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On Zweier I-Convergent Triple Sequence Spaces

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ABSTRACT: *Present work is an investigation of Ideal convergent triple sequence spaces via Zweier operator over an admissible ideal of* $\mathbf{N}\times\mathbf{N}\times\mathbf{N}$ *and study some of their algebraic and topological properties like convergence free, symmetricity, solidity, monotonicity etc. Also, we have proved some inclusion relations of these spaces.*

Key Words: Zweier Space, Triple sequence, Lipschitz function, I-convergence field, Ideal, filter.

Contents

1. Introduction

A triple sequence (Real or complex) is a function $x:\ \mathbf{N}\times\mathbf{N}\times\mathbf{N}\rightarrow\mathbf{R}(\mathbf{C})$, where $\mathbf{N},\mathbf{R}$ and $\mathbf{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence spaces was introduced and investigated at the initial by Sahiner et al. [26,27], Datta et al. [5], Debnath et al. [6], Esi et al. [7,8,9] and many others.

Let ω^3 denote the class of all complex triple sequences (x_{nkl}) where $n,k,l\in\mathbf{N}$. Then the classes of triple sequences l_∞^3, c^3 and c_0^3 denote the triple sequence spaces which are bounded in Pringsheim's sense, convergent in Pringsheim's sense and convergent to zero in Pringsheim's sense respectively, normed by

$$\|x\|_\infty=\sup_{n,k,l}|x_{nkl}|,\qquad where\ \ n,k,l\in\mathbf{N}.$$

In 1900, Pringsheim introduced the notion of convergence of double sequences. Further, this concept has been studied by many authors(see for instance [24,32,33]). The idea of statistical convergence was first presented by Fast [10] and Schoenberg [29] independently. Also, Steinhaus [31] studied asymptotic convergence in 1951. The notion of I-convergence is a generalization of statistical convergence which was introduced by Kostyrko, Salat and Wilczynski [19]. Later on it was studied by Tripathy, Salat and Ziman [28] and many other researchers. [3,13,14]

A sequence space λ with linear topology is called a K-space provided each map $p_i\rightarrow\mathbf{C}$ defined by $p_i(x)=x_i$ is continuous for all $i\in\mathbf{N}$. A K-space λ is called an FK-space provided λ is a complete linear metric space. An FK-space whose topology is normable is called a BK-space. [18,21,22]

Let λ and μ be two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,k\in\mathbf{N}$. Then we say that A defines a matrix mapping from λ to μ, and we denote it by writting $A:\lambda\rightarrow\mu$.

If for every sequence $x=(x_k)\in\lambda$ the sequence $Ax=\{(Ax)_n\}$, the A transform of x is in μ, where

$$(Ax)_n=\sum_k a_{nk}x_k,\quad (n\in\mathbf{N}). \tag{1.1}$$

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By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (1.1) converges for each $n \in \mathbf{N}$ and every $x \in \lambda$.

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently been employed by Mursaleen, Altay and Başar [1], Altay and Başar [2], Malkowsky [20], Ng and Lee [23] and Wang [34].

Şengönül [30] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$ i.e.,

$$y_i = px_i + (1-p)x_{i-1}$$

where $x{-}1 = 0$, $1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, & (i = k), \\ 1-p, & (i-1 = k); (i, k \in \mathbf{N}), \\ 0, & \text{otherwise.} \end{cases}$$

Following Başar and Altay [2], Şengönül [30] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows:

$$\mathcal{Z} = \{x = (x_k) \in \omega : Z^p x \in c\}$$
$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}$$

Here we list below some basic definitions and examples used in this article.

Definition 1.1 [25] A family $I \subset 2^{\mathbf{Y}\times\mathbf{Y}\times\mathbf{Y}}$ of subsets of a non empty set Y is said to be an ideal in Y if

(i) $\emptyset \in I$,

(ii) $A, B \in I$ implies $A \cup B \in I$,

(iii) $A \in I, B \subset A$ imples $B \in I$.

Definition 1.2 [25] A non-empty family of sets $F \subset 2^{\mathbf{X}\times\mathbf{X}\times\mathbf{X}}$ is a filter on X if and only if

(i) $\emptyset \notin F$,

(ii) $A, B \in F$ implies $A \cap B \in F$,

(iii) $A \in F$ and $A \subset B$ implies $B \in F$.

Definition 1.3 [25] An ideal I is called a non-trivial ideal if $I \neq \emptyset$ and $X \notin I$. Clearly $I \subset 2^{\mathbf{X}\times\mathbf{X}\times\mathbf{X}}$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X.

Definition 1.4 [25] A non-trivial ideal $I \subset 2^{\mathbf{X}\times\mathbf{X}\times\mathbf{X}}$ is called

(i) admissible if and only if $\{\{x\} : x \in X\} \subset I$,

(ii) maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

Definition 1.5 [26] A triple sequence (x_{nkl}) is said to be convergent to L in pringsheim's sense if for every $\epsilon > 0$, there exists $N(\epsilon) \in \mathbf{N}$ such that

$$|x_{nkl} - L| < \epsilon \ \ whenever \ \ n \geq N, \ \ k \geq N, \ \ l \geq N.$$

Example 1.1 [26] Let

$$x_{nkl} = \begin{cases} kl, & n = 3 \\ nl, & k = 5 \\ nk, & l = 7 \\ 8, & \text{otherwise.} \end{cases}$$

Then $(x_{nkl}) \to 8$ in Pringsheim's sense.

Note: A triple sequence convergent in Pringsheim's sense is not necessarily bounded [4].

Example 1.2 [4] consider the sequence (x_{nkl}) defined by

$$x_{nkl} = \begin{cases} n, & \text{for all } n \in \mathbf{N},\ k = 1 = l \\ \frac{1}{n+k+l}, & \text{otherwise.} \end{cases}$$

Then $x_{nkl} \to 0$ in Pringsheim's sense but is unbounded.

Definition 1.6 [26] A triple sequence (x_{nkl}) is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $N(\epsilon) \in \mathbf{N}$ such that

$$|x_{nkl} - x_{pqr}| < \epsilon \ \ whenever \ \ n \geq p \geq N, \ \ k \geq q \geq N, \ \ l \geq r \geq N.$$

Definition 1.7 [26] A triple sequence (x_{nkl}) is said to be bounded if there exists $M > 0$ such that $|x_{nkl}| < M$ for all n,k,l.

Definition 1.8 [4] A triple sequence (x_{nkl}) is said to be I-convergent to a number L if for every $\epsilon > 0$, such that

$$\{(n,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |x_{nkl} - L| \geq \epsilon\} \in I.$$

in this case we write $I - \lim x_{nkl} = L$.

Definition 1.9 [4] A triple sequence (x_{nkl}) is said to be I-null if $L = 0$. In this case we write $I - \lim x_{nkl} = 0$.

Definition 1.10 [4] A triple sequence (x_{nkl}) is said to be I-Cauchy if for every $\epsilon > 0$, there exists $p = p(\epsilon), q = q(\epsilon)$ and $r = r(\epsilon)$ such that

$$\{(n,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |x_{nkl} - x_{pqr}| \geq \epsilon\} \in I.$$

Definition 1.11 [4] A triple sequence (x_{nkl}) is said to be I-bounded if there exists $M > 0$ such that

$$\{(n,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |x_{nkl}| > M\} \in I.$$

Definition 1.12 [4] A triple sequence space E is said to be solid if $(\alpha_{nkl} x_{nkl}) \in E$ whenever $(x_{nkl}) \in E$ and for all sequences (α_{nkl}) of scalars with $|\alpha_{nkl}| \leq 1$, for all $n, k, l \in \mathbf{N}$.

Definition 1.13 [4] A triple sequence space E is said to be symmetric if $(x_{nkl}) \in E$ implies $(x_{\pi(n,k,l)}) \in E$, where π is a permutation of $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$.

Definition 1.14 [4] A triple sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

Definition 1.15 [4] A triple sequence space E is said to be convergence free if $(y_{nkl}) \in E$, whenever $(x_{nkl}) \in E$ and $x_{nkl} = 0$ implies $y_{nkl} = 0$.

Definition 1.16 [4] A triple sequence space E is said to be sequence algebra if $(x_{nkl} \star y_{nkl}) \in E$, whenever $(x_{nkl}) \in E$ and $y_{nkl} \in E$.

The following Lemmas will be used for establishing some results of this article:

Lemma 1.1 *Let E be a sequence space. If E is solid then E is monotone.(see [12,13,15,11])*

Lemma 1.2 *If $I \subset 2^{\mathbf{N}}$ and $M \subseteq \mathbf{N}$. If $M \notin I$, Then $M \cap \mathbf{N} \notin I$.(see [14,16,17])*

2. Main Results

In this section we introduce the following classes of sequence spaces.

$$_3\mathcal{Z}^I = \{x = (x_{nkl}) \in \omega^3 : \{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : I - \lim Z^p x = L, \quad for\ some\ \ L\}\} \in I$$

$$_3\mathcal{Z}_0^I = \{x = (x_{nkl}) \in \omega^3 : \{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : I - \lim Z^p x = 0\}\} \in I$$

$$_3\mathcal{Z}_\infty^I = \{x = (x_{nkl}) \in \omega^3 : \{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : \sup_{n,k,l} |Z^p x| < \infty\}\} \in I$$

we also denote by,

$$_3m_{\mathcal{Z}}^I = {}_3\mathcal{Z}_\infty \cap {}_3\mathcal{Z}^I \ \ and\ \ {}_3m_{\mathcal{Z}_0}^I = {}_3\mathcal{Z}_\infty \cap {}_3\mathcal{Z}_0^I$$

Theorem 2.1 *The classes of sequences* $_3\mathcal{Z}^I, {}_3\mathcal{Z}_0^I, {}_3m_{\mathcal{Z}}^I$ *and* $_3m_{\mathcal{Z}_0}^I$ *are linear spaces.*

Proof: We shall prove the result for the space $_3\mathcal{Z}^I$.
The proof for the other spaces will follow similarly.
Let $(x_{nkl}), (y_{nkl}) \in {}_3\mathcal{Z}^I$ and α, β be scalars. Then

$$I - \lim |x_{nkl} - L_1| = 0, \quad for\ some\ \ L_1 \in \mathbf{C};$$

$$I - \lim |y_{nkl} - L_2| = 0, \quad for\ some\ \ L_2 \in \mathbf{C};$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : |x_{nkl} - L_1| > \frac{\epsilon}{2}\} \in I, \qquad [1]$$

$$A_2 = \{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : |y_{nkl} - L_2| > \frac{\epsilon}{2}\} \in I. \qquad [2]$$

We have

$$|(\alpha x_{nkl} + \beta y_{nkl}) - (\alpha L_1 + \beta L_2)| \le |\alpha|(|x_{nkl} - L_1|) + |\beta|(|y_{nkl} - L_2|) \le |x_{nkl} - L_1| + |y_{nkl} - L_2|.$$

Now, by [1] and [2]

$$\{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : |(\alpha x_{nkl} + \beta y_{nkl}) - (\alpha L_1 + \beta L_2)| > \epsilon\} \subset A_1 \cup A_2.$$

Therefore $(\alpha x_{nkl} + \beta y_{nkl}) \in {}_3\mathcal{Z}^I$. Hence $_3\mathcal{Z}^I$ is a linear space.

Theorem 2.2 *The spaces* $_3m_{\mathcal{Z}}^I$ *and* $_3m_{\mathcal{Z}_0}^I$ *are normed linear spaces, normed by*

$$\|x_{nkl}\|_\star = \sup_{n,k,l} |x_{nkl}|.$$

Proof: The proof is straight forward in the view of Theorem 2.1 thus omitted.

Theorem 2.3 *A sequence* $x = (x_{nkl}) \in {}_3m_{\mathcal{Z}}^I$ *I-converges if and only if for every* $\epsilon > 0$ *there exists* $N_\epsilon = (r,s,t) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N}$ *such that*

$$\{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : |x_{nkl} - x_{N_\epsilon}| < \epsilon\} \in {}_3m_{\mathcal{Z}}^I$$

Proof: Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : |x_{nkl} - L| < \frac{\epsilon}{2} \in {}_3m_{\mathcal{Z}}^I \ \ for\ all\ \epsilon > 0.$$

Fix an $N_\epsilon = (n,k,l) \in B_\epsilon$. Then we have

$$|x_{N_\epsilon} - x_{nkl}| \leq |x_{N_\epsilon} - L| + |L - x_{nkl}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $(n,k,l) \in B_\epsilon$.
Hence $\{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : |x_{nkl} - x_{N_\epsilon}| < \epsilon\} \in {}_3m^I_{\mathcal{Z}}$.

conversly, suppose that $\{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : |x_{nkl} - x_{N_\epsilon}| < \epsilon\} \in {}_3m^I_{\mathcal{Z}}$.
That is $\{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : |x_{nkl} - x_{N_\epsilon}| < \epsilon\} \in {}_3m^I_{\mathcal{Z}}$ for all $\epsilon > 0$.
Then the set

$$C_\epsilon = \{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : x_{nkl} \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in {}_3m^I_{\mathcal{Z}} \;\; for\ all\ \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$.
If we fix an $\epsilon > 0$ then we have $C_\epsilon \in {}_3m^I_{\mathcal{Z}}$ as well as $C_{\frac{\epsilon}{2}} \in {}_3m^I_{\mathcal{Z}}$.
Hence, $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in {}_3m^I_{\mathcal{Z}}$.
This implies that

$$J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : x_{nkl} \in J\} \in {}_3m^I_{\mathcal{Z}}$$

that is

$$diam J \leq diam J_\epsilon$$

where the diam of J denotes the length of the interval J.
In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = J_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diam I_k \leq \frac{1}{2} diam I_{k-1}$ for $(k = 2,3,4,...)$ and $\{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : x_{nkl} \in I_k\} \in {}_3m^I_{\mathcal{Z}}$ for $(k = 1,2,3,4,.....)$.
Then there exists a $\xi \in \cap I_k$ where $(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N}$ such that $\xi = I - \lim x$, that is $L = I - \lim x$.

Theorem 2.4 *Let I be an admissible ideal. Then the following are equivalent.*

(a) $(x_{nkl}) \in {}_3m^I_{\mathcal{Z}}$;

(b) *there exists $(y_{nkl}) \in {}_3\mathcal{Z}$ such that $x_{nkl} = y_{nkl}$, for a.a.k.r.I;*

(c) *there exists $(y_{nkl}) \in {}_3\mathcal{Z}$ and $(z_{nkl}) \in {}_3\mathcal{Z}^I_0$ such that $x_{nkl} = y_{nkl} + z_{nkl}$ for all $(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N}$ and $\{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : |y_{nkl} - L| \geq \epsilon\} \in I$;*

(d) *there exists a subset $K = \{k_1 < k_2 ...\}$ of $\mathbf{N}$ such that $k \in F(I)$ and $\lim_{n\to\infty} |x_{k_n} - L| = 0$.*

Proof: (a) implies (b). Let $(x_{nkl}) \in {}_3m^I_{\mathcal{Z}}$. Then there exists $L \in \mathbf{C}$ such that

$$\{(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : |y_{nkl} - L| \geq \epsilon\} \in I.$$

Let (p_t, q_t, r_t) be an increasing sequence with $(p_t, q_t, r_t) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N}$ such that

$$\{(n,k,l) \leq (p_t, q_t, r_t) : |x_{nkl}| - L \geq t^{-1}\} \in I.$$

Define a sequence (y_{nkl}) by

$$y_{nkl} = x_{nkl}, \;\; for\ all\ (n,k,l) \geq (p_1, q_1, r_1).$$

For $(p_t, q_t, r_t) < (n, k, l) \leq (p_{t+1}, q_{t+1}, r_{t+1})$ for $t \in \mathbf{N}$.

$$y_{nkl} = \begin{cases} x_{nkl}, & \text{if } |x_{nkl} - L| < t^{-1}, \\ L, & \text{otherwise.} \end{cases}$$

Then $y_{nkl} \in {}_3\mathcal{Z}$ and form the following inclusion

$$\{(n, k, l) \leq (p_t, q_t, r_t) : x_{nkl} \neq y_{nkl}\} \subseteq \{(n, k, l) \leq (p_t, q_t, r_t) : |x_{nkl} - L| \geq \epsilon\} \in I.$$

we get $x_{nkl} = y_{nkl}$, for a.a.k.r.I.

(b) implies (c). For $x_{nkl} \in {}_3m^I_{\mathcal{Z}}$, there exists $(y_{nkl}) \in {}_3\mathcal{Z}$ such that $x_{nkl} = y_{nkl}$, for a.a.k.r.I. Let $K = \{(n, k, l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : x_{nkl} \neq y_{nkl}\}$, then $K \in I$

Define a sequence (z_{nkl}) as

$$z_{nkl} = \begin{cases} x_{nkl} - y_{nkl}, & \text{if } (n, k, l) \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Then $z_{nkl} \in {}_3\mathcal{Z}^I_0$ and $y_{nkl} \in {}_3\mathcal{Z}$.

(c)implies (d). Let $P_1 = \{(n, k, l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |z_{nkl}| \geq \epsilon\} \in I$ and

$$K = P_1^c = \{(n_1, k_1, l_1) < (n_2, k_2, l_2) < ...\} \in F(I).$$

Then we have $\lim_{m\to\infty} |x_{(n_m,k_m,l_m)} - L| = 0.$

(d) implies (a). Let

$$K = P_1^c = \{(n_1, k_1, l_1) < (n_2, k_2, l_2) < ...\} \in F(I) \text{ and } \lim_{m\to\infty} |x_{(n_m,k_m,l_m)} - L| = 0.$$

Then for any $\epsilon > 0$, and Lemma,we have

$$\{(n, k, l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |z_{nkl}| \geq \epsilon\} \subseteq K^c \cup \{(n, k, l) \in K : |z_{nkl}| \geq \epsilon\}.$$

Thus $(x_{nkl}) \in {}_3\mathcal{Z}^I$.

Theorem 2.5 *The inclusions* ${}_3\mathcal{Z}^I_0 \subset {}_3\mathcal{Z}^I \subset {}_3\mathcal{Z}^I_\infty$ *hold and are proper.*

Proof: Let $(x_{nkl}) \in {}_3\mathcal{Z}^I$. Then there exists $L \in \mathbf{C}$ such that

$$I - \lim |x_{nkl} - L| = 0$$

We have $|x_{nkl}| \leq \frac{1}{2}|x_{nkl} - L| + \frac{1}{2}|L|$.
Taking the supremum over (n, k, l) on both sides we get $(x_{nkl}) \in {}_3\mathcal{Z}^I_\infty$.
The inclusion ${}_3\mathcal{Z}^I_0 \subset {}_3\mathcal{Z}^I$ is obvious.
To prove that the inclusion is proper, we have the followig counter example.

Example 2.1 Consider the sequence (X_{nkl}) be defined by:
For $(n + k + l)$ even ,

$$X_{nkl}(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1 \\ -t + 2, & \text{if } 1 \leq t \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

For $(n + k + l)$ odd,

$$X_{nkl}(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1 \\ -t + 2, & \text{if } 1 \leq t \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $X_{nkl} \in \mathcal{Z}^I_\infty$, but neither belongs to ${}_3\mathcal{Z}^I_0$ nor to ${}_3\mathcal{Z}^I$.

Theorem 2.6 *The function* $\hbar : {}_3m^I_{\mathcal{Z}} \to R$ *Lipschitz function, where* ${}_3m^I_{\mathcal{Z}} = {}_3\mathcal{Z}^I \cap {}_3\mathcal{Z}_\infty$*, and hence uniformly continuous.*

Proof: Let $x, y \in {}_3m^I_{\mathcal{Z}}$, $x \neq y$. Then the sets

$$A_x = \{(n,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |x_{nkl} - \hbar(x)| \geq ||x-y||_*\} \in I,$$

$$A_y = \{(n,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |y_{nkl} - \hbar(y)| \geq ||x-y||_*\} \in I.$$

Thus the sets,

$$B_x = \{(n,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |x_{nkl} - \hbar(x)| < ||x-y||_*\} \in F(I),$$

$$B_y = \{(n,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |y_{nkl} - \hbar(y)| < ||x-y||_*\} \in F(I).$$

Hence also $B = B_x \cap B_y \in F(I)$, so that $B \neq \phi$. Now taking (n,k,l) in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_{nkl}| + |x_{nkl} - y_{nkl}| + |y - \hbar(y)| \leq 3||x-y||_*.$$

Thus $\hbar$ is a Lipschitz function.
For ${}_3m^I_{\mathcal{Z}_0}$ the result can be proved similarly.

Theorem 2.7 *If* $x, y \in {}_3m^I_{\mathcal{Z}}$*, then* $(x.y) \in {}_3m^I_{\mathcal{Z}}$ *and* $\hbar(xy) = \hbar(x)\hbar(y)$*.*

Proof: For $\epsilon > 0$

$$B_x = \{(n,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |x - \hbar(x)| < \epsilon\} \in \mathcal{F}(I),$$

$$B_y = \{(n,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |y - \hbar(y)| < \epsilon\} \in \mathcal{F}(I).$$

Now,

$$\begin{aligned} |x.y - \hbar(x)\hbar(y)| &= |x.y - x\hbar(y) + x\hbar(y) - \hbar(x)\hbar(y)| \\ &\leq |x||y - \hbar(y)| + |\hbar(y)||x - \hbar(x)| \end{aligned} \quad [3]$$

As ${}_3m^I_{\mathcal{Z}} \subseteq {}_3\mathcal{Z}_\infty$, there exists an $M \in R$ such that $\hbar|x| < M$ and $|\hbar(y)| < M$.
Using eqn[3] we get

$$|x.y - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $(n,k,l) \in B_x \cap B_y \in \mathcal{F}(I)$.
Hence $(x.y) \in {}_3m^I_{\mathcal{Z}}$ and $\hbar(xy) = \hbar(x)\hbar(y)$.
For ${}_3m^I_{\mathcal{Z}_0}$ the result can be proved similarly.

Theorem 2.8 *The spaces* ${}_3\mathcal{Z}^I_0$ *and* ${}_3m^I_{\mathcal{Z}_0}$ *are solid and monotone .*

Proof: We shall prove the result for ${}_3\mathcal{Z}^I_0$. Let $(x_{nkl}) \in \mathcal{Z}^I_0$. Then

$$I - \lim_k |x_{nkl}| = 0 \quad [4]$$

Let (α_{nkl}) be a sequence of scalars with $|\alpha_{nkl}| \leq 1$ for all $(n,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N}$. Then the result follows from [4] and the following inequality

$$|\alpha_{nkl}x_{nkl}| \leq |\alpha_{nkl}||x_{nkl}| \leq |x_{nkl}| \ for\ all\ (n,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N}.$$

That the space ${}_3\mathcal{Z}^I_0$ is monotone follows from the Lemma 1.1.
For ${}_3m^I_{\mathcal{Z}_0}$ the result can be proved similarly.

Theorem 2.9 *If* I *is not maximal, then the space* ${}_3\mathcal{Z}^I$ *is neither solid nor monotone.*

Proof: Here we give a counter example. Let $(x_{nkl}) = 1$ for all $(n,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N}$. Then $(x_{nkl}) \in {}_3\mathcal{Z}^I$. Let $K \subseteq \mathbf{N}\times\mathbf{N}\times\mathbf{N}$ be such that $K \notin I$ and $\mathbf{N}\times\mathbf{N}\times\mathbf{N} - K \notin I$.
Define the sequence

$$(y_{nkl}) = \begin{cases} (x_{nkl}), & \text{if } (n,k,l) \in K, \\ 0, & \textit{otherwise}. \end{cases}$$

Then (y_{nkl}) belongs to the canonical preimage of K-step space of ${}_3\mathcal{Z}^I$ but $(y_{nkl}) \notin {}_3\mathcal{Z}^I$.
Hence ${}_3\mathcal{Z}^I$ is not monotone.

Theorem 2.10 *The spaces ${}_3\mathcal{Z}^I$ and ${}_3\mathcal{Z}_0^I$ are sequence algebras.*

Proof: We prove that ${}_3\mathcal{Z}_0^I$ is a sequence algebra.
Let $(x_{nkl}), (y_{nkl}) \in {}_3\mathcal{Z}_0^I$.
Then

$$I - \lim |x_{nkl}| = 0 \qquad and \qquad I - \lim |y_{nkl}| = 0$$

Then we have $I - \lim |(x_{nkl}.y_{nkl})| = 0$. Thus $(x_{nkl}.y_{nkl}) \in {}_3\mathcal{Z}_0^I$.
Hence ${}_3\mathcal{Z}_0^I$ is a sequence algebra.
For the space ${}_3\mathcal{Z}^I$, the result can be proved similarly.

Theorem 2.11 *The spaces ${}_3\mathcal{Z}^I$ and ${}_3\mathcal{Z}_0^I$ are not convergence free in general.*

Proof:
Here we give a counter example.
Let $I = I_f$. Consider the sequence (x_{nkl}) and (y_{nkl}) defined by

$$x_{nkl} = (n.k.l)^{-1} \quad \text{and} \quad y_{nkl} = n.k.l \quad \text{for all} \quad \text{(n,k,l)} \in \mathbf{N}\times\mathbf{N}\times\mathbf{N}$$

Then $(x_{nkl}) \in {}_3\mathcal{Z}^I$ and ${}_3\mathcal{Z}_0^I$, but $(y_{nkl}) \notin {}_3\mathcal{Z}^I$ and ${}_3\mathcal{Z}_0^I$.
Hence the spaces ${}_3\mathcal{Z}^I$ and ${}_3\mathcal{Z}_0^I$ are not convergence free.

Theorem 2.12 *If I is not maximal and $I \neq I_f$, then the spaces ${}_3\mathcal{Z}^I$ and ${}_3\mathcal{Z}_0^I$ are not symmetric.*

Proof: Let $A \in I$ be infinite. If

$$x_{nkl} = \begin{cases} 1, & \text{for } n,k,l \in A, \\ 0, & \textit{otherwise}. \end{cases}$$

Then $x_{nkl} \in {}_3\mathcal{Z}_0^I \subset {}_3\mathcal{Z}^I$. Let $K \subset \mathbf{N}$ be such that $K \notin I$ and $\mathbf{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbf{N} - K \to \mathbf{N} - A$ be bijections, then the map $\pi : \mathbf{N} \to \mathbf{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & \textit{otherwise}. \end{cases}$$

is a permutation on $\mathbf{N}$, but $x_{(\pi(m)\pi(n))} \notin {}_3\mathcal{Z}^I$ and $x_{(\pi(m)\pi(n))} \notin {}_3\mathcal{Z}_0^I$.
Hence ${}_3\mathcal{Z}^I$ and ${}_3\mathcal{Z}_0^I$ are not symmetric.

Theorem 2.13 *The sequence spaces ${}_3\mathcal{Z}^I$ and ${}_3\mathcal{Z}_0^I$ are linearly isomorphic to the spaces ${}_3c^I$ and ${}_3c_0^I$ respectively, i.e ${}_3\mathcal{Z}^I \cong {}_3c^I$ and ${}_3\mathcal{Z}_0^I \cong {}_3c_0^I$.*

Proof: We shall prove the result for the space ${}_3\mathcal{Z}^I$ and ${}_3c^I$. The proof for the other spaces will follow similarly. We need to show that there exists a linear bijection between the spaces ${}_3\mathcal{Z}^I$ and ${}_3c^I$. Define a map $T : {}_3\mathcal{Z}^I \longrightarrow {}_3c^I$ such that $x \to x' = Tx$

$$T(x_{nkl}) = px_{nkl} + (1-p)x_{(n-1)(k-1)(l-1)} = x'_{nkl}$$

where $x_{-1} = 0, p \neq 1,\ 1 < p < \infty$. Clearly T is linear. Further, it is trivial that $x = 0 = (0, 0, 0, ...)$ whenever $Tx = 0$ and hence injective. Let $x'_{nkl} \in\ _3c^I$ and define the sequence $x = x_{nkl}$ by

$$x_{nkl} = M \sum_{r=0}^{n} \sum_{s=0}^{k} \sum_{t=0}^{l} (-1)^{(n-r)(k-s)(l-t)} N^{(n-r)(k-s)(l-t)} x'_{nkl}$$

for $(n, k, l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N}$ and where $M = \frac{1}{p}$ and $N = \frac{1-p}{p}$. Then we have

$$\lim_{(n,k,l)\to\infty} px_{nkl} + (1-p)x_{(n-1)(k-1)(l-1)} = p \lim_{(n,k,l)\to\infty} M \sum_{r=0}^{n} \sum_{s=0}^{k} \sum_{t=0}^{l} (-1)^{(n-r)(k-s)(l-t)} N^{(n-r)(k-s)(l-t)} x'_{nkl}$$

$$+(1-p) \lim_{(n,k,l)\to\infty} M \sum_{r=0}^{n-1} \sum_{s=0}^{k-1} \sum_{t=0}^{l-1} (-1)^{(n-1-r)(k-1-s)(l-1-t)} N^{(n-1-r)(k-1-s)(l-1-t)} x'_{n-1k-1l-1}$$

$$= \lim_{(n,k,l)\to\infty} x'_{nkl}$$

which shows that $x \in\ _3\mathcal{Z}^I$. Hence T is a linear bijection.
Also we have $||x||_* = ||Z^p x||_c$.
Therefore

$$||x||_* = \sup_{(n,k,l)\in\mathbf{N}\times\mathbf{N}\times\mathbf{N}} |px_{nkl} + (1-p)x_{(n-1)(k-1)(l-1)}|$$

$$= \sup_{(n,k,l)\in\mathbf{N}\times\mathbf{N}\times\mathbf{N}} |pM \sum_{r=0}^{n} \sum_{s=0}^{k} \sum_{t=0}^{l} (-1)^{(n-r)(k-s)(l-t)} N^{(n-r)(k-s)(l-t)} x'_{nkl}$$

$$+(1-p)M \sum_{r=0}^{n-1} \sum_{s=0}^{k-1} \sum_{t=0}^{l-1} (-1)^{(n-1-r)(k-1-s)(l-1-t)} N^{(n-1-r)(k-1-s)(l-1-t)} x'_{(n-1)(k-1)(l-1)}|$$

$$= \sup_{(n,k,l)\in\mathbf{N}\times\mathbf{N}\times\mathbf{N}} |x'_{nkl}| = ||x'||_{\ _3c^I}.$$

Hence $_3\mathcal{Z}^I \cong\ _3c^I$.

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