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On a solvable bidimensional system of rational difference equations via Jacobsthal numbers

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ABSTRACT: In this paper, we are interested in the closed-form solution of the following bidimensional system of rational of $(m+1)$ −order,

$$p_{n+1} = \frac{1}{7+8q_{n-m}}, q_{n+1} = \frac{1}{7+8p_{n-m}}, n, m \in \mathbb{N}_0,$$

and the initial values p_{-j} and q_{-j}, $j \in \{0, 1, ..., m\}$ are real numbers do not equal $-7/8$. We show that the solutions of this bidimensional system are associated with Jacobsthal numbers. As a consequence, these solutions are also associated with Jacobsthal-Lucas numbers. It is shown that the global stability of positive solutions of this system holds. Our results are illustrated via numerical examples.

Key Words: Stability, Jacobsthal numbers, Jacobsthal-Lucas numbers, Binet formula, system of difference equations.

Contents

1. Introduction

Much interest has been given lately to the solvability and stability of difference equations in the context of a rational system of difference equations. Whereas the following homogeneous linear difference equations of 2nd-order were solved by De Moivre,

$$r_{n+1} = \alpha r_n + \beta r_{n-1}, n \geq 1,$$

where $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$ such that $\beta \neq 0$, in particular, we give information about the Jacobsthal sequence that establishes a significant part of our study, defined as follows

$$J_{n+2} = J_{n+1} + 2J_n,\ n \geq 1,$$

with initial conditions $J_0 = 0$ and $J_1 = 1$. The following Binet formula of the Jacobsthal numbers gives, $J_n = (a^n - b^n) / (a - b)$, where $a = 2$ and $b = -1$, and the closed-form expression for the Jacobsthal-Lucas numbers are $j_n = a^n + b^n$. The search for solutions in the closed form of difference equations and/or systems has attracted the attention of many mathematicians (see., [1]−[10], [16]−[26], [29]−[37]). So, in this paper, we seek to provide a class of system of rational difference equations which can be solved in explicit form, but the solutions are expressed by Jacobsthal numbers, is the following bidimensional system of rational difference equations,

$$p_{n+1} = \frac{1}{7+8q_{n-m}}, q_{n+1} = \frac{1}{7+8p_{n-m}}, n, m \in \mathbb{N}_0, \tag{1.0}$$

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and the initial values $p_{-m}, ..., p_0$, $q_{-m}, ..., q_0$ are real numbers do not equal $-7/8$. Additionally, systems of difference equations have found applications in control theory, mathematical biology, and particularly time series analysis (see., [11]−[15], [20], [27]).

2. Main results

To solve system (2.4) we require to utilize the following lemmas.

Lemma 2.1 *Consider the homogeneous linear difference equation with constant coefficients*

$$r_{n+1} - 7r_n - 8r_{n-1} = 0, n \geq 0, \tag{2.0}$$

with initial conditions r_0, $r_{-1} \in \mathbb{R}$. *Then,*

$$\forall n \geq 0,\ r_n = \frac{r_0}{3} J_{3(n+1)} + \frac{8r_{-1}}{3} J_{3n},$$

where $(J_n, n \geq 0)$ *is the Jacobsthal sequence.*

Proof. Difference equation (2.0) is ordinarily solved by using the following characteristic polynomial,

$$\lambda^2 - 7\lambda - 8 = (\lambda - 8)(\lambda + 1) = 0,$$

roots of this equation are

$$\lambda_1 = a^3, \lambda_2 = b^3.$$

These roots are linked to the roots of the Jacobsthal number sequence. Then the closed form of the general solution of the equation (2.0) is

$$\forall n \geq -1,\ r_n = c_1 a^{3n} + c_2 b^{3n},$$

where r_0, r_{-1} are initial values such that

$$\begin{cases} r_0 = c_1 + c_2 \\ r_{-1} = \dfrac{c_1}{a^3} + \dfrac{c_2}{b^3} \end{cases},$$

and we have

$$c_1 = \frac{a^3(r_0 + r_{-1})}{9}, c_2 = \frac{-b^3 r_0 - a^3 r_{-1}}{9},$$

after some calculations, we get

$$\begin{aligned} r_n &= \frac{a^3(r_0 + r_{-1})}{9} a^{3n} + \frac{-b^3 r_0 - a^3 r_{-1}}{9} b^{3n} \\ &= \frac{r_0}{3}\left(\frac{a^{3(n+1)} - b^{3(n+1)}}{a - b}\right) + \frac{8r_{-1}}{3}\left(\frac{a^{3n} - b^{3n}}{a - b}\right). \end{aligned}$$

The lemma is proved.

Lemma 2.2 *Consider the homogeneous linear difference equation with constant coefficients*

$$s_{n+1} + 7s_n - 8s_{n-1} = 0, n \geq 0, \tag{2.1}$$

with initial conditions s_0, $s_{-1} \in \mathbb{R}$. *Then,*

$$\forall n \geq 0,\ s_n = (-1)^n \left(\frac{s_0}{3} J_{3(n+1)} - \frac{8s_{-1}}{3} J_{3n}\right),$$

where $(J_n, n \geq 0)$ *is the Jacobsthal sequence.*

Proof. Difference equation (2.1) is ordinarily solved by using the following characteristic polynomial, $\lambda^2 + 7\lambda - 8 = 0$, roots of this equation are

$$\lambda_1 = -a^3, \lambda_2 = -b^3.$$

These roots are linked to the roots of the Jacobsthal number sequence. Then the closed form of the general solution of the equation (2.1) is

$$\forall n \geq -1,\ s_n = (-1)^n \left(\widetilde{c}_1 a^{3n} + \widetilde{c}_2 b^{3n}\right),$$

where s_0, s_{-1} are initial values such that

$$\begin{cases} s_0 = \widetilde{c}_1 + \widetilde{c}_2 \\ s_{-1} = -\dfrac{\widetilde{c}_1}{a^3} - \dfrac{\widetilde{c}_2}{b^3} \end{cases},$$

and we have

$$\widetilde{c}_1 = \frac{a^3 (s_0 - s_{-1})}{9}, \widetilde{c}_2 = \frac{-b^3 s_0 + a^3 s_{-1}}{9},$$

after some calculations, we get

$$\begin{aligned} s_n &= (-1)^n \left(\frac{a^3 (s_0 - s_{-1})}{9} a^{3n} + \frac{-b^3 s_0 + a^3 s_{-1}}{9} b^{3n} \right) \\ &= (-1)^n \left(\frac{s_0}{3} \left(\frac{a^{3(n+1)} - b^{3(n+1)}}{a-b} \right) - \frac{8 s_{-1}}{3} \left(\frac{a^{3n} - b^{3n}}{a-b} \right) \right). \end{aligned}$$

The lemma is proved.

Lemma 2.3 *Consider the following system of difference equations*

$$\begin{cases} u_{n+1} = 7v_n + 8u_{n-1} \\ v_{n+1} = 7u_n + 8v_{n-1} \end{cases}, n \geq 0, \tag{2.2}$$

with initial conditions $u_{-1}, u_0, v_{-1}, v_0 \in \mathbb{R}$. *Then,*

$$\begin{aligned} u_{2n} &= \tfrac{u_0}{3} J_{3(2n+1)} + \tfrac{8v_{-1}}{3} J_{6n}, & v_{2n} &= \tfrac{v_0}{3} J_{3(2n+1)} + \tfrac{8u_{-1}}{3} J_{6n}, \\ u_{2n+1} &= \tfrac{v_0}{3} J_{6(n+1)} + \tfrac{8u_{-1}}{3} J_{3(2n+1)}, & v_{2n+1} &= \tfrac{u_0}{3} J_{6(n+1)} + \tfrac{8v_{-1}}{3} J_{3(2n+1)}. \end{aligned}$$

Proof. From system (2.2), we get the following system

$$\begin{cases} u_{n+1} + v_{n+1} = 7(u_n + v_n) + 8(u_{n-1} + v_{n-1}) \\ u_{n+1} - v_{n+1} = -7(u_n - v_n) + 8(u_{n-1} - v_{n-1}) \end{cases}, n \geq 0, \tag{2.3}$$

Using the change of variables $r_n = u_n + v_n$ and $s_n = u_n - v_n$, we can write (2.3) as

$$\begin{cases} r_{n+1} = 7r_n + 8r_{n-1} \\ s_{n+1} = -7s_n + 8s_{n-1} \end{cases}, n \geq 0,$$

by Lemmas 2.1 – 2.2, we have

$$\begin{aligned} &\forall n \geq 0,\ r_n = \frac{r_0}{3} J_{3(n+1)} + \frac{8r_{-1}}{3} J_{3n}, \\ &\forall n \geq 0,\ s_n = (-1)^n \left(\frac{s_0}{3} J_{3(n+1)} - \frac{8s_{-1}}{3} J_{3n} \right), \end{aligned}$$

hence, the closed form of general solution of the system (2.2) is $(u_n, v_n) = \left(\frac{r_n + s_n}{2}, \frac{r_n - s_n}{2}\right)$, $n \geq 0$. The lemma is proved.

2.1. On the system (2.4)

In this subsection, we consider the following system of difference equations of 1st-order,

$$p_{n+1}=\frac{1}{7+8q_n},q_{n+1}=\frac{1}{7+8p_n},n\in\mathbb{N}_0. \tag{2.4}$$

To find the closed form of the solutions of the system (2.4) we consider the following change variables

$$p_n=\frac{v_{n-1}}{u_n},q_n=\frac{u_{n-1}}{v_n},$$

then the system (2.4) becomes

$$\begin{cases} u_{n+1}=7v_n+8u_{n-1} \\ v_{n+1}=7u_n+8v_{n-1} \end{cases},n\geq 0.$$

By Lemma 2.3, the closed form of general solution of the equation (2.4) is easily obtained, in the following Theorem

Theorem 2.1 *Let $\{p_n,q_n,n\geq 0\}$ be a solution of equation (2.4). Then,*

$$p_{2n}=\frac{J_{6n}+8p_0J_{3(2n-1)}}{J_{3(2n+1)}+8p_0J_{6n}},$$
$$p_{2n+1}=\frac{J_{3(2n+1)}+8q_0J_{6n}}{J_{6(n+1)}+8q_0J_{3(2n+1)}},$$

$$q_{2n}=\frac{J_{6n}+8q_0J_{3(2n-1)}}{J_{3(2n+1)}+8q_0J_{6n}},$$
$$q_{2n+1}=\frac{J_{3(2n+1)}+8p_0J_{6n}}{J_{6(n+1)}+8p_0J_{3(2n+1)}},$$

where $(J_n,n\geq 0)$ is the Jacobsthal sequence.

Proof. Straightforward and hence omitted.

2.2. On the system (1.0)

In this paper, we study the System (1.0), which is an extension of System (2.4). Therefore, the System (1.0) can be written as follows

$$p_{(m+1)(n+1)-t}=\frac{1}{7+8q_{(m+1)n-t}},q_{(m+1)(n+1)-t}=\frac{1}{7+8p_{(m+1)n-t}},$$

for $t\in\{0,1,...,m\}$ and $n\in\mathbb{N}$. Now, using the following notation,

$$p_{n,t}=p_{(m+1)n-t},q_{n,t}=q_{(m+1)n-t},t\in\{0,1,...,m\},$$

we can get $(m+1)$ −systems similar to System (2.4),

$$p_{n+1,t}=\frac{1}{7+8q_{n,t}},q_{n+1,t}=\frac{1}{7+8p_{n,t}},n\in\mathbb{N}_0,$$

for $t\in\{0,1,...,m\}$. Through the above discussion, we can introduce the following Theorem

Theorem 2.2 *Let $\{p_n,q_n,n\geq -m\}$ be a solution of equation (1.0). Then, for $t\in\{0,1,...,m\}$,*

$$p_{2(m+1)n-t}=\frac{J_{6n}+8p_{-t}J_{3(2n-1)}}{J_{3(2n+1)}+8p_{-t}J_{6n}},$$
$$p_{(m+1)(2n+1)-t}=\frac{J_{3(2n+1)}+8q_{-t}J_{6n}}{J_{6(n+1)}+8q_{-t}J_{3(2n+1)}},$$

$$q_{2(m+1)n-t} = \frac{J_{6n} + 8q_{-t}J_{3(2n-1)}}{J_{3(2n+1)} + 8q_{-t}J_{6n}},$$
$$q_{(m+1)(2n+1)-t} = \frac{J_{3(2n+1)} + 8p_{-t}J_{6n}}{J_{6(n+1)} + 8p_{-t}J_{3(2n+1)}},$$

where $(J_n, n \geq 0)$ *is the Jacobsthal sequence.*

Proof. The proof of Theorem 2.2 is based on Theorem 2.1 for $(m+1)$ −systems (1.0).

Corollary 2.1 *Let* $\{p_n, q_n, n \geq -m\}$ *be a solution of equation* (1.0)*. Then, for* $t \in \{0, 1, ..., m\}$,

$$p_{2(m+1)n-t} = \frac{j_{6n} - 2 + 8\left(j_{3(2n-1)} + 2\right)p_{-t}}{j_{3(2n+1)} + 2 + 8\left(j_{6n} - 2\right)p_{-t}},$$
$$p_{(m+1)(2n+1)-t} = \frac{j_{3(2n+1)} + 2 + 8\left(j_{6n} - 2\right)q_{-t}}{j_{6(n+1)} - 2 + 8\left(j_{3(2n+1)} + 2\right)q_{-t}},$$

$$q_{2(m+1)n-t} = \frac{j_{6n} - 2 + 8\left(j_{3(2n-1)} + 2\right)q_{-t}}{j_{3(2n+1)} + 2 + 8\left(j_{6n} - 2\right)q_{-t}},$$
$$q_{(m+1)(2n+1)-t} = \frac{j_{3(2n+1)} + 2 + 8\left(j_{6n} - 2\right)p_{-t}}{j_{6(n+1)} - 2 + 8\left(j_{3(2n+1)} + 2\right)p_{-t}},$$

where $(j_n, n \geq 0)$ *is the Jacobsthal-Lucas sequence.*

Proof. We see that it suffices to remark

$$j_n = 3J_n + 2(-1)^n \text{ (see., [28]).}$$

Remark 2.1 *There are many systems whose solutions can be expressed by Jacobsthal and Jacobsthal-Lucas numbers, which are*

$$p_{n+1} = \frac{1}{c_k + d_k q_{n-m}}, q_{n+1} = \frac{1}{c_k + d_k p_{n-m}}, n, m \in \mathbb{N}_0, k \geq 1,$$

where $(c_k, d_k) = \left(a^k + b^k, -(ab)^k\right) \in \{(1,2); (5,-4); (17,-16); ...\}, k \geq 1$*. Using the results of Theorem 2.2, we get*

$$p_{2(m+1)n-t} = \frac{J_{2kn} + 8p_{-t}J_{k(2n-1)}}{J_{k(2n+1)} + 8p_{-t}J_{2kn}},$$
$$p_{(m+1)(2n+1)-t} = \frac{J_{k(2n+1)} + 8q_{-t}J_{2kn}}{J_{2k(n+1)} + 8q_{-t}J_{k(2n+1)}},$$
$$q_{2(m+1)n-t} = \frac{J_{2kn} + 8q_{-t}J_{k(2n-1)}}{J_{k(2n+1)} + 8q_{-t}J_{2kn}},$$
$$q_{(m+1)(2n+1)-t} = \frac{J_{k(2n+1)} + 8p_{-t}J_{2kn}}{J_{2k(n+1)} + 8p_{-t}J_{k(2n+1)}}, k \geq 1.$$

3. Global stability of positive solutions of (1.0)

In the following, we will study the global stability character of the solutions of system (1.0). Obviously, the positive equilibrium of system (1.0) is

$$U = (\overline{p}, \overline{q}) = a^{-3}(1,1), \ E = b^{-3}(1,1).$$

Let the functions $f_1, f_2 : (0, +\infty)^{2(m+1)} \to (0, +\infty)$ defined by

$$f_1\left(\underline{p}'_{0:m}, \underline{q}'_{0:m}\right) = \frac{1}{7 + 8q_{n-m}}, f_2\left(\underline{p}'_{0:m}, \underline{q}'_{0:m}\right) = \frac{1}{7 + 8p_{n-m}},$$

where $\underline{u}_{0:m} = (u_0, u_1, ..., u_m)'$. Now, it is usually useful to linearize the system (1.0) around the equilibrium point U in order to facilitate its study. For this purpose, introducing the vectors $\underline{X}'_n := \left(\underline{P}'_n, \underline{Q}'_n\right)$ where $\underline{P}'_n = (p_n, p_{n-1}, ..., p_{n-m})$ and $\underline{Q}'_n = (q_n, q_{n-1}, ..., q_{n-m})$. With these notations, we obtain the following representation

$$\underline{X}_{n+1} = F_m \underline{X}_n, \tag{3.1}$$

where

$$F_m = \begin{pmatrix} \underline{O}'_{(m-1)} & 0 & \underline{O}'_{(m-1)} & -\frac{8}{49} \\ I_{(m-1)} & \underline{O}_{(m-1)} & O_{(m-1)} & \underline{O}_{(m-1)} \\ \underline{O}'_{(m-1)} & -\frac{8}{49} & O_{(m-1)} & 0 \\ O_{(m-1)} & \underline{O}_{(m-1)} & I_{(m-1)} & \underline{O}_{(m-1)} \end{pmatrix},$$

with $O_{(k,l)}$ denotes the matrix of order $k \times l$ whose entries are zeros, for simplicity, we set $O_{(k)} := O_{(k,k)}$ and $\underline{O}_{(k)} := O_{(k,1)}$ and $I_{(m)}$ is the $m \times m$ identity matrix. We summarize the above discussion in the following theorem

Theorem 3.1 *The positive equilibrium point U is locally asymptotically stable.*

Proof. After some preliminary calculations, the characteristic polynomial of F_m is

$$P_{F_m}(\lambda) = \det\left(F_m - \lambda I_{(2(m+1))}\right) = \Lambda_1(\lambda) - \Lambda_2(\lambda),$$

where $\Lambda_1(\lambda) = \lambda^{2(m+1)}$ and $\Lambda_2(\lambda) = \left(\frac{8}{49}\right)^2$, then $|\Lambda_2(\lambda)| < |\Lambda_1(\lambda)|, \forall \lambda : |\lambda| = 1$. So, according to Rouche's Theorem, all zeros of $\Lambda_1(\lambda) - \Lambda_2(\lambda) = 0$ lie in the unit disc $|\lambda| < 1$. Thus, the positive equilibrium point U is locally asymptotically stable.

Corollary 3.1 *For every well defined solution of system (1.0), we have $\lim p_n = \lim q_n = a^{-3}$.*

Proof. From Theorem 2.2, we have

$$\begin{aligned} \lim p_{2(m+1)n-t} &= \lim \frac{J_{6n} + 8p_{-t}J_{3(2n-1)}}{J_{3(2n+1)} + 8p_{-t}J_{6n}} \\ &= \lim \frac{1 + 8p_{-t}\frac{J_{3(2n-1)}}{J_{6n}}}{\frac{J_{3(2n+1)}}{J_{6n}} + 8p_{-t}} \\ &= \frac{1 + 8p_{-t}a^{-3}}{a^3 + 8p_{-t}} = a^{-3}, \end{aligned}$$

$$\begin{aligned} \lim p_{(m+1)(2n+1)-t} &= \lim \frac{J_{3(2n+1)} + 8q_{-t}J_{6n}}{J_{6(n+1)} + 8q_{-t}J_{3(2n+1)}} \\ &= \lim \frac{1 + 8q_{-t}\frac{J_{6n}}{J_{3(2n+1)}}}{\frac{J_{6(n+1)}}{J_{3(2n+1)}} + 8q_{-t}} \\ &= \frac{1 + 8q_{-t}a^{-3}}{a^3 + 8q_{-t}} = a^{-3}. \end{aligned}$$

Rest of the proof of $\lim q_n$ is similar to the proof of $\lim p_n$, which completes the proof of Corollary 3.1. The following result is an immediate consequence of Theorem 3.1 and Corollary 3.1.

Corollary 3.2 *The positive equilibrium point U is globally asymptotically stable.*

4. Numerical Examples

In order to clarify and shore theoretical results of the previous section, we consider some interesting numerical examples in this section.

Example 4.1 *We consider interesting numerical example for the difference equations system* (1.0) *when* $m = 1$ *with the initial conditions* $p_{-1} = 2.3$, $p_0 = 4$, $q_{-1} = 0.4$ *and* $q_0 = -2.3$. *The plot of the solutions is shown in Figure 1.*

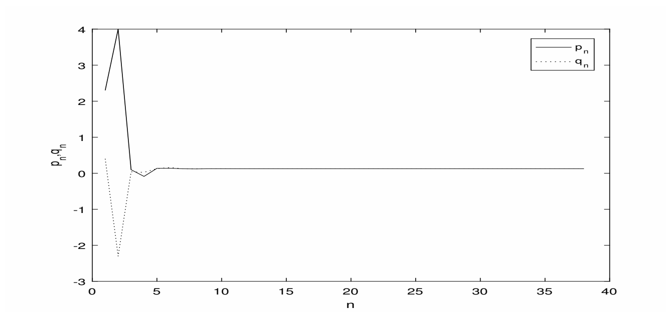

Figure 1: The plot of the solutions of system (1.0), when m=1 and we put the initial conditions $p_{-1} = 2.3, p_0 = 4, q_{-1} = 0.4$ and $q_0 = -2.3$.

Example 4.2 *We consider interesting numerical example for the difference equations system* (1.0) *when* $m = 2$ *with the initial conditions*

i	0	1	2
p_{-i}	0	4	1
q_{-i}	2	3	-0.6

Table 1. The initial conditions.

The plot of the solutions is shown in Figure 2.

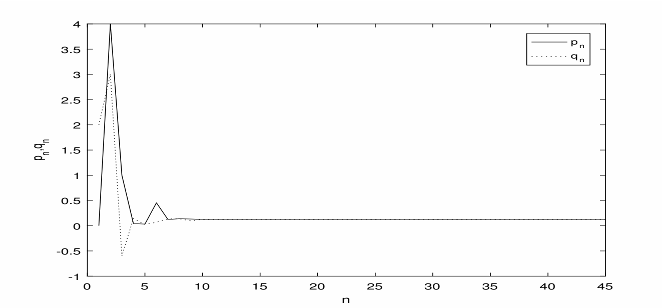

Figure 2: The plot of the solutions of system (1.0); when we put the initial conditions in Table 1.

Example 4.3 *We consider interesting numerical example for the difference equations system* (1.0) *when* $m = 3$ *with the initial conditions*

i	0	1	2	3
p_{-i}	3	1	2	0.2
q_{-i}	0.4	3	1	3.9

Table 2. The initial conditions.

The plot of the solutions is shown in Figure 3.

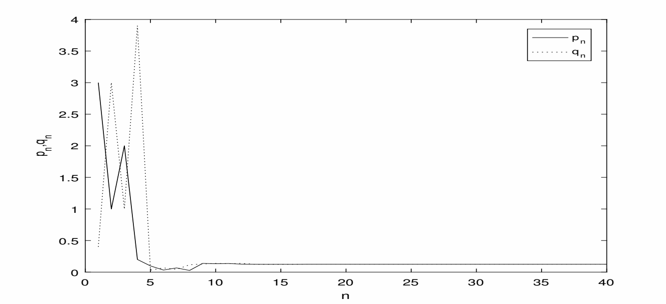

Figure 3: The plot of the solutions of system (1.0); when we put the initial conditions in Table 2.

In these examples, we show that the solutions of the system (1.0) for some cases are globally asymptotically stable.

Author contributions

All authors have contributed equally to the paper.

Data availability statement

This manuscript has no associated data or the data will not be deposited. [Authors'comment: The numerical data generated and analyzed in this paper is available from the corresponding author on reasonable request .]

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