
11. H. Z. Hdeib, *ω-continuous functions*, Dirasat J., **16** (1989), 136-153.
12. Higgins, Phillips J. An Introduction to Topological Groups. London: Cambridge University Press, 1974.
13. M. Hussain, M. Khan and C. Ozel, *On generalized topological groups*, Filomat, **27** (2013), 567–575.
14. M. Hussain, M. Khan and C. Ozel, *More on generalized topological groups*, Creative Mathematics and Informatics, **22** (2013), 47–51.

Ahmad Al-Omari,
Al al-Bayt University,
Department of Mathematics,
Jordan.
E-mail address: omarimutah1@yahoo.com