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Skew-product and peripheral local spectrum preservers

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ABSTRACT: Consider two infinite-dimentional complex Hilbert spaces, denoted as $\mathcal{H}$ and $\mathcal{K}$. Choose two nonzero vectors $h_0 \in \mathcal{H}$ and $k_0 \in \mathcal{K}$. Let $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{K})$ represent the algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{K}$, respectively. Additionally, $\gamma_T(x)$ stands for the peripheral local spectrum of an operator T at x, and $\mathcal{F}_n(\mathcal{K})$ represents the ideal of operators in $\mathcal{L}(\mathcal{K})$ with a rank at most n. Our goal is to demonstrate that if the maps $\phi_1 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ and $\phi_2 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ satisfy the condition

$$\gamma_{TS^*}(h_0) = \gamma_{\phi_1(T)\phi_2(S)^*}(k_0)$$

for all $T, S \in \mathcal{L}(\mathcal{H})$ and their ranges contain $\mathcal{F}_2(\mathcal{K})$, then there exist bijective linear operators $U : \mathcal{H} \longrightarrow \mathcal{K}$ and $V : \mathcal{K} \longrightarrow \mathcal{H}$ such that $\phi_1(T) = UTV$ and $\phi_2(T) = V^{-1}TU^{-1}$ for all $T \in \mathcal{L}(\mathcal{H})$. Moreover, we derive some interesting results in this direction.

Key Words: Peripheral local spectrum, nonlinear preserver problem, skew-product, finite rank operators.

Contents

1. Introduction

The study of linear and nonlinear local spectra preserver problems has attracted a lot of authors. The first result is due to Bourhim and Ransford [11]. It characterizes tha map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ verifying

$$\sigma_{\phi(T)}(x) = \sigma_T(x) \quad (T \in \mathcal{B}(X)), \quad x \in X.$$

During the last years others authors consider the more general problem of characterization of linear or nonlinear maps on matrices or operators preserving different local spectral sets and quantities such as the local spectrum, the peripheral local spectrum, the local spectral radius and the local inner spectral radius; see for instance [2,3,4,5,7,8,10,13,15,16].

In [8] and [10], Bourhim and Mashreghi have described all surjective maps on $B(X)$ preserving the local spectrum at a nonzero fixed vector of product and triple product of operators. In [3], Bendaoud, Jabbar and Sarih described maps preserving the local spectrum of the product and the triple product of matrices. In [5], Bourhim, Jari and Mashreghi characterized maps preserving the peripheral local spectrum at a nonzero fixed vector of double and triple product of operators. In [17], Parvinianzadeh, Asadipour and Pazhman characterized maps preserving local spectrum of skew product of operators. Many other results on this research area can be found on [9].

Recently, Bourhim and Lee provided a comprehensive analysis in [6] concerning the structure of surjective maps ϕ_1 and ϕ_2 on $B(X)$ that satisfy the condition of having the same local spectrum of $\phi_1(T)\phi_2(S)$ and TS at a nonzero fixed vector x_0 in X, for all T and S in $B(X)$. Building upon the insights presented by Bourhim and Lee in [6], this paper describes the complete form of maps ϕ_1 and ϕ_2 from $\mathcal{L}(\mathcal{H})$ into $\mathcal{L}(\mathcal{K})$, ensuring that the peripheral local spectrum of TS^* and $\phi_1(T)\phi_2(S)^*$ remains

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the same at a nonzero fixed vector, for every T and S in $\mathcal{L}(\mathcal{H})$. Consequently, our work extends their findings comprehensively. Specifically, our results offer a generalization by considering cases where the peripheral local spectrum forms a proper subset of the local spectrum. It is worth noting that, without imposing any restrictions on the range of the maps ϕ_1 and ϕ_2, our findings do not hold.

2. Main result

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ denote infinite-dimentional complex Hilbert spaces. As per usual $\mathcal{L}(\mathcal{H},\mathcal{K})$ denotes the space of all bounded linear maps from $\mathcal{H}$ into $\mathcal{K}$. When $\mathcal{H}=\mathcal{K}$, we simply write $\mathcal{L}(\mathcal{H})$ instead of $\mathcal{L}(\mathcal{H},\mathcal{H})$. The local resolvent set, $\rho_T(x)$, of an operator $T\in\mathcal{L}(\mathcal{H})$ at a point $x\in\mathcal{H}$ is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $\psi: U\rightarrow\mathcal{H}$ such that $(T-\lambda)\psi(\lambda)=x$, $(\lambda\in U)$. The local spectrum of an operator $T\in\mathcal{L}(\mathcal{H})$ at x is defined as follows

$$\sigma_T(x)=\mathbb{C}\backslash\rho_T(x),$$

and is clearly a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T. The local spectral radius of T at x is given by $r_T(x):=\limsup_{n\to+\infty}\|T^nx\|^{\frac{1}{n}}$. If the operator T has the single-valued extension property (SVEP), $r_T(x)$ coincides with maximum modulus of $\sigma_T(x)$. Let us recall that $T\in\mathcal{L}(\mathcal{H})$ is said to have SVEP, if for every open subset U of $\mathbb{C}$, the equation $(T-\lambda)\psi(\lambda)=0$, $(\lambda\in U)$ has no nontrivial analytic solution ψ. Every operator $T\in\mathcal{L}(\mathcal{H})$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty verified this property. For more information on the local spectral theory, we can consult the books by P. Aiena [1] and by K. B. Laursen and M. M. Neumann [14].
Let

$$\gamma_T(x):=\{\lambda\in\sigma_T(x):|\lambda|=r_T(x)\}$$

be the peripheral local spectrum of T at x, and note that $\gamma_T(x)=\emptyset$ provided that $max\{|\lambda|:\lambda\in\sigma_T(x)\}<r_T(x)$. Clearly

$$\gamma_T(x)\subseteq\sigma_T(x).$$

The goal of this work is to prove the following result.

Theorem 2.1 *Consider two nonzero vectors $h_0\in\mathcal{H}$ and $k_0\in\mathcal{K}$. Let $\phi_1:\mathcal{L}(\mathcal{H})\rightarrow\mathcal{L}(\mathcal{K})$ and $\phi_2:\mathcal{L}(\mathcal{H})\rightarrow\mathcal{L}(\mathcal{K})$ be two maps such that their range containing $\mathcal{F}_2(\mathcal{K})$. If ϕ_1 and ϕ_2 satisfying the condition*

$$\gamma_{TS^*}(h_0)=\gamma_{\phi_1(T)\phi_2(S)^*}(k_0),\quad (T,S\in\mathcal{L}(\mathcal{H})), \tag{2.1}$$

then there exist bijective linear operators U from $\mathcal{H}$ into $\mathcal{K}$ and V from $\mathcal{K}$ into $\mathcal{H}$ such that $\phi_1(T)=UTV$ and $\phi_2(T)=V^{-1}TU^{-1}$ for all $T\in\mathcal{L}(\mathcal{H})$.

3. Preliminaries

In this section, we will recall some basic notions on the local spectral theory that needed later, as well as some results that necessary to prove the main result. The following lemma gathers some elementary properties of the local spectrum which will be used later.

Lemma 3.1 *Let $T\in\mathcal{L}(\mathcal{H})$ be an operator. For vectors $x,y\in\mathcal{H}$ and a non zero scalar $\alpha\in\mathbb{C}$, the following statements hold.*

1. *$\sigma_T(\alpha x)=\sigma_T(x)$ and $\sigma_{\alpha T}(x)=\alpha\sigma_T(x)$.*
2. *If T has a SVEP, $x\neq 0$ and $Tx=\lambda x$ for some $\lambda\in\mathbb{C}$, then $\sigma_T(x)=\{\lambda\}$.*

Proof: See for instance [1,14]. □

For every nonzero vectors $x,y\in\mathcal{H}$, let $x\otimes y$ denote the rank one operator on $\mathcal{H}$ defined by

$$(x\otimes y)z=<z,y>x,\quad \text{for all } z\in\mathcal{H}.$$

Recall that every rank one operator in $\mathcal{L}(\mathcal{H})$ can be written in this way and that every finite rank operator is defined as a finite sum of rank one operators.

The following lemma is a restatement of Bourhim and Mashreghi [8].

Lemma 3.2 *(See [8, Lemma 2.2]) Let $h_0 \in \mathcal{H} \setminus \{0\}$. For two vectors $x, y \in \mathcal{H}$, the following statements hold.*

1.
$$\gamma_{x\otimes y}(h_0) = \begin{cases} \{0\} & \text{if} \quad \langle h_0, y\rangle = 0 \\ \{< x, y >\} & \text{if} \quad \langle h_0, y\rangle \neq 0 \end{cases}$$

2. *For all rank one operator $R \in \mathcal{L}(\mathcal{H})$ and all $A, B \in \mathcal{L}(\mathcal{H})$, we have*
$$\gamma_{(A+B)R}(h_0) = \gamma_{AR}(h_0) + \gamma_{BR}(h_0).$$

The following lemma, which is a restatement of Bourhim and Mashreghi [8], gives the necessary and sufficient conditions for two operators to be identical in term of the peripheral local spectrum.

Lemma 3.3 *(See [8, Theorem 3.2]) For a nonzero vector $h_0 \in \mathcal{H}$ and two operators $A, B \in \mathcal{L}(\mathcal{H})$, the following statements are equivalent.*

1. *$A = B$.*
2. *$\gamma_{AT^*}(h_0) = \gamma_{BT^*}(h_0)$ for all $T \in \mathcal{L}(\mathcal{H})$.*
3. *$\gamma_{AT^*}(h_0) = \gamma_{BT^*}(h_0)$ for all $T \in \mathcal{F}_1(\mathcal{H})$.*

The following result is a reformulation of Bourhim and Mashregh [8], and which characterizes all rank one operators in term of the peripheral local spectrum.

Lemma 3.4 *(See [8, Theorem 4.1]) If h_0 is a nonzero vector in $\mathcal{H}$ and R is a nonzero operator in $\mathcal{L}(\mathcal{H})$, then following statements are equivalent.*

1. *R is a rank one operator.*
2. *$\gamma_{RT^*}(h_0)$ is a singleton for all $T \in \mathcal{L}(\mathcal{H})$.*
3. *$\gamma_{RT^*}(h_0)$ is a singleton for all $T \in \mathcal{F}_2(\mathcal{H})$.*

We present an essential theorem needed to complement the proof of our main results. It is important to observe that when $h : \mathbb{C} \longrightarrow \mathbb{C}$ is a ring homomorphism, an h-quasilinear operator $A : \mathcal{H} \longrightarrow \mathcal{H}$ is defined as an additive map with the property $A(\alpha x) = h(\alpha)x$, where $(x \in \mathcal{H}, \quad \alpha \in \mathbb{C})$. The following theorem, cited from [16], will be utilized in establishing the proof for Theorem 2.1.

Theorem 3.1 *(See [16]) Let $\mathcal{H}$ be a infinite-dimentional complex Hilbert space. Suppose that $\phi : \mathcal{F}(\mathcal{H}) \longrightarrow \mathcal{F}(\mathcal{H})$ is an additive bijective mapping preserving operators of rank one in both directions. Then there exists a ring automorphism $h : \mathbb{C} \longrightarrow \mathbb{C}$ and there are either h-quasilinear bijective mappings $A : \mathcal{H} \longrightarrow \mathcal{H}$ and $B : \mathcal{H} \longrightarrow \mathcal{H}$ such that*
$$\phi(x \otimes y) = Ax \otimes By, \quad \text{for all} x, y \in \mathcal{H}$$
or h-quasilinear bijective mappings $C : \mathcal{H} \longrightarrow \mathcal{H}$ and $D : \mathcal{H} \longrightarrow \mathcal{H}$ such that
$$\phi(x \otimes y) = Cy \otimes Dx, \quad \text{for all} x, y \in \mathcal{H}.$$

Proof: See Theorem 3.3 in [16]. □

4. Proof of the main result

In this section, we proceed to prove our main result, i.e. Theorem 2.1. The proof is long, we divided it into eight steps.

Step 1. *ϕ_1 is injective and $\phi_1(0) = 0$.*
If $\phi_1(A) = \phi_1(B)$ for some $A, B \in \mathcal{L}(\mathcal{H})$, we obtain

$$\begin{aligned}\gamma_{AT^*}(h_0) &= \gamma_{\phi_1(A)\phi_2(T)^*}(k_0)\\ &= \gamma_{\phi_1(B)\phi_2(T)^*}(k_0)\\ &= \gamma_{BT^*}(h_0)\end{aligned}$$

for all $T \in \mathcal{L}(\mathcal{H})$. By Lemma 3.3, we deduce that $A = B$ and thus ϕ_1 is injective.

For the second part, let $T \in \mathcal{L}(\mathcal{K})$ be an operator, we have

$$\begin{aligned}\gamma_{\phi_1(0)\phi_2(T)^*}(k_0) &= \gamma_{0T^*}(h_0)\\ &= \{0\}\\ &= \gamma_{0\phi_2(T)^*}(k_0)\end{aligned}$$

Using Lemma 3.3 and the fact that the range of ϕ_2 contains all rank one operators, we get that $\phi_1(0) = 0$.

Step 2. *ϕ_1 preserves rank one operators in both directions.*
We claim that

$$\sigma_T(h_0) = 0 \Leftrightarrow \gamma_T(h_0) = 0 \Leftrightarrow \gamma_{\phi_1(T)}(k_0) = 0 \Leftrightarrow \sigma_{\phi_1(T)}(k_0) = 0$$

for all $T \in \mathcal{L}(\mathcal{H})$. Let $R \in \mathcal{L}(\mathcal{H})$ be a rank one operator. Note that $\phi_1(R) \neq 0$. Thus Lemma 3.4 tell us that

$$\gamma_{RT^*}(h_0) = \gamma_{\phi_1(R)\phi_2(T)^*}(k_0)$$

is a singleton for all $T \in \mathcal{L}(\mathcal{H})$. Moreover, since the range of ϕ_2 contains $\mathcal{F}_2(\mathcal{K})$, then $\gamma_{\phi_1(R)S^*}(k_0)$ is a singleton for all $S \in \mathcal{F}_2(\mathcal{K})$. Lemma 3.4 implies that $\phi_1(R)$ has rank one.

Conversely, assume that $\phi_1(R)$ has rank one for some operator $R \in \mathcal{L}(\mathcal{H})$. The fact that $R \neq 0$ and $\gamma_{\phi_1(R)\phi_2(T)^*}(k_0)$ is a singleton for all $T \in \mathcal{L}(\mathcal{H})$, we see that $\gamma_{RT^*}(h_0)$ is a singleton for all $T \in \mathcal{L}(\mathcal{H})$. From Lemma 3.4, we deduce that R has a rank one.

Step 3. *ϕ_1 is linear.*
Firstly, let us show that ϕ_1 is homogeneous. For every $\lambda \in \mathbb{C}$ and $T \in \mathcal{L}(\mathcal{H})$, we have

$$\begin{aligned}\gamma_{\lambda\phi_1(T)\phi_2(R)^*}(k_0) &= \lambda\gamma_{\phi_1(T)\phi_2(R)^*}(k_0)\\ &= \lambda\gamma_{TR^*}(h_0)\\ &= \gamma_{(\lambda T)R^*}(h_0)\\ &= \gamma_{\phi_1(\lambda T)\phi_2(R)^*}(k_0).\end{aligned}$$

Lemma 3.3 shows that $\phi_1(\lambda T) = \lambda\phi_1(T)$.

To complete the proof and show that ϕ_1 is additive, let R be a rank one operator in $\mathcal{L}(\mathcal{H})$. Then, Step 2 implies that $\phi_1(R)$ has a rank one. Let $T, S \in \mathcal{L}(\mathcal{H})$, Lemma 3.2 shows that

$$\begin{aligned}\gamma_{\phi_1(T+S)\phi_2(R)^*}(k_0) &= \gamma_{(T+S)R^*}(h_0)\\ &= \gamma_{TR^*}(h_0) + \gamma_{SR^*}(h_0)\\ &= \gamma_{\phi_1(T)\phi_2(R)^*}(k_0) + \gamma_{\phi_1(S)\phi_2(R)^*}(k_0)\\ &= \gamma_{(\phi_1(T)+\phi_1(S))\phi_2(R))^*}(k_0)\end{aligned}$$

for all rank one operators $R \in \mathcal{L}(\mathcal{H})$. Now using Lemma 3.3, we have $\phi_1(T+S) = \phi_1(T) + \phi_1(S)$ and thus ϕ_1 is additive. Hence, ϕ_1 is linear.

Step 4. *There are bijective linear mappings A and B from $\mathcal{H}$ into $\mathcal{K}$ such that: $\phi_1(x \otimes y) = Ax \otimes By$ for all $x, y \in \mathcal{H}$.*

On Step 2, ϕ_1 restricted to $\mathcal{F}(\mathcal{H})$ is bijective linear maps which preserves rank one operators in both directions. By Theorem 3.1, either there are bijective linear mappings $A, B : \mathcal{H} \to \mathcal{K}$ such that

$$\phi_1(x \otimes y) = Ax \otimes By, \quad (x, y \in \mathcal{H}), \tag{4.1}$$

or there are bijective linear mappings $C, D : \mathcal{H} \to \mathcal{K}$ such that

$$\phi_1(x \otimes y) = Cy \otimes Dx, \quad (x, y \in \mathcal{H}). \tag{4.2}$$

Suppose by the way of contradiction that ϕ_1 take the second form (4.2). We have $\gamma_{R_1R_2^*}(h_0) = \gamma_{\phi_1(R_1)\phi_2(R_2)^*}(k_0)$ for all rank one operators $R_1, R_2 \in \mathcal{F}_1(\mathcal{H})$. Let $z \in \mathcal{H}$ a nonzero vector such that $\langle k_0, \phi_2(\mathbf{1})(z)\rangle = \langle \phi_2(\mathbf{1})^*(k_0), z\rangle = 0$ and $x = D^{-1}z$. Since x and h_0 are nonzero vectors, there exists a nonzero $y \in \mathcal{H}$ such that $\langle h_0, y\rangle \neq 0$ and $\langle x, y\rangle \neq 0$.
So we have

$$\begin{aligned}
\{0\} &= \gamma_{(Cy \otimes z)\phi_2(\mathbf{1})^*}(k_0) \\
&= \gamma_{(Cy \otimes Dx)\phi_2(\mathbf{1})^*}(k_0) \\
&= \gamma_{\phi_1(x \otimes y)\phi_2(\mathbf{1})^*}(k_0) \\
&= \gamma_{x \otimes y}(h_0) \\
&= \{\langle x, y\rangle\}.
\end{aligned}$$

This is a contradiction and thus ϕ_1 takes the form (4.1) when it is restricted on $\mathcal{F}(\mathcal{H})$.

Step 5. *For any $x, y \in \mathcal{H}$, we have $\langle x, y\rangle = \langle Ax, \phi_2(\mathbf{1})(By)\rangle$.*

Let $x, y \in \mathcal{H}$ be two vectors. By the previous step and $(2, 1)$, we have

$$\gamma_{x \otimes y}(h_0) = \gamma_{\phi_1(x \otimes y)\phi_2(\mathbf{1})^*}(k_0) = \gamma_{(Ax \otimes By)\phi_2(\mathbf{1})^*}(k_0)$$

Case 1: If $\langle h_0, y\rangle \neq 0$, then by Lemma 3.2, we have:

$$\begin{aligned}
\{\langle x, y\rangle\} &= \gamma_{x \otimes y}(h_0) \\
&= \gamma_{\phi_1(x \otimes y)\phi_2(\mathbf{1})^*}(k_0) \\
&= \gamma_{(Ax \otimes By)\phi_2(\mathbf{1})^*}(k_0) \\
&= \gamma_{(Ax \otimes \phi_2(\mathbf{1})By)}(k_0) \\
&= \{\langle Ax, \phi_2(\mathbf{1})(By)\rangle\}.
\end{aligned}$$

This shows that $\langle k_0, \phi_2(\mathbf{1})(By)\rangle \neq 0$ and $\langle Ax, \phi_2(\mathbf{1})(By)\rangle = \langle x, y\rangle$.

Case 2: If $\langle h_0, y\rangle = 0$, we pick up easily a vector v in $\mathcal{H}$ such that $\langle h_0, v\rangle \neq 0$ and $\langle h_0, y + v\rangle \neq 0$. Using the same approach of the first case, we find that $\langle x, v\rangle = \langle Ax, \phi_2(\mathbf{1})(Bv)\rangle$ and $\langle x, y + v\rangle = \langle Ax, \phi_2(\mathbf{1})(B(y + v))\rangle$ Hence,

$$\begin{aligned}
\langle x, y\rangle + \langle x, v\rangle &= \langle x, y + v\rangle \\
&= \langle Ax, \phi_2(\mathbf{1})(B(y + v))\rangle \\
&= \langle Ax, \phi_2(\mathbf{1})(By)\rangle + \langle Ax, \phi_2(\mathbf{1})(Bv)\rangle \\
&= \langle Ax, \phi_2(\mathbf{1})(By)\rangle + \langle x, v\rangle.
\end{aligned}$$

Therefore, $\langle x, y\rangle = \langle Ax, \phi_2(\mathbf{1})(By)\rangle$ in this case too.

Step 6. *A is continuous and $\phi_2(\mathbf{1})^*$ is invertible.*

Firstly, let us show that $\phi_2(\mathbf{1})^*$ is injective. Suppose by the way of contradiction that $\phi_2(\mathbf{1})^*$ is not injective, and let $x \in \mathcal{H}$ and $y \in \mathcal{K}$ be a nonzero vectors such that $\phi_2(\mathbf{1})^* y = 0$ and $x = A^{-1}y$. Take $v \in \mathcal{H}$ a vector such that $\langle x, v\rangle \neq 0$. By the previous steps, we have

$$\begin{aligned} 0 = \langle \phi_2(\mathbf{1})^* y, Bv\rangle &= \langle y, \phi_2(\mathbf{1})(B(v))\rangle \\ &= \langle Ax, \phi_2(\mathbf{1})(Bv)\rangle \\ &= \langle x, v\rangle \neq 0. \end{aligned}$$

This contradiction asserts that $\phi_2(\mathbf{1})^*$ is injective.

Secondly, let us now show that A is continuous. Let $(x_n)_n$ be a sequence in $\mathcal{H}$ converging to $x \in \mathcal{H}$, and let $y \in \mathcal{K}$ such that $\lim_{n\to+\infty} Ax_n = y$. For every $v \in \mathcal{H}$, we have

$$\begin{aligned} \langle y, \phi_2(\mathbf{1})(B(v))\rangle &= \lim_{n\to+\infty} \langle Ax_n, \phi_2(\mathbf{1})(B(v))\rangle \\ &= \lim_{n\to+\infty} \langle x_n, v\rangle \\ &= \langle x, v\rangle \\ &= \langle Ax, \phi_2(\mathbf{1})(B(v))\rangle. \end{aligned}$$

Given that $\phi_2(\mathbf{1})^*$ is an injective mapping and $v \in \mathcal{H}$ is an arbitrary vector, the closed graph theorem provides evidence of A's continuity. Consequently, we observe that $\langle x, A^*\phi_2(\mathbf{1})B(y)\rangle = \langle Ax, \phi_2(\mathbf{1})B(y)\rangle = \langle x, y\rangle$ for all $x, y \in \mathcal{H}$. From this, we can deduce that $A^*\phi_2(\mathbf{1})B = \mathbf{1}$. As a result, $\phi_2(\mathbf{1})^*$ is invertible.

Step 7. *$Ah_0 = \alpha k_0$ for some nonzero scalar $\alpha \in \mathbb{C}$.*

Suppose for the sake of contradiction that Ah_0 and k_0 are linearly independent and choose a nonzero vector $y \in \mathcal{H}$ such that $\langle h_0, y\rangle = 1$ and $\langle A^{-1}k_0, y\rangle = 0$. Therefore,

$$(h_0 \otimes y)h_0 = h_0 \quad \text{and} \quad A(h_0 \otimes y)(A^{-1})k_0 = 0,$$

then

$$\gamma_{h_0\otimes y}(h_0) = \{1\} \quad \text{and} \quad \gamma_{A(h_0\otimes y)(A^{-1})}(k_0) = \{0\}.$$

Hence,

$$\begin{aligned} \{1\} &= \gamma_{h_0\otimes y}(h_0) \\ &= \gamma_{\phi_1(h_0\otimes y)\phi_2(\mathbf{1})^*}(k_0) \\ &= \gamma_{(Ah_0\otimes By)\phi_2(\mathbf{1})^*}(k_0) \\ &= \gamma_{A(h_0\otimes y)A^{-1}}(k_0) \\ &= \{0\}. \end{aligned}$$

This contradiction establishes the claim.

Step 8. *ϕ_1 and ϕ_2 takes the desired forms.*

Choose two bijective linear maps U and V such that $U = \alpha^{-1}A$ for some nonzero scalar α in $\mathbb{C}$ and $V = (\phi_2(\mathbf{1})^* U)^{-1}$. By Step 6, we have $\phi_2(\mathbf{1})^{-1} = BA^*$, and thus $B = \phi_2(\mathbf{1})^{-1}(A^*)^{-1}$. From this, it follows that

$$\begin{aligned} \phi_1(x\otimes y) &= Ax \otimes By \\ &= A(x\otimes y)B^* \\ &= A(x\otimes y)(\phi_2(\mathbf{1})^* A)^{-1} \\ &= \alpha U(x\otimes y)(\phi_2(\mathbf{1})^* \alpha U)^{-1} \\ &= U(x\otimes y)V \end{aligned}$$

for all $x, y \in \mathcal{H}$. Hence,

$$\begin{aligned}\gamma_{U(x\otimes y)V\phi_2(T)^*}(k_0) &= \gamma_{\phi_1(x\otimes y)\phi_2(T)^*}(k_0)\\ &= \gamma_{(x\otimes y)T^*}(h_0)\\ &= \gamma_{(x\otimes y)T^*}(U^{-1}k_0)\\ &= \gamma_{U(x\otimes y)T^*U^{-1}}(k_0)\\ &= \gamma_{U(x\otimes y)VV^{-1}T^*U^{-1}}(k_0),\end{aligned}$$

for any $x, y \in \mathcal{H}$ and $T \in \mathcal{L}(\mathcal{H})$. Lemma 3.3 tells us that $\phi_2(T)^* = V^{-1}T^*U^{-1}$ for every $T \in \mathcal{L}(\mathcal{H})$.

Now, observe that

$$\begin{aligned}\gamma_{UTV\phi_2(S)^*}(k_0) &= \gamma_{UTVV^{-1}S^*U^{-1}}(k_0)\\ &= \gamma_{TS^*}(U^{-1}k_0)\\ &= \gamma_{TS^*}(h_0)\\ &= \gamma_{\phi_1(T)\phi_1(S)^*}(k_0),\end{aligned}$$

for all $T, S \in \mathcal{L}(\mathcal{H})$. Again, by Lemma 3.3, we see that $\phi_1(T) = UTV$ for every $T \in \mathcal{L}(\mathcal{H})$. The proof is then complete. ■

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