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# Fixed point theorems for generalized weak integral contractive condition

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ABSTRACT: In the present work, we establish the existence of common fixed points for pairs of subsequential continuous mappings and variants of sequential continuous mappings as well as variants of $R$-weakly commuting mappings, non-compatible and faintly compatible mappings satisfying a generalized $(\psi, \phi)$-weak integral contractive condition involving cubic terms of distance functions. We also discuss the existence and uniqueness of common solutions for a system of functional equations arising in dynamic programming.

Key Words: Generalized $(\psi, \phi)$-weak integral contractive condition, faintly compatible mappings, $R$-weakly commuting mappings, subsequential continuous mappings, variants of sequential continuous mappings, functional equations.

## Contents



## 1. Introduction and preliminaries

Banach contraction principle [5] ensures the existence and uniqueness of fixed point for every contraction mapping defined on a complete metric space. Over a hundred years, researchers have made efforts to extend, generalize and improve the Banach fixed point theorem in various directions. Jungck [21] was the first to prove a common fixed point theorem for a pair of commuting mappings. This theorem paved the path of generalization of the Banach contraction principle for a pair/pairs of mappings satisfying a set of minimal commutative conditions.

Now, we recall some basic definitions that are useful for this paper.
Let $(E, d)$ be a metric space and $(S, T)$ be a pair of self mappings of $E$. Let $\mathcal{X}_n$ denotes a collection of all sequences $\{u_n\}$ of $E$ such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$.

**Definition 1.1** The pair $(S, T)$ is said to be

(i) compatible [22] if and only if $\lim_{n\to\infty} d(STu_n, TSu_n) = 0$, for any sequence $\{u_n\} \in \mathcal{X}_n$.

(ii) non-compatible [22] if there exists a sequence $\{u_n\} \in \mathcal{X}_n$ such that $\lim_{n\to\infty} d(STu_n, TSu_n)$ is either non zero or does not exist.

(iii) weakly compatible [23] if the pair commutes on the set of coincidence points, i.e., $STu = TSu$, whenever $Su = Tu$ for some $u \in E$.

(iv) occasionally weakly compatible [3], if there exists a coincidence point $u \in E$ such that $Su = Tu$ implies $STu = TSu$.

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In 1994, Pant [33] introduced the notion of $R-$weak commutativity in metric spaces. In 1997, Pathak *et al.* [37] gave the generalization of $R$-weakly commuting in the form of $R$-weakly commuting of type $(A_S)$ and type $(A_T)$. Further, Kumar and Garg [28] introduced the notion of $R$-weakly commuting of type $(P)$.

**Definition 1.2** The pair $(S, T)$ is said to be

(i) $R$-weakly commuting [33] if there exists $R > 0$ such that $d(STu, TSu) \leq Rd(Su, Tu)$,

(ii) $R$-weakly commuting of type $(A_S)$ [37] if there exists $R > 0$ such that $d(STu, TTu) \leq Rd(Su, Tu)$,

(iii) $R$-weakly commuting of type $(A_T)$ [37] if there exists $R > 0$ such that $d(TSu, SSu) \leq Rd(Su, Tu)$,

(iv) $R$-weakly commuting of type $(P)$ [28] if there exists $R > 0$ such that $d(SSu, TTu) \leq Rd(Su, Tu)$,

for all $u \in E$.

**Remark 1.1** Notions of $R$-weakly commuting and $R$-weakly commuting of type $(A_S)$ or type $(A_T)$ are independent to each other,(see, [37]). But at a coincidence point, $R$-weakly commuting and all the notions analogous to $R$-weakly commuting are equivalent and imply commutativity, (see, [1] ). Both compatible and non-compatible mappings can imply $R$-weakly commuting of type $(A_S)$ or $(A_T)$,(see, [32,36] ).

In 2010, Pant *et. al* [31] redefined the concept of occasionally weakly compatible mappings in the form of conditional commuting mappings. In 2012, Pant and Bisht [35] introduced a concept of conditional compatible mappings and proved that conditional compatibility is independent of compatibility condition. They also proved that conditional compatible mappings need not commute at the coincidence points. In 2013, Bisht and Sahhzad [10] introduced the notion of conditionally compatible mappings in a different setting and coined it as faintly compatible mappings.

**Definition 1.3** The pair $(S, T)$ is said to be

(i) conditionally commuting [31] if the pair commutes on a non empty subset of the set of coincidence points whenever the set of coincidences is non empty.

(ii) conditionally compatible [35] if and only if there exists a sequence $\{v_n\} \in \mathcal{X}_n$ such that

$$\lim_{n\to\infty} d(STv_n, TSv_n) = 0,$$

whenever the set $\mathcal{X}_n \neq \emptyset$

(iii) faintly compatible [10] if and only if the pair $(S, T)$ is conditionally compatible and $S$ and $T$ commute on a non empty subset of coincidence points whenever the set of coincidences is non empty.

For a comparative discussion on Definition 1.3 along with Definition 1.1, we present the following examples.

(a) Conditional compatibility does not imply compatibility, see the example given below

**Example 1.1** Let $E = [2, 10]$ and $d$ be a usual metric. Let $S, T : E \to E$ be two mappings defined as $Su = 3$, $Tu = 6 - u$, $u \in [2, 3]$ and $Su = 4$, $Tu = 10$, $u \in (3, 10]$. If we consider the constant sequence $\{u_n\}$, where $u_n = 3$, for each $n$, then $\lim_{n\to\infty} Su_n = 3$, $\lim_{n\to\infty} Tu_n = 3$ and $\lim_{n\to\infty} d(STu_n, TSu_n) = 0$, but if we consider the sequence $\{3 - \frac{1}{n}\}$, then $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = 3$ and $\lim_{n\to\infty} d(STu_n, TSu_n) = 1 \neq 0$. Thus $S$ and $T$ are conditionally compatible but not compatible.

(b) Faintly compatibility and non-compatibility are independent of each other, see examples given below

**Example 1.2** Let $E = [0, 10]$ and $d$ be a usual metric. Let $S, T : E \to E$ be two mappings defined as $Su = 10$, $Tu = 0$, $u \in [0, 6)$ and $Su = 0$, $Tu = u - 6$, $u \in [6, 10]$. If we consider the sequence $\{u_n\}$, where $u_n = 6 + \frac{1}{n}$, for each $n$, then $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = 0$ and $\lim_{n\to\infty} d(STu_n, TSu_n) = 10 \neq 0$.Thus $S$ and $T$ are non-compatible but not faintly compatible.

(c) Weakly compatibility implies faintly compatibility but the converse may not be true.

**Example 1.3** Let $E = [0, \frac{4}{5}]$ and $d$ be a usual metric. Define self mappings $S$ and $T$ as $Tu = \frac{4-5u}{10}$ and $Su = \frac{2}{5} - \left|\frac{2}{5} - u\right|$, for all u $\in E$. Consider a constant sequence $\{u_n = \frac{4}{15}\}$ and a coincidence point $u = \frac{4}{15}$ for which the mappings $S$ and $T$ are commuting, therefore, the pair $(S, T)$ is faintly compatible. But the mappings do not commute at the coincidence point $u = \frac{4}{5}$, hence, pair $(S, T)$ is not weakly compatible.

In 1998, Pant [34] introduced the concept of reciprocal continuity and initiated the study of fixed points for discontinuous mappings. In 2009, Bouhadjera and Godet-Thobie [11] introduced the concept of subsequential continuous mappings. In 2012, Gopal *et al.* [17] gave the concept of variants of sequential continuous mappings ( type $(A_S)$, type $(A_T)$). Motivated by Gopal *et al.* [17], Chauhan *et al.* [15] presented the concept of sequential continuous mappings of type $(P)$.

**Definition 1.4** The pair $(S, T)$ is said to be

(i) reciprocally continuous [34] if $\lim_{n\to\infty} STu_n = Sz$ and $\lim_{n\to\infty} TSu_n = Tz$, for any sequence $\{u_n\} \in \mathcal{X}_n$.

(ii) subsequential continuous [11], if there exists a sequence $\{u_n\} \in \mathcal{X}_n$ such that $\lim_{n\to\infty} STu_n = Sz$ and $\lim_{n\to\infty} TSu_n = Tz$.

(iii) sequentially continuous of type $(A_S)$ [17] if and only if there exists a sequence $\{u_n\} \in \mathcal{X}_n$ such that $\lim_{n\to\infty} STu_n = Sz$ and $\lim_{n\to\infty} TTu_n = Tz$.

(iv) sequentially continuous of type $(A_T)$ [17] if and only if there exists a sequence $\{u_n\} \in \mathcal{X}_n$ such that $\lim_{n\to\infty} TSu_n = Tz$ and $\lim_{n\to\infty} SSu_n = Sz$.

(v) sequentially continuous of type $(P)$ [15] if and only if there exists a sequence $\{u_n\} \in \mathcal{X}_n$ such that $\lim_{n\to\infty} SSu_n = Sz$ and $\lim_{n\to\infty} TTu_n = Tz$.

**Remark 1.2** If the pair $(S, T)$ is continuous then it is reciprocally continuous and subsequential continuous also, but the converse is not true in general (see [26, Example 2.2]).

**Remark 1.3** Subsequential continuity and reciprocal continuity are independent from each other(see [26, Examples 2.3- 2.4]). Also, it may be noted that the concept of subsequential continuity and sequential continuity of type $(A_S)$ and type $(A_T)$ are independent of each other(see, [17, Example 2.1- 2.2]).

Khan *et al.* [27] gave the idea of altering the distance/control function as follows. An altering distance is an increasing continuous function $\phi : [0, \infty) \to [0, \infty)$ which vanishes only at zero. Many researchers generalized the Banach contraction principle using control functions. In this direction, Boyd and Wong [12] introduced the concept of $\phi$ contraction mappings as follows. A self mapping $T$ defined on a complete metric space $E$ is said to be $\phi$ contraction mapping if $d(Tu, Tv) \leq \phi(d(u, v))$, for all $u, v \in E$, where $\phi : [0, \infty) \to [0, \infty)$ is an upper semi continuous function from right such that $0 \leq \phi(t) < t$ for all $t > 0$.

Alber and Guerre- Delabriere [2] generalized $\phi$ contraction to $\phi-$weak contraction in Hilbert spaces, and further, Rhoades [39] proved some results of $\phi-$weak contraction in the setting of complete metric spaces.

A self mapping $T$ of a complete metric space is said to be a $\phi-$ weak contraction, if for each $u, v \in E$, there exists a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) > 0$, for all $t > 0$ and $\phi(0) = 0$ such that $d(Tu, Tv) \leq d(u, v) - \phi(d(u, v))$.

In 2013, Murthy and Prasad [30] introduced a new weak contraction involving cubic terms of distance functions.

**Theorem 1.1** *[30] Let $T$ be a self mapping on a complete metric space $E$ satisfying*

$$\begin{aligned}[1 + pd(u, v)]d^2(Tu, Tv) \leq p \max\Big\{&\frac{1}{2}[d^2(u, Tu)d(v, Tv) + d(u, Tu)d^2(v, Tv)],\\ &d(u, Tu)d(u, Tv)d(v, Tu), d(u, Tv)d(v, Tu)d(v, Tv)\Big\} + m(u, v) - \phi(m(u, v)), \quad (1.1)\end{aligned}$$

*where*

$$m(u,v) = \max\Big\{d^2(u,v), d(u,Tu)d(v,Tv), d(u,Tv)d(v,Tu), \frac{1}{2}[d(u,Tu)d(u,Tv) + d(v,Tu)d(v,Tv)]\Big\}, \quad (1.2)$$

*$p \geq 0$ is a real number and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) > 0$, for each $t > 0$. Then $T$ has a unique fixed point in $E$.*

In 2022, Kavita and Kumar [25] generalized the weak contraction (1.1) by introducing a generalized $(\psi,\phi)-$weak contraction involving cubic terms of distance functions.

In 2002, Branciari [13] introduced an integral version of the Banach contraction principle and obtained a fixed point theorem for a contractive mapping of integral type in metric spaces as follows.

**Theorem 1.2** *[13] Let $(E,d)$ be a complete metric space and $f : E \to E$ be a mapping such that for each $u,v \in E$ and $c \in (0,1)$*

$$\int_0^{d(fu,fv)} \zeta(t)dt < c \int_0^{d(u,v)} \zeta(t)dt,$$

*where $\zeta : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable function, which is summable on each compact subset of $[0,\infty)$ such that for each $\epsilon > 0$, $\int_0^{\epsilon} \zeta(t)\,dt > 0$. Then $f$ has a unique fixed point $z \in E$ and $\lim_{n\to\infty} f^n u = z$, for each $u \in E$.*

In recent years, researchers have been fascinated with the study of common fixed points for pairs of mappings satisfying contractive conditions of integral type and proved several fixed point theorems for more general integral contractive mappings, see, for example, [4,6,15,16,29,40,41].

The present work aims to establish the existence of a common fixed point for pairs of subsequential continuous mappings and variants of sequential continuous mappings as well as variants of $R$-weakly commuting mappings, faintly compatible mappings satisfying a generalized $(\psi,\phi)$-weak integral contractive condition involving cubic terms of distance functions that improve the Theorem 1.1 and the results of Branciari [13], Jain *et al.* [18,20], Kang *et al.* [24] and Murthy and Prasad [30], Kumar *et al.* [29] and many more results cited in the literature.

## 2. Main results

Let $\Psi$ be a collection of all non decreasing functions $\psi : [0,\infty)^4 \to [0,\infty)$ such that $\psi$ is an upper semi continuous in each coordinate variables and for each $t > 0$,

$$\Delta(t) = \max\{\psi(t,t,0,0), \psi(0,0,0,t), \psi(0,0,t,0), \psi(t,t,t,t)\} \leq t.$$

Let $\Phi$ be a collection of all continuous functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) > 0$ and $\phi(0) = 0$. Let $\mathcal{A}$ be a collection of all sequences $\{u_n\}$ of $E$ such that $\lim_{n\to\infty} fu_n = \lim_{n\to\infty} Su_n = x$, for some $x \in E$ and $\mathcal{B}$ be a collection of all sequences $\{v_n\}$ of $E$ such that $\lim_{n\to\infty} gv_n = \lim_{n\to\infty} Tv_n = y$, for some $y \in E$, where $f, g, S$ and $T$ are self mappings of a metric space $E$.

Now, we establish the existence of a fixed point for pairs of subsequential continuous mappings and variants of sequential continuous mappings as well as variants of $R$-weakly commuting mappings.

**Theorem 2.1** *Let $(E,d)$ be a metric space and $f$, $g$, $S$ and $T$ be self mappings defined on $E$ such that for all $u,v\in E$, there exists a function $\phi\in\Phi$, a function $\psi\in\Psi$ and a real number $p\geq 0$ such that*

$$\begin{aligned}\Big[1+p\int_0^{d(fu,gv)}\zeta(t)dt\Big]\Big(\int_0^{d(Su,Tv)}\zeta(t)dt\Big)^2 \leq p\psi\Bigg(&\Big(\int_0^{d(fu,Su)}\zeta(t)dt\Big)^2\cdot\int_0^{d(gv,Tv)}\zeta(t)dt,\\ &\int_0^{d(fu,Su)}\zeta(t)dt\cdot\Big(\int_0^{d(gv,Tv)}\zeta(t)dt\Big)^2,\\ &\int_0^{d(fu,Su)}\zeta(t)dt\cdot\int_0^{d(fu,Tv)}\zeta(t)dt\cdot\int_0^{d(gv,Su)}\zeta(t)dt,\\ &\int_0^{d(fu,Tv)}\zeta(t)dt\cdot\int_0^{d(gv,Su)}\zeta(t)dt\cdot\int_0^{d(gv,Tv)}\zeta(t)dt\Bigg)\\ &+m(fu,gv)-\phi(m(fu,gv)),\end{aligned}\tag{2.1}$$

*where*

$$\begin{aligned}m(fu,gv)=\max\Big\{&\Big(\int_0^{d(fu,gv)}\zeta(t)dt\Big)^2,\int_0^{d(fu,Su)}\zeta(t)dt\cdot\int_0^{d(gv,Tv)}\zeta(t)dt,\\ &\int_0^{d(fu,Tv)}\zeta(t)dt\cdot\int_0^{d(gv,Su)}\zeta(t)dt,\\ &\frac{1}{2}[\int_0^{d(fu,Su)}\zeta(t)dt\cdot\int_0^{d(fu,Tv)}\zeta(t)dt+\\ &\int_0^{d(gv,Su)}\zeta(t)dt\cdot\int_0^{d(gv,Tv)}\zeta(t)dt]\Big\},\end{aligned}\tag{2.2}$$

*and $\zeta:[0,\infty)\to[0,\infty)$ is a Lebesgue integrable function which is summable on each compact subset of $[0,\infty)$ such that for each $\epsilon>0$, $\int_0^{\epsilon}\zeta(t)\,dt>0$. Suppose the pairs $(f,S)$ and $(g,T)$ are either of the followings*

*(i) subsequential continuous as well as R-weakly commuting.*

*(ii) sequential continuous of type $(A_f)$ and $(A_g)$ as well as R-weakly commuting of type $(A_f)$ and $(A_g)$ respectively.*

*(iii) sequential continuous of type $(A_S)$ and $(A_T)$ as well as R-weakly commuting of type $(A_S)$ and $(A_T)$ respectively.*

*(iv) sequential continuous of type $(P)$ as well as R-weakly commuting of type $(P)$.*

*Then $f,g,S$ and $T$ have a unique common fixed point in $E$.*

**Proof:** Case $(i)$ Suppose the pair $(f,S)$ is subsequential continuous as well as $R-$weakly commuting. Then there exists a sequence $\{u_n\}$ in $\mathcal{A}$ such that $\lim_{n\to\infty} fSu_n=fz$ and $\lim_{n\to\infty} Sfu_n=Sz$, for some $z\in E$. Also, $\lim_{n\to\infty} d(fSu_n,Sfu_n)\leq Rd(fu_n,Su_n)=0$, which implies that $d(fz,Sz)=\lim_{n\to\infty} d(fSu_n,Sfu_n)=0$. Therefore, we have $fz=Sz$, i.e., $z$ is a coincidence point of the pair $(f,S)$.

As the pair $(g,T)$ is subsequential continuous as well as $R-$weakly commuting, there exists a sequence $\{v_n\}\in\mathcal{B}$ such that $\lim_{n\to\infty} gTv_n=gw$ and $\lim_{n\to\infty} Tgv_n=Tw$, for some $w\in E$. Also, $\lim_{n\to\infty} d(gTv_n,Tgv_n)\leq Rd(gv_n,Tv_n)=0$ implies that $\lim_{n\to\infty} d(gTv_n,Tgv_n)=0$, i.e., $d(gw,Tw)=0$, i.e., $gw=Tw$. Thus, $w$ is a coincidence point of the pair $(g,T)$.

Now, we claim that $z=w$. Taking $u=u_n$ and $v=v_n$ in (2.1) and (2.2) and letting $n\to\infty$, we have

$$\begin{aligned}\Big[1+p\int\limits_0^{d(z,w)}\zeta(t)dt\Big]\Big(\int\limits_0^{d(z,w)}\zeta(t)dt\Big)^2 \le p\psi\Bigg(&\Big(\int\limits_0^{d(z,z)}\zeta(t)dt\Big)^2\cdot\int\limits_0^{d(w,w)}\zeta(t)dt,\\ &\int_0^{d(z,z)}\zeta(t)dt\cdot\Big(\int_0^{d(w,w)}\zeta(t)dt\Big)^2,\\ &\int_0^{d(z,z)}\zeta(t)dt\cdot\int_0^{d(z,w)}\zeta(t)dt\cdot\int_0^{d(w,z)}\zeta(t)dt,\\ &\int_0^{d(z,w)}\zeta(t)dt\cdot\int_0^{d(w,z)}\zeta(t)dt\cdot\int_0^{d(w,w)}\zeta(t)dt\Bigg)\\ &+m(z,w)-\phi(m(z,w)),\end{aligned}$$

where

$$\begin{aligned}m(z,w)=\max\Big\{&\Big(\int_0^{d(z,w)}\zeta(t)dt\Big)^2,\int_0^{d(z,z)}\zeta(t)dt\cdot\int_0^{d(w,w)}\zeta(t)dt,\\ &\int_0^{d(z,w)}\zeta(t)dt\cdot\int_0^{d(w,z)}\zeta(t)dt,\\ &\frac{1}{2}[\int_0^{d(z,z)}\zeta(t)dt\cdot\int_0^{d(z,w)}\zeta(t)dt+\\ &\int_0^{d(w,z)}\zeta(t)dt\cdot\int_0^{d(w,w)}\zeta(t)dt]\Big\}\\ &=\Big(\int_0^{d(z,w)}\zeta(t)dt\Big)^2.\end{aligned}$$

After simplification, we get

$$p\Big(\int\limits_0^{d(z,w)}\zeta(t)dt\Big)^3+\phi\Big(\Big(\int_0^{d(z,w)}\zeta(t)dt\Big)^2\Big)\le 0,$$

which is true only if $d(z,w)=0$, i.e., $z=w$.

Next, we claim that $Sz=z$. Substituting $u=z$ and $v=v_n$ in (2.1) and (2.2) and letting $n\to\infty$, we have

$$\begin{aligned}\Big[1+p\int\limits_0^{d(fz,z)}\zeta(t)dt\Big]\Big(\int\limits_0^{d(Sz,z)}\zeta(t)dt\Big)^2 \le p\psi\Bigg(&\Big(\int\limits_0^{d(fz,Sz)}\zeta(t)dt\Big)^2\cdot\int\limits_0^{d(z,z)}\zeta(t)dt,\\ &\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\Big(\int_0^{d(z,z)}\zeta(t)dt\Big)^2,\\ &\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(fz,z)}\zeta(t)dt\cdot\int_0^{d(z,Sz)}\zeta(t)dt,\\ &\int_0^{d(fz,z)}\zeta(t)dt\cdot\int_0^{d(z,Sz)}\zeta(t)dt\cdot\int_0^{d(z,z)}\zeta(t)dt\Bigg)\\ &+m(fz,z)-\phi(m(fz,z)),\end{aligned}$$

where

$$m(fz,z)=\max\Big\{\Big(\int_0^{d(fz,z)}\zeta(t)dt\Big)^2,\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(z,z)}\zeta(t)dt,$$
$$\int_0^{d(fz,z)}\zeta(t)dt\cdot\int_0^{d(z,Sz)}\zeta(t)dt,$$
$$\frac{1}{2}[\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(fz,z)}\zeta(t)dt+$$
$$\int_0^{d(z,Sz)}\zeta(t)dt\cdot\int_0^{d(z,z)}\zeta(t)dt]\Big\}.$$

On solving, we conclude that

$$p\Big(\int_0^{d(Sz,z)}\zeta(t)dt\Big)^3+\phi\Big(\Big(\int_0^{d(Sz,z)}\zeta(t)dt\Big)^2\Big)\leq 0.$$

which holds only for $d(Sz,z)=0$, i.e., $Sz=z$.

Next, we prove that $gz=z$. For this, taking $u=v=z$ in (2.1) and (2.2), we get

$$\Big[1+p\int_0^{d(fz,gz)}\zeta(t)dt\Big]\Big(\int_0^{d(Sz,Tz)}\zeta(t)dt\Big)^2\leq p\psi\Bigg(\Big(\int_0^{d(fz,Sz)}\zeta(t)dt\Big)^2\cdot\int_0^{d(gz,Tz)}\zeta(t)dt,$$
$$\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\Big(\int_0^{d(gz,Tz)}\zeta(t)dt\Big)^2,$$
$$\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(fz,Tz)}\zeta(t)dt\cdot\int_0^{d(gz,Sz)}\zeta(t)dt,$$
$$\int_0^{d(fz,Tz)}\zeta(t)dt\cdot\int_0^{d(gz,Sz)}\zeta(t)dt\cdot\int_0^{d(gz,Tz)}\zeta(t)dt\Bigg)$$
$$+m(fz,gz)-\phi(m(fz,gz)),$$

where

$$m(fz,gz)=\max\Big\{\Big(\int_0^{d(fz,gz)}\zeta(t)dt\Big)^2,\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(gz,Tz)}\zeta(t)dt,$$
$$\int_0^{d(fz,Tz)}\zeta(t)dt\cdot\int_0^{d(gz,Sz)}\zeta(t)dt,$$
$$\frac{1}{2}[\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(fz,Tz)}\zeta(t)dt+\int_0^{d(gz,Sz)}\zeta(t)dt\cdot\int_0^{d(gz,Tz)}\zeta(t)dt]\Big\}.$$

Using the facts $fz=Sz=z$, $z=w$ and $gw=Tw$, the above inequality reduces to

$$p\Big(\int_0^{d(z,gz)}\zeta(t)dt\Big)^3+\phi\Big(\Big(\int_0^{d(z,gz)}\zeta(t)dt\Big)^2\Big)\leq 0,$$

which is true only for $d(z,gz)=0$, i.e., $gz=z$. Thus, $z$ is a common fixed point of $f,g,S$ and $T$.

Suppose mappings $f,g,S$ and $T$ have two common fixed points, say $z$ and $x$.

Putting $u = z$ and $v = x$ in (2.1) and (2.2), we have

$$\Big[1 + p\int_0^{d(z,x)} \zeta(t)dt\Big]\Big(\int_0^{d(z,x)} \zeta(t)dt\Big)^2 \leq p\psi\Bigg(\Big(\int_0^{d(z,z)} \zeta(t)dt\Big)^2 \cdot \int_0^{d(x,x)} \zeta(t)dt, \int_0^{d(z,z)} \zeta(t)dt \cdot \Big(\int_0^{d(x,x)} \zeta(t)dt\Big)^2, \int_0^{d(z,z)} \zeta(t)dt \cdot \int_0^{d(z,x)} \zeta(t)dt \cdot \int_0^{d(x,z)} \zeta(t)dt, \int_0^{d(z,x)} \zeta(t)dt \cdot \int_0^{d(x,z)} \zeta(t)dt \cdot \int_0^{d(x,x)} \zeta(t)dt\Bigg) + m(z,x) - \phi(m(z,x)),$$

where

$$m(z,x) = \max\Big\{\Big(\int_0^{d(z,x)} \zeta(t)dt\Big)^2, \int_0^{d(z,z)} \zeta(t)dt \cdot \int_0^{d(x,x)} \zeta(t)dt, \int_0^{d(z,x)} \zeta(t)dt \cdot \int_0^{d(x,z)} \zeta(t)dt, \frac{1}{2}[\int_0^{d(z,z)} \zeta(t)dt \cdot \int_0^{d(z,x)} \zeta(t)dt + \int_0^{d(x,z)} \zeta(t)dt \cdot \int_0^{d(x,x)} \zeta(t)dt]\Big\} = \Big(\int_0^{d(z,x)} \zeta(t)dt\Big)^2.$$

Solving the above inequality, we have

$$p\Big(\int_0^{d(z,x)} \zeta(t)dt\Big)^3 + \phi\Big(\Big(\int_0^{d(z,x)} \zeta(t)dt\Big)^2\Big) \leq 0.$$

The hypotheses of $p$ and $\phi$ gives $d(x,z) = 0$, i.e., $z = x$. Hence, $z$ is a unique common fixed point of $f, g, S$ and $T$.

Case $(ii)$ Assume that the pair $(f, S)$ is sequential continuous of type $(A_f)$ as well as $R-$weakly commuting of type $(A_f)$. Then there exists a sequence $\{u_n\} \in \mathcal{A}$ such that $\lim_{n\to\infty} fSu_n = fz$ and $\lim_{n\to\infty} SSu_n = Sz$, for some $z \in E$. Also, $d(fSu_n, SSu_n) \leq Rd(fu_n, Su_n)$. Letting $n$ tends to $\infty$, we have $d(fz, Sz) = \lim_{n\to\infty} d(fSu_n, SSu_n) \leq R \cdot 0$, which is true only for $d(fz, Sz) = 0$, i.e., $fz = Sz$. This implies that $z$ is a coincidence point of $f$ and $S$.

As the pair $(g, T)$ is sequential continuous of type $(A_g)$ as well as $R-$weakly commuting of type $(A_g)$, therefore, for some $w \in E$ there exists a sequence $\{v_n\} \in \mathcal{B}$ such that $\lim_{n\to\infty} gTv_n = gw$, $\lim_{n\to\infty} TTv_n = Tw$. Also, $d(gTv_n, TTv_n) \leq Rd(gv_n, Tv_n)$. Taking limit as $n \to \infty$, we have $d(gw, Tw) = \lim_{n\to\infty} d(gTv_n, TTv_n)$ $\leq R \cdot 0$, which is possible only if $d(gw, Tw) = 0$, i.e., $gw = Tw$, which implies that $w$ is a coincidence point of $g$ and $T$. The rest of the proof follows from case $(i)$.

Case $(iii)$ Suppose the pair $(f, S)$ is sequential continuous of type $(A_S)$ as well as $R-$weakly commuting of type $(A_S)$.Then, for some $z \in E$, there exists a sequence $\{u_n\} \in \mathcal{A}$ such that $\lim_{n\to\infty} Sfu_n = Sz$ and $\lim_{n\to\infty} ffu_n = fz$. Also, $d(Sfu_n, ffu_n) \leq Rd(fu_n, Su_n)$. As $n \to \infty$, we have $d(Sz, fz) = \lim_{n\to\infty} d(Sfu_n, ffu_n) \leq R \cdot 0$, which holds only if $d(Sz, fz) = 0$, i.e., $fz = Sz$. This implies that $z$ is a coincidence point of $f$ and $S$.

Similarly, the pair $(g, T)$ is sequential continuous of type $(A_T)$ as well as $R-$weakly commuting of type $(A_T)$, so, for some $w \in E$, there exists a sequence $\{v_n\} \in \mathcal{B}$ satisfying $\lim_{n\to\infty} Tgv_n = Tw$ and $\lim_{n\to\infty} ggv_n = gw$. Also, $d(Tgv_n, ggv_n) \leq Rd(gv_n, Tv_n)$. Proceeding with $n \to \infty$, we have $d(Tw, gw) = \lim_{n\to\infty} d(Tgv_n, ggv_n) \leq R \cdot 0$, which is possible only if $d(Tw, gw) = 0$, i.e., $gw = Tw$, which implies that $w$ is a coincidence point of $g$ and $T$. The rest of the proof follows on the similar lines of case $(i)$.

Case $(iv)$ Let the pair $(f, S)$ is sequential continuous of type $(P)$ as well as $R-$weakly commuting of type $(P)$, for some $z \in E$, there exists a sequence $\{u_n\} \in \mathcal{A}$ such that $\lim_{n\to\infty} ffu_n = fz$ and $\lim_{n\to\infty} SSu_n = Sz$. Also, $d(ffu_n, SSu_n) \leq Rd(fu_n, Su_n)$. Letting $n \to \infty$, we have $d(fz, Sz) = \lim_{n\to\infty} d(ffu_n, SSu_n) \leq R \cdot 0$, which holds for $d(fz, Sz) = 0$, i.e., $fz = Sz$.Thus, $z$ is a coincidence point of $f$ and $S$.

Similarly, sequential continuity of type $(P)$ as well as $R-$weakly commuting of type $(P)$ of the pair $(g, T)$ implies that for some $w \in E$, there exists a sequence $\{v_n\} \in \mathcal{B}$ such that $\lim_{n\to\infty} ggv_n = gw$ and $\lim_{n\to\infty} TTv_n = Tw$. Also, $d(ggv_n, TTv_n) \leq Rd(gv_n, Tv_n)$. Letting $n \to \infty$, we have $d(gw, Tw) = \lim_{n\to\infty} d(ggv_n, TTv_n) \leq R \cdot 0$, which is possible only for $d(gw, Tw) = 0$, i.e., $gw = Tw$, which implies that $w$ is a coincidence point of $g$ and $T$. The rest of the proof follows from case $(i)$. □

Now, we present the existence of a unique common fixed point for pairs of faintly compatible mappings.

**Theorem 2.2** *Let $(E, d)$ be a metric space. Let $f, g, S$ and $T$ be continuous self mappings of $E$ such that $S(E) \subset g(E)$, $T(E) \subset f(E)$. If $(f, S)$ and $(g, T)$ are pairs of non-compatible as well as faintly compatible mappings satisfying (2.1) and (2.2), then the mappings $f, g, S$ and $T$ have a unique common fixed point in $E$.*

**Proof:** Since the pair $(f, S)$ is non-compatible, therefore, there exists some sequences $\{u_n\} \in \mathcal{A}$ such that $\lim_{n\to\infty} d(fSu_n, Sfu_n)$ is either non zero or does not exist. Also, the pair $(f, S)$ is faintly compatible and $\mathcal{A} \neq \emptyset$, therefore, there exists a sequence $\{x_n\} \in \mathcal{A}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in E$ and

$$\lim_{n\to\infty} d(fSx_n, Sfx_n) = 0. \tag{2.3}$$

Suppose $S$ is continuous, then

$$\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} Sfx_n = Sz, \text{ for some } z \in E. \tag{2.4}$$

Using equations (2.3) and (2.4), we have $\lim_{n\to\infty} fSx_n = \lim_{n\to\infty} Sfx_n = Sz$, for some $z \in E$. Since $S(E) \subset g(E)$, therefore, there exists some $w \in E$ such that $Sz = gw$ and hence, $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} Sfx_n = Sz = gw$.

Similarly, non-compatibility of the pair $(g, T)$ implies that there exists a sequence $\{v_n\} \in \mathcal{B}$ such that $\lim_{n\to\infty} d(gTv_n, Tgv_n)$ is either non zero or does not exist. Also, the pair $(g, T)$ is faintly compatible and $\mathcal{B} \neq \emptyset$, therefore, there exists a sequence $\{y_n\}$ in $\mathcal{B}$ such that $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} gy_n = y$, for some $y \in E$ and

$$\lim_{n\to\infty} d(gTy_n, Tgy_n) = 0. \tag{2.5}$$

Next, suppose that $T$ is continuous, then

$$\lim_{n\to\infty} TTy_n = \lim_{n\to\infty} Tgy_n = Ty, \text{ for some } y \in E. \tag{2.6}$$

Using equations ( 2.5 ) and ( 2.6), we have $\lim_{n\to\infty} gTy_n = \lim_{n\to\infty} Tgy_n = Ty$.

Since $T(E) \subset f(E)$ implies that there exists $x \in E$ such that $Ty = fx$ and $\lim_{n\to\infty} TTy_n = \lim_{n\to\infty} Tgy_n = \lim_{n\to\infty} gTy_n = fx$.

We claim that $z = y$. Taking $u = x_n$, $v = y_n$ in (2.1) and (2.2) and letting $n \to \infty$, we get

$$\begin{aligned}\Big[1 + p\int_0^{d(z,y)} \zeta(t)dt\Big]\Big(\int_0^{d(z,y)} \zeta(t)dt\Big)^2 \le p\psi\Bigg(&\Big(\int_0^{d(z,z)} \zeta(t)dt\Big)^2 \cdot \int_0^{d(y,y)} \zeta(t)dt,\\ &\int_0^{d(z,z)} \zeta(t)dt \cdot \Big(\int_0^{d(y,y)} \zeta(t)dt\Big)^2,\\ &\int_0^{d(z,z)} \zeta(t)dt \cdot \int_0^{d(z,y)} \zeta(t)dt \cdot \int_0^{d(y,z)} \zeta(t)dt,\\ &\int_0^{d(z,y)} \zeta(t)dt \cdot \int_0^{d(y,z)} \zeta(t)dt \cdot \int_0^{d(y,y)} \zeta(t)dt\Bigg)\\ &+ m(z,y) - \phi(m(z,y)),\end{aligned}$$

where

$$\begin{aligned}m(z,y) = \max\Big\{&\Big(\int_0^{d(z,y)} \zeta(t)dt\Big)^2, \int_0^{d(z,z)} \zeta(t)dt \cdot \int_0^{d(y,y)} \zeta(t)dt,\\ &\int_0^{d(z,y)} \zeta(t)dt \cdot \int_0^{d(y,z)} \zeta(t)dt,\\ &\frac{1}{2}[\int_0^{d(z,z)} \zeta(t)dt \cdot \int_0^{d(z,y)} \zeta(t)dt +\\ &\int_0^{d(y,z)} \zeta(t)dt \cdot \int_0^{d(y,y)} \zeta(t)dt]\Big\} = \Big(\int_0^{d(z,y)} \zeta(t)dt\Big)^2.\end{aligned}$$

Solving the above inequality, we have

$$p\Big(\int_0^{d(z,y)} \zeta(t)dt\Big)^3 + \phi\Big(\Big(\int_0^{d(z,y)} \zeta(t)dt\Big)^2\Big) \le 0.$$

The hypothesis of $p$ and $\phi$ yields $d(z, y) = 0$, i.e., $z = y$. Therefore, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} gy_n = z$.

Further, continuity of the mappings $f$ and $g$ along with equations (2.3) and (2.5) imply that

$$\begin{aligned}\lim_{n\to\infty} Sfx_n = \lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fSx_n = fz\\ \lim_{n\to\infty} Tgy_n = \lim_{n\to\infty} ggy_n = \lim_{n\to\infty} gTy_n = gz.\end{aligned}$$

Next, we claim that $fz = gz$. Substituting $u = fx_n$, $v = gy_n$ in (2.1) and (2.2) and letting $n \to \infty$, we get

$$\begin{aligned}\Big[1 + p\int_0^{d(fz,gz)} \zeta(t)dt\Big]\Big(\int_0^{d(fz,gz)} \zeta(t)dt\Big)^2 \le p\psi\Bigg(&\Big(\int_0^{d(fz,fz)} \zeta(t)dt\Big)^2 \cdot \int_0^{d(gz,gz)} \zeta(t)dt,\\ &\int_0^{d(fz,fz)} \zeta(t)dt \cdot \Big(\int_0^{d(gz,gz)} \zeta(t)dt\Big)^2,\\ &\int_0^{d(fz,fz)} \zeta(t)dt \cdot \int_0^{d(fz,gz)} \zeta(t)dt \cdot \int_0^{d(gz,fz)} \zeta(t)dt,\\ &\int_0^{d(fz,gz)} \zeta(t)dt \cdot \int_0^{d(gz,fz)} \zeta(t)dt \cdot \int_0^{d(gz,gz)} \zeta(t)dt\Bigg)\\ &+ m(fz,gz) - \phi(m(fz,gz)),\end{aligned}$$

where

$$m(fz,gz)=\max\Big\{\Big(\int_0^{d(fz,gz)}\zeta(t)dt\Big)^2,\int_0^{d(fz,fz)}\zeta(t)dt\cdot\int_0^{d(gz,gz)}\zeta(t)dt,$$
$$\int_0^{d(fz,gz)}\zeta(t)dt\cdot\int_0^{d(gz,fz)}\zeta(t)dt,$$
$$\frac{1}{2}[\int_0^{d(fz,fz)}\zeta(t)dt\cdot\int_0^{d(fz,gz)}\zeta(t)dt+$$
$$\int_0^{d(gz,fz)}\zeta(t)dt\cdot\int_0^{d(gz,gz)}\zeta(t)dt]\Big\}=\Big(\int_0^{d(fz,gz)}\zeta(t)dt\Big)^2.$$

Simplifying the above inequality, we have

$$p\Big(\int_0^{d(fz,gz)}\zeta(t)dt\Big)^3+\phi\Big(\Big(\int_0^{d(fz,gz)}\zeta(t)dt\Big)^2\Big)\leq 0.$$

Using the hypothesis of $p$ and $\phi$, we have $d(fz,gz)=0$, i.e., $fz=gz$. Next, we claim that $gz=Sz$. For this putting $u=z$, $v=gy_n$ in (2.1) and (2.2) and letting $n\to\infty$, we get

$$\Big[1+p\int_0^{d(fz,gz)}\zeta(t)dt\Big]\Big(\int_0^{d(Sz,gz)}\zeta(t)dt\Big)^2\leq p\psi\Bigg(\Big(\int_0^{d(fz,Sz)}\zeta(t)dt\Big)^2\cdot\int_0^{d(gz,gz)}\zeta(t)dt,$$
$$\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\Big(\int_0^{d(gz,gz)}\zeta(t)dt\Big)^2,$$
$$\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(fz,gz)}\zeta(t)dt\cdot\int_0^{d(gz,Sz)}\zeta(t)dt,$$
$$\int_0^{d(fz,gz)}\zeta(t)dt\cdot\int_0^{d(gz,Sz)}\zeta(t)dt\cdot\int_0^{d(gz,gz)}\zeta(t)dt\Bigg)$$
$$+m(fz,gz)-\phi(m(fz,gz)),$$

where

$$m(fz,gz)=\max\Big\{\Big(\int_0^{d(fz,gz)}\zeta(t)dt\Big)^2,\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(gz,gz)}\zeta(t)dt,$$
$$\int_0^{d(fz,gz)}\zeta(t)dt\cdot\int_0^{d(gz,Sz)}\zeta(t)dt,$$
$$\frac{1}{2}[\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(fz,gz)}\zeta(t)dt+$$
$$\int_0^{d(gz,Sz)}\zeta(t)dt\cdot\int_0^{d(gz,gz)}\zeta(t)dt]\Big\}=0.$$

Simplifying the above inequality, we have

$$\Big(\int_0^{d(Sz,gz)}\zeta(t)dt\Big)^2\leq 0,$$

which is possible only if $d(Sz,gz)=0$,i.e., $Sz=gz$. Therefore, we have $fz=gz=Sz$. Next, we prove that $Sz=Tz$. Taking $u=v=z$ in (2.1) and (2.2), we get

$$\Big[1+p\int_0^{d(fz,gz)}\zeta(t)dt\Big]\Big(\int_0^{d(Sz,Tz)}\zeta(t)dt\Big)^2\leq p\psi(0,0,0,0)+m(fz,gz)-\phi(m(fz,gz)),$$

where

$$m(fz,gz)=\max\Big\{\Big(\int_0^{d(fz,gz)}\zeta(t)dt\Big)^2,\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(gz,Tz)}\zeta(t)dt,$$
$$\int_0^{d(fz,Tz)}\zeta(t)dt\cdot\int_0^{d(gz,Sz)}\zeta(t)dt,$$
$$\frac{1}{2}\Big[\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(fz,Tz)}\zeta(t)dt+$$
$$\int_0^{d(gz,Sz)}\zeta(t)dt\cdot\int_0^{d(gz,Tz)}\zeta(t)dt\Big]\Big\}=0.$$

After simplification, we get

$$\Big(\int_0^{d(Sz,Tz)}\zeta(t)dt\Big)^2\le 0,$$

which holds only for $d(Sz,Tz)=0$,i.e., $Sz=Tz$. Therefore, $z$ is a coincidence point of $f,g,S$, and $T$. It remains to prove that $z$ is a common fixed point. For this, putting $u=z$, $v=y_n$ in (2.1) and (2.2) and letting $n\to\infty$, we get

$$\Big[1+p\int_0^{d(fz,z)}\zeta(t)dt\Big]\Big(\int_0^{d(Sz,z)}\zeta(t)dt\Big)^2\le p\psi(0,0,0,0)+m(fz,z)-\phi(m(fz,z)),$$

where

$$m(fz,z)=\max\Big\{\Big(\int_0^{d(fz,gz)}\zeta(t)dt\Big)^2,\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(z,z)}\zeta(t)dt,$$
$$\int_0^{d(fz,z)}\zeta(t)dt\cdot\int_0^{d(z,fz)}\zeta(t)dt,$$
$$\frac{1}{2}[\int_0^{d(fz,Sz)}\zeta(t)dt\cdot\int_0^{d(fz,z)}\zeta(t)dt+$$
$$\int_0^{d(z,Sz)}\zeta(t)dt\cdot\int_0^{d(z,z)}\zeta(t)dt]\Big\}=\Big(\int_0^{d(fz,z)}\zeta(t)dt\Big)^2.$$

After simplifying the above inequality,

$$p\Big(\int_0^{d(fz,z)}\zeta(t)dt\Big)^3+\phi\Big(\Big(\int_0^{d(fz,z)}\zeta(t)dt\Big)^2\Big)\le 0.$$

The hypothesis of $p$ and $\phi$ yields $d(fz,z)=0$, which implies that $fz=z$. Hence, $z$ is a common fixed point of $f,g,S$, and $T$. The uniqueness follows easily. □

## 3. Consequences and Example

Setting $\zeta(t)=1$ in Theorems 2.1 and 2.2, we have following results.

**Corollary 3.1** *Let $(E,d)$ be a metric space and $f$, $g$, $S$ and $T$ be self mappings defined on $E$ such that for all $u,v\in E$, there exists a function $\phi\in\Phi$, a function $\psi\in\Psi$ and a real number $p\ge 0$ such that*

$$\begin{aligned}[1+pd(fu,gv)]d^2(Su,Tv)\le p\,\psi\Big(&d^2(fu,Su)d(gv,Tv),d(fu,Su)d^2(gv,Tv),\\ &d(fu,Su)d(fu,Tv)d(gv,Su),\\ &d(fu,Tv)d(gv,Su)d(gv,Tv)\Big)\\ &+m(fu,gv)-\phi(m(fu,gv)),\end{aligned}\tag{3.1}$$

*where*

$$m(fu,gv) = \max\Big\{d^2(fu,gv), d(fu,Su)d(gv,Tv), d(fu,Tv)d(gv,Su), \frac{1}{2}[d(fu,Su)d(fu,Tv)+d(gv,Su)d(gv,Tv)]\Big\}. \tag{3.2}$$

*If the pairs $(f,S)$ and $(g,T)$ are either of the followings*

*(i) subsequential continuous as well as R-weakly commuting.*

*(ii) sequential continuous of type $(A_f)$ and $(A_g)$ as well as R-weakly commuting of type $(A_f)$ and $(A_g)$ respectively.*

*(iii) sequential continuous of type $(A_S)$ and $(A_T)$ as well as R-weakly commuting of type $(A_S)$ and $(A_T)$ respectively.*

*(iv) sequential continuous of type $(P)$ as well as R-weakly commuting of type $(P)$.*

*Then $f,g,S,$ and $T$ have a unique common fixed point in $E$.*

**Corollary 3.2** *Let $(E,d)$ be a metric space and Let $f,g,S$ and $T$ be continuous self mappings of $E$ such that $S(E)\subset g(E)$, $T(E)\subset f(E)$. If pairs $(f,S)$ and $(g,T)$ are non-compatible as well as faintly compatible satisfying (3.1) and (3.2), then the mappings $f,g,S$ and $T$ have a unique common fixed point in $E$.*

By taking $f=g$ and $S=T$ in Theorems 2.1 and 2.2, one can deduce the following results for two self mappings.

**Corollary 3.3** *Let $(E,d)$ be a metric space and $f$ and $S$ be self mappings defined on $E$ such that for all $u,v\in E$, there exists a function $\phi\in\Phi$, a function $\psi\in\Psi$ and a real number $p\geq 0$ such that*

$$\begin{aligned}\Big[1+p\int_0^{d(fu,fv)}\zeta(t)dt\Big]\Big(\int_0^{d(Su,Sv)}\zeta(t)dt\Big)^2 \leq p\psi\Bigg(&\Big(\int_0^{d(fu,Su)}\zeta(t)dt\Big)^2\cdot\int_0^{d(fv,Sv)}\zeta(t)dt,\\ &\int_0^{d(fu,Su)}\zeta(t)dt\cdot\Big(\int_0^{d(fv,Sv)}\zeta(t)dt\Big)^2,\\ &\int_0^{d(fu,Su)}\zeta(t)dt\cdot\int_0^{d(fu,Sv)}\zeta(t)dt\cdot\int_0^{d(fv,Su)}\zeta(t)dt,\\ &\int_0^{d(fu,Sv)}\zeta(t)dt\cdot\int_0^{d(fv,Su)}\zeta(t)dt\cdot\int_0^{d(fv,Sv)}\zeta(t)dt\Bigg)\\ &+m(fu,fv)-\phi(m(fu,fv)),\end{aligned} \tag{3.3}$$

*where*

$$\begin{aligned}m(fu,fv)=\max\Big\{&\Big(\int_0^{d(fu,fv)}\zeta(t)dt\Big)^2, \int_0^{d(fu,Su)}\zeta(t)dt\cdot\int_0^{d(fv,Sv)}\zeta(t)dt,\\ &\int_0^{d(fu,Sv)}\zeta(t)dt\cdot\int_0^{d(fv,Su)}\zeta(t)dt, \frac{1}{2}[\int_0^{d(fu,Su)}\zeta(t)dt\cdot\int_0^{d(fu,Sv)}\zeta(t)dt+\\ &\int_0^{d(fv,Su)}\zeta(t)dt\cdot\int_0^{d(fv,Sv)}\zeta(t)dt]\Big\},\end{aligned} \tag{3.4}$$

*and $\zeta:[0,\infty)\to[0,\infty)$ is a Lebesgue integrable function which is summable on each compact subset of $[0,\infty)$ such that for each $\epsilon>0$, $\int_0^{\epsilon}\zeta(t)\,dt>0$. If $f$ and $S$ are either of the followings* *(i) subsequential continuous as well as R-weakly commuting.*

*(ii) sequential continuous of type* $(A_f)$ *as well as R-weakly commuting of type* $(A_f)$*.*

*(iii) sequential continuous of type* $(A_S)$ *as well as R-weakly commuting of type* $(A_S)$*.*

*(iv) sequential continuous of type* $(P)$ *as well as R-weakly commuting of type* $(P)$*.*

*Then* $f$ *and* $S$ *have a unique common fixed point in* $E$*.*

**Corollary 3.4** *Let* $(E, d)$ *be a metric space. Let* $f$ *and* $S$ *be two continuous self mappings of* $E$ *such that* $S(E) \subset f(E)$*. If* $S$ *and* $f$ *are non-compatible as well as faintly compatible mappings satisfying (3.3) and (3.4), then both mappings* $f$ *and* $S$ *have a unique common fixed point in* $E$*.*

The following example shows the validity of Corollary 3.2

**Example 3.1** Let $E = [0, 20]$ and $d$ be a usual metric. Let $f, g, S, T : E \to E$ be four mappings defined by $fu = 10$, $Su = \frac{-u+60}{5}$, $gu = \frac{4u+10}{5}$, for $u \in [0, 10]$, $fu = gu = Su = 20 - u$, for $u \in (10, 20]$ and $Tu = 20 - u$, for $u \in [0, 20]$. Let $p$ be a positive real number and $\phi : [0, \infty) \to [0, \infty)$ be a function defined by $\phi(t) = \frac{3}{2}t$, for $t \geq 0$ and $\psi : [0, \infty)^4 \to [0, \infty)$ be a function defined by $\psi(w_1, w_2, w_3, w_4) = \max\{w_1, w_2, w_3, w_4\}, w_i \geq 0, i = 1, 2, 3, 4$.

Consider a constant sequence $\{u_n\}$, where $u_n = 10$, for each $n$. For this, $fS(10) = Sf(10)$, therefore, the pairs $(f, S)$ and $(g, T)$ are faintly compatible mappings on $E$. Also, the pairs $(f, S)$ and $(g, T)$ are non-compatible mappings, for this consider a sequence $\{u_n = 20 - \frac{1}{n}\}$ in $E$ such that $\lim_{n\to\infty} u_n = 20$, then

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} Su_n = \lim_{n\to\infty} gu_n = \lim_{n\to\infty} Tu_n = 0$$

and

$$\lim_{n\to\infty} d(fSu_n, Sfu_n) \neq 0, \lim_{n\to\infty} d(gTu_n, Tgu_n) \neq 0$$

It is easy to verify that the mappings $f, g, S$ and $T$ are satisfying all the conditions of the Corollary 3.2 and $u = 10$ is the only common fixed point of $f, g, S$ and $T$.

## 4. Application

Let $U, V$ denote Banach spaces, $\hat{S} \subset U$, $D \subset V$ are state spaces and decision spaces respectively. Let $\mathbb{R}$ denotes the set of all real numbers and $C(\hat{S}) = \{h : \hat{S} \to \mathbb{R}, h \text{ is continuous }\}$. Let $d(h, k) = \sup\{|h(u) - k(u)| : u \in \hat{S}\}$, for any $h, k \in C(\hat{S})$. Obviously, $(C(\hat{S}), d)$ is a complete metric space.

Bellman and Lee [9] gave the basic form of functional equation as follows.

$$g(u) = \operatorname*{opt}_{v} G(u, v, g(\tau(u, v))),$$

where $u \in \hat{S}$, $v \in D$, $\tau$ is the transformation process, $g(u)$ is the optimal return with initial state $u$ and the opt denotes max or min.

Now, we discuss the existence of a common solution for the following functional equations that are arising in dynamic programming (see [7,8,9]):

$$f_i(u) = \sup_{v\in D} F_i(u, v, f_i(\tau(u, v))), u \in \hat{S} \quad (4.1)$$

$$g_i(u) = \sup_{v\in D} G_i(u, v, g_i(\tau(u, v))), u \in \hat{S}, \quad (4.2)$$

where $\tau : \hat{S} \times D \to \hat{S}$ and $F_i, G_i : \hat{S} \times D \times \mathbb{R} \to \mathbb{R}, i = 1, 2$.

**Theorem 4.1** *Let $F_i, G_i : \hat{S} \times D \times \mathbb{R} \to \mathbb{R}, i = 1, 2.$ be bounded. Define the mappings $P_i, Q_i : C(\hat{S}) \to C(\hat{S})$, as follows*

$$\begin{aligned} P_i h(u) &= \sup_{v \in D} F_i(u, v, h(\tau(u, v))), \\ Q_i k(u) &= \sup_{v \in D} G_i(u, v, k(\tau(u, v))), \end{aligned} \tag{4.3}$$

*for all $u \in \hat{S}$, $h, k \in C(\hat{S})$, $i = 1, 2$. Let $\mathcal{W}$ denotes the collection of all sequences $\{h_n\}$ of $C(\hat{S})$ such that $\lim_{n\to\infty} P_i h_n = \lim_{n\to\infty} Q_i h_n, i = 1, 2$. Suppose that the following conditions hold:*

*(a) for all $u, t \in \hat{S}, v \in D$, $h, k \in C(\hat{S})$,*

$$\begin{aligned} \big|F_1(u, v, h(t)) - F_2(u, v, k(t))\big|^2 \le M^{-1}\Big(& p\,\psi\big(d^2(Q_1 h, P_1 h) d(Q_2 k, P_2 k), \\ & d(Q_1 h, P_1 h) d^2(Q_2 k, P_2 k), \\ & d(Q_1 h, P_1 h) d(Q_1 h, P_2 k) d(Q_2 k, P_1 h), \\ & d(Q_1 h, P_2 k) d(Q_2 k, P_1 h) d(Q_2 k, P_2 k)\big) + \\ & m(Q_1 h, Q_2 k) - \phi(m(Q_1 h, Q_2 k))\Big), \end{aligned}$$

*where*

$$\begin{aligned} m(Q_1 h, Q_2 k) = \max\Big\{& d^2(Q_1 h, Q_2 k), d(Q_1 h, P_1 h) d(Q_2 k, P_2 k), d(Q_1 h, P_2 k) d(Q_2 k, P_1 h) \\ & \frac{1}{2}[d(Q_1 h, P_1 h) d(Q_1 h, P_2 k) + d(Q_2 k, P_1 h) d(Q_2 k, P_2 k)]\Big\}, \end{aligned}$$

*$M = 1 + pd(Q_1 h, Q_2 k)$, $\phi \in \Phi$, $\psi \in \Psi$, $p$ is a positive real number,*

*(b) for any $h \in C(\hat{S})$, there exists $k_1, k_2 \in C(\hat{S})$ such that $P_1 h(u) = Q_2 k_1(u)$, $P_2 h(u) = Q_1 k_2(u), u \in \hat{S}$,*

*(c)) for $i \in \{1, 2\}$, $(P_i, Q_i)$ is non-compatible and faintly compatible mappings.*

*Then the system of functional equations (4.1) and (4.2) has a unique common solution in $C(\hat{S})$.*

**Proof:** Since $F_i, G_i$, are continuous, for $i = 1, 2$, so, the mappings defined by (4.3) are continuous. By conditions $(b)$ and $(c)$, $P_1(C(\hat{S})) \subset Q_2(C(\hat{S}))$ and $P_2(C(\hat{S})) \subset Q_1(C(\hat{S}))$ and the pairs $(P_1, Q_1)$ and $(P_2, Q_2)$ are non-compatible and faintly compatible. For $\eta > 0$, $u \in \hat{S}$ and $k_1, k_2 \in C(\hat{S})$, there exists $v_1, v_2 \in D$ such that

$$P_i k_i(u) < F_i(u, v_i, k_i(u_i)) + \eta, \tag{4.4}$$

where $u_i = \tau(u, v_i), i = 1, 2$. Also, we have

$$P_1 k_1(u) \ge F_1(u, v_2, k_1(u_2)), \tag{4.5}$$

$$P_2 k_2(u) \ge F_2(u, v_1, k_2(u_1)). \tag{4.6}$$

From (4.4),(4.6) and condition $(a)$, we have

$$\begin{aligned} (P_1 k_1(u) - P_2 k_2(u))^2 <& (F_1(u, v_1, k_1(u_1)) - F_2(u, v_1, k_2(u_1)) + \eta)^2 \\ &= (F_1(u, v_1, k_1(u_1)) - F_2(u, v_1, k_2(u_1)))^2 + \xi, \\ &\le M^{-1}\Big(p\,\psi\big(d^2(Q_1 k_1, P_1 k_1) d(Q_2 k_2, P_2 k_2), \\ & d(Q_1 k_1, P_1 k_1) d^2(Q_2 k_2, P_2 k_2), \\ & d(Q_1 k_1, P_1 k_1) d(Q_1 k_1, P_2 k_2) d(Q_2 k_2, P_1 k_1), \\ & d(Q_1 k_1, P_2 k_2) d(Q_2 k_2, P_1 k_1) d(Q_2 k_2, P_2 k_2)\big) + \\ & m(Q_1 k_1, Q_2 k_2) - \phi(m(Q_1 k_1, Q_2 k_2))\Big) + \xi, \end{aligned} \tag{4.7}$$

where $\xi = \eta^2 + 2\eta(F_1 - F2)$.

From (4.4), (4.5) and condition $(a)$, we have

$$\begin{aligned}(P_1k_1(u) - P_2k_2(u))^2 >& (F_1(u, v_2, k_1(u_2)) - F_2(u, v_2, k_2(u_2)) - \eta)^2\\ &= (F_1(u, v_1, k_1(u_1)) - F_2(u, v_1, k_2(u_1)))^2 + \xi_1,\\ &\geq -M^{-1}\Big(p\,\psi\big(d^2(Q_1k_1, P_1k_1)d(Q_2k_2, P_2k_2),\\ &d(Q_1k_1, P_1k_1)d^2(Q_2k_2, P_2k_2),\\ &d(Q_1k_1, P_1k_1)d(Q_1k_1, P_2k_2)d(Q_2k_2, P_1k_1),\\ &d(Q_1k_1, P_2k_2)d(Q_2k_2, P_1k_1)d(Q_2k_2, P_2k_2)\big)+\\ &m(Q_1k_1, Q_2k_2) - \phi(m(Q_1k_1, Q_2k_2))\Big) - \xi,\end{aligned} \tag{4.8}$$

where $\xi_1 = \eta^2 - 2\eta(F_1 - F2) < \xi$.

From (4.7) and (4.8), we obtain

$$\begin{aligned}|P_1k_1(u) - P_2k_2(u)|^2 \leq & M^{-1}\Big(p\,\psi\big(d^2(Q_1k_1, P_1k_1)d(Q_2k_2, P_2k_2),\\ &d(Q_1k_1, P_1k_1)d^2(Q_2k_2, P_2k_2),\\ &d(Q_1k_1, P_1k_1)d(Q_1k_1, P_2k_2)d(Q_2k_2, P_1k_1),\\ &d(Q_1k_1, P_2k_2)d(Q_2k_2, P_1k_1)d(Q_2k_2, P_2k_2)\big)+\\ &m(Q_1k_1, Q_2k_2) - \phi(m(Q_1k_1, Q_2k_2))\Big) + \xi,\end{aligned} \tag{4.9}$$

As $\eta > 0$ is arbitrary, so $\xi$ is negligible and (4.9) is true for all $u \in \hat{S}$, taking supremum, we get

$$\begin{aligned}[1 + pd(Q_1k_1, Q_2k_2)]d^2(P_1k_1, P_2k_2) \leq & p\,\psi\big(d^2(Q_1k_1, P_1k_1)d(Q_2k_2, P_2k_2), d(Q_1k_1, P_1k_1)d^2(Q_2k_2, P_2k_2),\\ &d(Q_1k_1, P_1k_1)d(Q_1k_1, P_2k_2)d(Q_2k_2, P_1k_1),\\ &d(Q_1k_1, P_2k_2)d(Q_2k_2, P_1k_1)d(Q_2k_2, P_2k_2)\big)\\ &+ m(Q_1k_1, Q_2k_2) - \phi(m(Q_1k_1, Q_2k_2)).\end{aligned}$$

All the hypotheses of Corollary 3.2 are satisfied. So, $P_1, P_2, Q_1$ and $Q_2$ have a unique common fixed point $k^* \in C(\hat{S})$, i.e., $k^*(u)$ is a unique common solution of the system of functional equations (4.1) and (4.2). □

**Remark 4.1** Corollary 3.1 is also applicable, if the conditions $(b)$ and $(c)$ in the above Theorem 4.1 are replaced with either of the following conditions.

(d) There exists numbers $R, R' > 0$ such that $d(Q_1P_1h, P_1Q_1h) \leq Rd(Q_1h, P_1h)$ and $d(Q_2P_2h, P_2Q_2h) \leq R'd(Q_2h, P_2h)$, for all $h \in C(\hat{S})$.

(e) There exist numbers $R, R' > 0$ such that $d(Q_1P_1h, P_1P_1h) \leq Rd(Q_1h, P_1h)$ and $d(Q_2P_2h, P_2P_2h) \leq R'd(Q_2h, P_2h)$, for all $h \in C(\hat{S})$.

(f) There exist numbers $R, R' > 0$ such that $d(Q_1Q_1h, P_1Q_1h) \leq Rd(Q_1h, P_1h)$ and $d(Q_2Q_2h, P_2Q_2h) \leq R'd(Q_2h, P_2h)$, for all $h \in C(\hat{S})$.

(g) There exist numbers $R, R' > 0$ such that $d(Q_1Q_1h, P_1P_1h) \leq Rd(Q_1h, P_1h)$ and $d(Q_2Q_2h, P_2P_2h) \leq R'd(Q_2h, P_2h)$, for all $h \in C(\hat{S})$.

Taking the rest of the conditions as it is in the Theorem 4.1.

## 5. Conclusion

Theorem 2.1 and Corollary 3.1 are improved versions of the result of Jain *et al.* [19, Theorem 2] with the use of control function, generalized weak integral contraction condition and in the manner that containment, continuity of the mappings are relaxed. Theorems 2.1 and 2.2 and Corollaries 3.1 and 3.2 generalize the results of Jain *et al.* [18,20], Kang *et al.* [24], Murthy and Prasad [30] and Kumar *et al.* [29] in various aspects. Theorem 2.2 and Corollary 3.2 generalize the results of Bisht and Shahzad [10], Chandra *et al.* [14], Rani *et al.* [38] for pairs of faintly compatible mappings. Theorem 4.1 and Remark 4.1 show the applicability of obtained results in finding the common solution of certain systems of functional equations arising in dynamic programming.

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