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Schatten class Localization operators for Wigner Transform Associated with the Jacobi-Dunkl operator

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ABSTRACT: The main crux of this paper is to introduce a new integral transform called the Jacobi-Dunkl-Wigner transform and to give some new results related to this transform as inversion formula. Next, we introduce a new class of pseudo-differential operator $\mathscr{L}_{u,v}(\sigma)$ called localization operator which depend on a symbol σ and two admissible functions u and v, we give a criteria in terms of the symbol σ for its boundedness and compactness, we also show that these operators belongs to the Schatten-Von Neumann class S^p for all $p \in [1; +\infty]$ and we give a trace formula.

Key Words: Wigner transform, Localization operators, Jacobi-Dunkl operator, Schatten-von Neumann classes.

Contents

1. Introduction

The Wigner transform has a long story wich stared in 1932 with Eugene Wigner's as a probability quasi-distribution which allows expression of quantum mechanical expectation values in the same form as the averages of classical statistical mechanics.It is also used in signal processing as a transform in time-frequency analysis, for more information one can see [7,20]. A mathematical object closely related to the Wigner transform is the windowed Fourier transform used in signal theory and time-frequency analysis, using this connection we will define and study the localization operators for the Fourier-Wigner transform associated with the Jacobi-Dunkl operator.
The classical Fourier transform in $\mathbb{R}^d$ can be defined by many ways, its most basic formulation it is given by the integral transform

$$\mathcal{F}(f)(\lambda) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i<\lambda,x>} f(x)\mathrm{d}x.$$

Alternatively, one can rewrite this transform as

$$\mathcal{F}(f)(\lambda) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} K(\lambda,x) f(x)\mathrm{d}x,$$

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where $K(\lambda, x)$ is the unique solution of the system of partial differential equations

$$\begin{cases} \partial_{x_j} K(\lambda, x) & = -i\lambda_j K(\lambda, x), \quad \text{for} \quad j = 1, \ldots\ldots, d, \\ K(\lambda, 0) & = 1, \quad \lambda \in \mathbb{R}^d. \end{cases}$$

A lot of attention has been given to various generalization of the classical Fourier transform. This paper focuses on the generalized Fourier transform associated with the Jacobi-Dunkl operator called the Jacobi-Dunkl transform, more precisely we consider the differential-difference operator $\Delta_{\alpha,\beta}$ defined for $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq \beta \geq \frac{-1}{2}$ and $\alpha \neq \frac{-1}{2}$ on $\mathbb{R}$ by

$$\Delta_{\alpha,\beta} f(x) := \frac{\partial}{\partial x} f(x) + [(2\alpha+1)\coth x + (2\beta+1)\tanh x] \times \frac{f(x) - f(-x)}{2} \tag{1.1}$$

The eigenfunctions of this operator are related to the Jacobi functions and they satisfy a product formula which permits to develop a new harmonic analysis associated with this operator see [1,4]. One of the aims of the Fourier transform is the study of the theory of localization operators called also Gabor multipliers, Toeplitz operators or Anti-Wick operators, this theory was initiated by Daubechies in [6], developed and detailed in the book [21] by Wong. Wong was the first one who defined the localization operators on the Weyl Heisenberg group in [22], next Boggiatto and Wong have extended this results on $L^p\left(\mathbb{R}^d\right)$ in [2]. Then Wong studies the localization operators associated to left regular representation of localy compact and Hausdorff group G on $L^p(G)$ in [23]. Some results for wavelets multipliers which are localization operators associated to modulation on the additive group on $\mathbb{R}^d$ are given by Ma and Wong in [11]. The theory of localization operators associated with the Fourier-Wigner transform has been studied and known remarkable development in many settings for example in the Riemann-Liouville setting [12], in the spherical mean setting [14], in the Dunkl setting [13], in the Weinstein setting [17]. However, upto our knowledge, the localization operators have not been studied for the Jacobi-Dunkl transform, the main purpose of this paper is twofold on the one hand we introduce the Fourier-Wigner transform associated with the Jacobi-Dunkl operator and we give some new results related to this transform on the other hand we introduce the localization operator $\mathscr{L}_{u,v}(\sigma)$ associated with this transform and we give a criteria in terms of the symbol σ for its boundedness and compactness, we also show that these operators belongs to the Schatten-Von Neumann classes S^p for all $p \in [1; +\infty]$ and we give a trace formula.
The remainder of this paper is arranged as follows, in section 2 we recall the main results concerning the harmonic analysis associated with the Jacobi-Dunkl transform and Schatten-Von Neumann classes, in section 3 we will study the boundedness, compactness and the Schatten properties of the localization operator associated with the Jacobi-Dunkl-Wigner transform.

2. Harmonic Analysis Associated with the Jacobi-Dunkl operator

In this section we set some notations and we recall some results in harmonic analysis related to the Jacobi-Dunkl operator and the Schatten-Von Neumann classes, for more details we refer the reader to [1,4,15,21].
In the following we denote by
• $C_0(\mathbb{R})$, the space of continuous functions defined on $\mathbb{R}$ satisfying
$\lim\limits_{|x|\to+\infty} f(x) = 0$, and $\|f\|_{C_0} = \sup_{x\in\mathbb{R}} |f(x)| < \infty$.

• $\mathcal{S}(\mathbb{R})$, the usual Schwartz space of $\mathcal{C}^\infty$-functions on $\mathbb{R}$ rapidly decreasing together with their derivatives, equipped with the topology of semi-norms $q_{m,n}, (m,n) \in \mathbb{N}^2$, where

$$q_{m,n}(f) = \sup_{x\in\mathbb{R}, 0\leq k\leq n} \left[\left(1+x^2\right)^m \left|\frac{d^k}{dx^k} f(x)\right|\right] < +\infty.$$

• $\mathcal{S}_*(\mathbb{R}) = \left\{(\cosh(x))^{-2\rho} f; f \in \mathcal{S}(\mathbb{R})\right\}$ with $\rho \in \mathbb{R}$. and
α, β denote real numbers such that $\alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha \neq -\frac{1}{2}$. we put

$$\mathcal{A}_{\alpha,\beta}(x) = 2^{2\rho}(\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}, \quad \rho = \alpha+\beta+1 \tag{2.1}$$

- $L^p_{\alpha,\beta}(\mathbb{R}), p \geq 1$, the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha,\beta} = \begin{cases} \left(\int_{\mathbb{R}} |f(x)|^p d\mu_{\alpha,\beta}(x)\right)^{1/p} < +\infty & \text{si } 1 \leq p < +\infty, \\ \text{ess sup}_{x\in\mathbb{R}} |f(x)| < +\infty & \text{si } p = +\infty. \end{cases}$$

where $\mu_{\alpha,\beta}$ is the measure given by

$$d\mu_{\alpha,\beta}(x) = \mathcal{A}_{\alpha,\beta}(x)dx,$$

and $\mathcal{A}_{\alpha,\beta}(x)$ is given by (2.1).
For $p = 2$, $L^2_{\alpha,\beta}(\mathbb{R})$ is a Hilbert space with inner product defined for $f, g \in L^2_{\alpha,\beta}(\mathbb{R})$ by

$$\langle f, g\rangle_{\mu_{\alpha,\beta}} = \int_{\mathbb{R}} f(x)\overline{g(x)}d\mu_{\alpha,\beta}(x).$$

- $L^p_\sigma(\mathbb{R}), p \geq 1$, the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{p,\sigma} = \begin{cases} \left(\int_{\mathbb{R}} |f(\lambda)|^p d\sigma(\lambda)\right)^{1/p} < +\infty & \text{si } 1 \leq p < +\infty, \\ \underset{\lambda\in\mathbb{R}}{\text{ess sup}}|f(\lambda)| < +\infty & \text{si } p = +\infty. \end{cases}$$

where σ is the spectral measure supported in $\mathbb{R}\backslash]-\rho,\rho[$ given by

$$d\sigma(\lambda) = \frac{|\lambda|}{8\pi\sqrt{\lambda^2-\rho^2}\left|C_{\alpha,\beta}\left(\sqrt{\lambda^2-\rho^2}\right)\right|}1_{\mathbb{R}\backslash]-\rho,\rho[}(\lambda)d\lambda,$$

where

$$C_{\alpha,\beta}(\mu) = \frac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma\left(\frac{1}{2}(\rho+i\mu)\right)\Gamma\left(\frac{1}{2}(\alpha-\beta+1+i\mu)\right)}, \quad \mu \in \mathbb{C}\backslash(i\mathbb{N}),$$

is the Harish-Chandra function given explicitely in [9,10,16] and $1_{\mathbb{R}\backslash]-\rho,\rho[}$ is the characteristic function of $\mathbb{R}\backslash]-\rho,\rho[$.
In this paper, we consider the differential-difference operator $\Delta_{\alpha,\beta}$ given by (1.1), this operator is a particular case of the operator Δ_A given by

$$\Delta_A(f)(x) = \frac{\partial f(x)}{\partial x} + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right),$$

where $A(x) = \mathcal{A}_{\alpha,\beta}(x)$.
The operator $\Delta_{\alpha,\beta}$ is skew-adjoint i.e $\Delta^*_{\alpha,\beta} = -\Delta_{\alpha,\beta}$ that mean for two continuous functions f, g on $\mathbb{R}$ with at least one of them with compact support we have

$$\int_{\mathbb{R}} \Delta_{\alpha,\beta}f(x)g(x)d\mu_{\alpha,\beta}(x) = -\int_{\mathbb{R}} f(x)\Delta_{\alpha,\beta}g(x)d\mu_{\alpha,\beta}(x),$$

furthermore if one of them is even then we have

$$\Delta_{\alpha,\beta}(fg) = \Delta_{\alpha,\beta}(f)g + f\Delta_{\alpha,\beta}(g).$$

2.1. The eigenfunctions of the Jacobi-Dunkl operator

The main purpose of this subsection is to define the eigenfunctions of the Jacobi-Dunkl operator which will be used later to define the Jacobi-Dunkl transform, to do this we need to define first those of the Jacobi operator.

2.1.1. Jacobi kernels. Harmonic analysis associated with the Jacobi operator was firstly developed by Flensted-Jensen and Koorwinder see [9,10].
For $\alpha \geq \frac{-1}{2}, \beta \in \mathbb{R}$, the Jacobi operator $J_{\alpha,\beta}$ on $]0,+\infty[$ is given by

$$J_{\alpha,\beta} f(x) = \frac{\partial^2}{\partial x^2} f(x) + [(2\alpha+1)\coth x + (2\beta+1)\tanh x] \frac{\partial}{\partial x} f(x).$$

We point out that $J_{\alpha,\beta}$ is the radial part of the Laplace-Beltrami operator on a symetric space of rank one see [9,10].
The Jacobi function $\varphi_\mu^{\alpha,\beta}$, $\mu \in \mathbb{C}$ is defined on $\mathbb{R}$ by

$$\forall x \in \mathbb{R}, \quad \varphi_\mu^{\alpha,\beta}(x) = {}_2F_1\left(\frac{\rho+i\mu}{2}, \frac{\rho-i\mu}{2}; \alpha+1; -(\sinh x)^2\right), \tag{2.2}$$

where ${}_2F_1$ is the Gauss hyper-geometric function given by

$${}_2F_1(a,b,c,z) = \sum_{l=0}^{\infty} \frac{(a)_l (b)_l}{(c)_l l!} z^l, \quad |z| < 1,$$

with $a, b, c \in \mathbb{C}$ and $c \notin -\mathbb{N}$ and $(a)_0 = 1, (a)_l = a(a+1)\cdots(a+l-1) = \frac{\Gamma(a+l)}{\Gamma(a)}$.
For a special values of α and β the Jacobi functions (2.2) are interpreted as spherical functions on non-compact Riemannian symmetric spaces of rank one [9].
From [16], the Jacobi function (2.2) satisfies the following properties.

Proposition 2.1
(1) $\varphi_\mu^{\alpha,\beta}$ is the unique solution on $[0,+\infty[$ of the differential equation

$$\begin{cases} J_{\alpha,\beta}(f) = -\left(\mu^2+\rho^2\right) f \\ f(0) = 1, f'(0) = 0 \end{cases}$$

(2) $\varphi_\mu^{\alpha,\beta}$ is even, infinitely differentiable on $[0,+\infty[$ and we have

$$\frac{\partial}{\partial x} \varphi_\mu^{\alpha,\beta}(x) = -\frac{\mu^2+\rho^2}{4(\alpha+1)} \sinh(2x) \varphi_\mu^{\alpha+1,\beta+1}(x). \tag{2.3}$$

We will use this function to define the eigenfunctions of the Jacobi-Dunkl operator.

2.1.2. The Jacobi-Dunkl kernels. Now we consider the Jacobi-Dunkl operator given by (1.1) and we determine the eigenfunctions $\psi_\lambda^{\alpha,\beta}$ of this operator called the Jacobi-Dunkl kernels associated with the eigenvalue $(i\lambda)$, $\lambda \in \mathbb{C}$ and equal to 1 for $x = 0$.
For $\lambda \in \mathbb{C}, x \in \mathbb{R}$ we put

$$\forall x \in \mathbb{R}, \quad \psi_\lambda^{\alpha,\beta}(x) := \begin{cases} \varphi_\mu^{\alpha,\beta}(x) - \frac{i}{\lambda} \frac{\partial}{\partial x} \varphi_\mu^{\alpha,\beta}(x) & \text{if } \lambda \in \mathbb{C}^*, \\ 1 & \text{if } \lambda = 0, \end{cases} \tag{2.4}$$

where $\varphi_\mu^{\alpha,\beta}$ is the Jacobi function given by (2.2), the function $\psi_\lambda^{\alpha,\beta}$ is called the Jacobi-Dunkl kernel and we have the following results for the proofs we refer the reader to [1,4,15].

Proposition 2.2
(1) The Jacobi-Dunkl kernel $\psi_\lambda^{\alpha,\beta}$ is the unique $\mathcal{C}^\infty$-solution on $\mathbb{R}$ of the differential-difference equation

$$\begin{cases} \Delta_{\alpha,\beta}(u) = -i\lambda(u), \lambda \in \mathbb{C}, \\ f(0) = 1. \end{cases}$$

Furthermore it is infinitely differentiable and we have

$$|\frac{\partial^n}{\partial\lambda^n}\psi_\lambda^{\alpha,\beta}(x)| \leq |x|^n e^{|Im(\lambda)||x|}.$$

In particular we have the following important result

$$\forall x,\lambda \in \mathbb{R} \quad |\psi_\lambda^{\alpha,\beta}(x)| \leq 1. \tag{2.5}$$

Remark 2.1 Using the relation (2.3), the Jacobi-Dunkl kernel (2.4) can be written as

$$\psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + i\frac{\lambda}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x), \quad x \in \mathbb{R}. \tag{2.6}$$

We will use the Jacobi-Dunkl kernel (2.6) to define the Jacobi-Dunkl transform

2.2. The Jacobi-Dunkl transform

Definition 2.1 *The Jacobi-Dunkl transform $\mathcal{F}_{\alpha,\beta}$ defined on $L^1_{\alpha,\beta}(\mathbb{R})$ by*

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = \int_{\mathbb{R}} \psi_\lambda^{\alpha,\beta}(x)f(x)d\mu_{\alpha,\beta}(x) \quad for \quad \lambda \in \mathbb{R}.$$

Some basic properties of this transform are as follows, for the proofs, we refer the reader to [1,4,15].

Proposition 2.3

(1) (Riemann-Lebesgue) For all $f \in L^1_{\alpha,\beta}(\mathbb{R})$, the function $\mathcal{F}_{\alpha,\beta}(f)$ belongs to $C_0(\mathbb{R})$ and we have

$$\|\mathcal{F}_\alpha(f)\|_{\infty,\sigma} \leq \|f\|_{1,\alpha,\beta}. \tag{2.7}$$

(2)(Inversion formula) For all $f \in L^1_{\alpha,\beta}(\mathbb{R})$ such that $\mathcal{F}_{\alpha,\beta}(f) \in L^1_\sigma(\mathbb{R})$ we have

$$f(x) = \int_{\mathbb{R}} \psi_{-\lambda}^{\alpha,\beta}(x)\mathcal{F}_{\alpha,\beta}(f)(\lambda)d\sigma(\lambda), \qquad a.e \quad x \in \mathbb{R}. \tag{2.8}$$

(3) (Plancherel theorem) The Jacobi-Dunkl transform is a topological isomorphism from $S_*(\mathbb{R})$ onto $S(\mathbb{R})$ and extends uniquely to a unitary isomorphism from $L^2_{\alpha,\beta}(\mathbb{R})$ onto $L^2_\sigma(\mathbb{R})$ and for all $f \in L^2_{\alpha,\beta}(\mathbb{R})$ we have

$$\int_{\mathbb{R}} |f(x)|^2 d\mu_{\alpha,\beta}(x) = \int_{\mathbb{R}} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda). \tag{2.9}$$

2.3. The translation operator associated with the Jacobi-Dunkl transform

From [1,4,18], the Jacobi-Dunkl kernel $\psi_\lambda^{\alpha,\beta}$ is multiplicative on $\mathbb{R}$ in the sense

$$\psi_\lambda^{\alpha,\beta}(x)\psi_\lambda^{\alpha,\beta}(y) = \int_{\mathbb{R}} \psi_\lambda^{\alpha,\beta}(z)d\delta_{x,y}^{\alpha,\beta}(z), \tag{2.10}$$

where $\delta_{x,y}^{\alpha,\beta}$ is a real uniformly bounded measure with compact support wich may not be positive given explicitly in [1,4,19].
The product formula (2.10) permits to define a translation operator, a convolution product and to develop a new harmonic analysis associated with the Jacobi-Dunkl operator (1.1).

Definition 2.2 *Let $x, y, z \in \mathbb{R}$ and f be a measurable function on $\mathbb{R}$ the translation operator associated with the Jacobi-Dunkl transform is defined by:*

$$\mathcal{T}^x_{\alpha,\beta}(f)(y) = \int_{\mathbb{R}} f(z)d\delta_{x,y}^{\alpha,\beta}(z).$$

The following proposition summarizes some properties of the Jacobi-Dunkl translation operator, for the proofs we refer the reader to [1,4,19].

Proposition 2.4 *For all $x, y, z \in \mathbb{R}$, f a measurable function on $\mathbb{R}$ we have*

(1)

$$\mathcal{T}^x_{\alpha,\beta}(f)(y) = \mathcal{T}^x_{\alpha,\beta}(f)(x). \tag{2.11}$$

(2)

$$\mathcal{T}^x_{\alpha,\beta}(\psi^{\alpha,\beta}_\lambda(.))(y) = \psi^{\alpha,\beta}_\lambda(x)\psi^{\alpha,\beta}_\lambda(y).$$

(3)

$$\int_{\mathbb{R}} \mathcal{T}^x_{\alpha,\beta}(f)(y)d\mu_{\alpha,\beta}(x) = \int_{\mathbb{R}} f(y)d\mu_{\alpha,\beta}(x). \tag{2.12}$$

(4) for $f \in L^p_{\alpha,\beta}(\mathbb{R})$ with $p \in [1;+\infty]$, $\mathcal{T}^x_{\alpha,\beta}(f) \in L^p_{\alpha,\beta}(\mathbb{R})$ and we have

$$\|\mathcal{T}^x_{\alpha,\beta}(f)\|_{p,\alpha,\beta} \leq \|f\|_{p,\alpha,\beta}. \tag{2.13}$$

By using the generalized translation, we define the generalized convolution product of $f, g \in \mathcal{S}_*(\mathbb{R})$ and $x \in \mathbb{R}$ by

$$(f * g)(x) = \int_{\mathbb{R}} \mathcal{T}^x_{\alpha,\beta}(f)(-y)g(y)d\mu_{\alpha,\beta}(y).$$

This convolution is commutative, associative and its satisfies the following properties, for the proofs we refer the reader to [1,4,15,19].

Proposition 2.5

(1)(Young's inequality) for all $p, q, r \in [1;+\infty]$ such that: $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and for all $f \in L^p_{\alpha,\beta}(\mathbb{R}), g \in L^q_{\alpha,\beta}(\mathbb{R})$ the function $f *_\alpha g$ belongs to the space $L^r_{\alpha,\beta}(\mathbb{R})$ and we have

$$\|f *_\alpha g\|_{r,\alpha,\beta} \leq \|f\|_{p,\alpha,\beta}\|g\|_{q,\alpha,\beta}. \tag{2.14}$$

(2) For $f, g \in L^2_{\alpha,\beta}(\mathbb{R})$ the function $f *_\alpha g$ belongs to $L^2_{\alpha,\beta}(\mathbb{R})$ if and only if the function $\mathcal{F}_{\alpha,\beta}(f)\mathcal{F}_{\alpha,\beta}(g)$ belongs to $L^2_\sigma(\mathbb{R})$ and in this case we have

$$\mathcal{F}_{\alpha,\beta}(f * g) = \mathcal{F}_{\alpha,\beta}(f)\mathcal{F}_{\alpha,\beta}(g).$$

and

$$\int_{\mathbb{R}} |f * g(x)|^2 d\mu_{\alpha,\beta}(x) = \int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(\lambda)|^2 |\mathcal{F}_\alpha(g)(\lambda)|^2 d\sigma(\lambda). \tag{2.15}$$

2.4. The Schatten-Von Neumann classes

Notation: we denote by

- $l^p(\mathbb{N}), 1 \leq p \leq \infty$, the set of all infinite sequences of real (or complex) numbers $u := (u_j)_{j\in\mathbb{N}}$, such that

$$\|u\|_p := \left(\sum_{j=1}^{\infty} |u_j|^p\right)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \leq p < \infty,$$

$$\|u\|_\infty := \sup_{j\in\mathbb{N}} |u_j| < \infty, \quad \text{if} \quad p = +\infty.$$

- $B\left(L^p_{\alpha,\beta}(\mathbb{R})\right), 1 \leq p \leq \infty$, the space of bounded operators from $L^p_{\alpha,\beta}(\mathbb{R})$ into itself.

For $p = 2$, we define the space $S_\infty := B\left(L^2_{\alpha,\beta}(\mathbb{R})\right)$, equipped with the norm,

$$\|A\|_{S_\infty} := \sup_{v\in L^2_{\alpha,\beta}(\mathbb{R}):\|v\|_{2,\alpha,\beta}=1} \|Av\|_{2,\alpha,\beta}. \tag{2.16}$$

Definition 2.3
(1) The singular values $(s_n(A))_{n\in\mathbb{N}}$ *of a compact operator* A *in* $B\left(L^p_{\alpha,\beta}(\mathbb{R})\right)$ *are the eigenvalues of the positive self-adjoint operator* $|A| = \sqrt{A^*A}$.

(2) For $1 \leq p < \infty$, *the Schatten class* S_p *is the space of all compact operators whose singular values lie in* $l^p(\mathbb{N})$. *The space* S_p *is equipped with the norm*

$$\|A\|_{S_p} := \left(\sum_{n=1}^{\infty} (s_n(A))^p\right)^{\frac{1}{p}}.$$

Remark 2.2 We note that the space S_2 is the space of Hilbert-Schmidt operators, and S_1 is the space of trace class operators.

Definition 2.4 *The trace of an operator* A *in* S_1 *is defined by*

$$\operatorname{tr}(A) = \sum_{n=1}^{\infty} \langle A\phi_n, \phi_n\rangle_{\mu_{\alpha,\beta}}, \tag{2.17}$$

where $(\phi_n)_n$ *is any orthonormal basis of* $L^2_{\alpha,\beta}(\mathbb{R})$.

Remark 2.3 If A is positive, then

$$\operatorname{tr}(A) = \|A\|_{S_1}. \tag{2.18}$$

Moreover, a compact operator A on the Hilbert space $L^2_{\alpha,\beta}(\mathbb{R})$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class S_1. Then

$$\|A\|^2_{HS} := \|A\|^2_{S_2} = \|A^*A\|_{S_1} = \operatorname{tr}(A^*A) = \sum_{n=1}^{\infty} \|A\phi_n\|^2_{2,\alpha,\beta}, \tag{2.19}$$

for any orthonormal basis $(\phi_n)_n$ of $L^2_{\alpha,\beta}(\mathbb{R})$.
For more informations about the Schatten-Von neumann classes one can see [3,5,21].

2.5. Fourier-Wigner transform associated with the Jacobi-Dunkl operator

In this section we define and give some results for the Wigner transform associated with the Jacobi-Dunkl operator.
Notation : we denote by
• $\mathcal{S}(\mathbb{R}^2)$ the generlaized Schwartz space defined on $\mathbb{R}^2$ equipped with it s usual topology.
• $L^p_\theta(\mathbb{R}^2)$,$1 \leq p \leq +\infty$ the space of measurable functions on $\mathbb{R}^2$ satisfying

$$\|f\|_{p,\theta} := \begin{cases} \left(\int_{\mathbb{R}^2} |f(x,\lambda)|^p \mathrm{d}\theta_{\alpha,\beta}(x,\lambda)\right)^{\frac{1}{p}}, & \text{if } p \in [1,+\infty[, \\ \operatorname*{ess\,sup}_{(x,\lambda)\in\mathbb{R}^2} |f(x,\lambda)|, & \text{if } p = +\infty. \end{cases}$$

where $\theta_{\alpha,\beta}$ is the measure defined on $\mathbb{R}^2$ by

$$\mathrm{d}\theta_{\alpha,\beta}(x,\lambda) := d\sigma(\lambda) \otimes d\mu_{\alpha,\beta}(x)$$

for all $x, \lambda \in \mathbb{R}$.

Definition 2.5 *The Wigner transform associated with the Jacobi-Dunkl operator is defined on* $\mathcal{S}_*(\mathbb{R}) \times \mathcal{S}_*(\mathbb{R})$ *by*

$$\mathcal{W}(f,g)(x,\lambda) := \int_{\mathbb{R}} f(y)\mathcal{T}^x_{\alpha,\beta}(g)(-y)\psi^{\alpha,\beta}_\lambda(y)d\mu_{\alpha,\beta}(y). \tag{2.20}$$

Remark 2.4 the transform $\mathcal{W}$ is a bilinear mapping from $\mathcal{S}_*(\mathbb{R}) \times \mathcal{S}_*(\mathbb{R})$ into $\mathcal{S}(\mathbb{R}^2)$ and can be written as

$$\mathcal{W}(f,g)(x,\lambda) = (g * f\psi_\lambda^{\alpha,\beta}(.)(x) \tag{2.21}$$

$$= \mathcal{F}_\alpha(f\widetilde{\mathcal{T}_{\alpha,\beta}^x(g)})(\lambda, m). \tag{2.22}$$

where $\widetilde{h}(x) = h(-x)$.
We have the following results.

Proposition 2.6 *Let $f, g \in L^2_{\alpha,\beta}(\mathbb{R})$ then $\mathcal{W}(f,g)$ is well defined and belongs to $L^2_\theta(\mathbb{R}^2) \cap L^\infty_\theta(\mathbb{R}^2)$ and we have*

$$\|\mathcal{W}(f,g)\|_{2,\theta} \leq \|f\|_{2,\alpha,\beta}\|g\|_{2,\alpha,\beta}, \tag{2.23}$$

and

$$\|\mathcal{W}(f,g)\|_{\infty,\theta} \leq \|f\|_{2,\alpha,\beta}\|g\|_{2,\alpha,\beta}. \tag{2.24}$$

Proof: Let $f, g \in L^2_{\alpha,\beta}(\mathbb{R})$, $x, \lambda \in \mathbb{R}$ by using the relations (2.9),(2.11), (2.22) and Fubini's theorem we find that

$$\|\mathcal{W}(f,g)\|^2_{2,\theta} = \int_\mathbb{R} [\int_\mathbb{R} |f(y)|^2 |\mathcal{T}^{-y}_{\alpha,\beta}(g)(x)|^2 \mathrm{d}\mu_{\alpha,\beta}(y)]\mathrm{d}\mu_{\alpha,\beta}(x),$$

by using the relation (2.13) we get

$$\|\mathcal{W}(f,g)\|^2_{2,\theta} = \|f\|^2_{2,\alpha,\beta}\|\mathcal{T}^{-y}_{\alpha,\beta}(g)\|^2_{2,\alpha,\beta} \leq \|f\|^2_{2,\alpha,\beta}\|g\|^2_{2,\alpha,\beta}$$

which give the result.
On the other hand we have

$$\|\mathcal{W}(f,g)\|_{\infty,\theta} = \|\mathcal{F}_{\alpha,\beta}(f\widetilde{\mathcal{T}^x_{\alpha,\beta}(g)})\|_{\infty,\sigma},$$

by the Riemann-Lebesgue result (2.7), Hölder's inequality and the relation (2.13) we find (2.24). □

Remark 2.5 For $p, q, r \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$ and for $f \in L^p_{\alpha,\beta}(\mathbb{R})$, $g \in L^q_{\alpha,\beta}(\mathbb{R})$ we define the Wigner transform $\mathcal{W}(f,g)$ by the relation (2.21), then we have the following result.

Proposition 2.7 *Let $p, q, r \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$ and for $f \in L^p_{\alpha,\beta}(\mathbb{R})$, $g \in L^q_{\alpha,\beta}(\mathbb{R})$ and $\lambda \in \mathbb{R}$ then the function*

$$x \longmapsto \mathcal{W}(f,g)(x,\lambda)$$

belongs to $L^r_{\alpha,\beta}(\mathbb{R})$ and we have

$$\|\mathcal{W}(f,g)(.,\lambda)\|_{r,\alpha,\beta} \leq \|f\|_{p,\alpha,\beta}\|g\|_{q,\alpha,\beta}. \tag{2.25}$$

Proof: By the relation (2.21) we have

$$\mathcal{W}(f,g)(.,\lambda) = g * f\psi_\lambda^{\alpha,\beta}(.),$$

by Young's inequality (2.14) we find that

$$\|\mathcal{W}(f,g)(.,\lambda)\|_{r,\alpha,\beta} \leq \|g\|_{q,\alpha,\beta}\|f\psi_\lambda^{\alpha,\beta}(.)\|_{p,\alpha,\beta},$$

the relation (2.5) gives the disered result. □

We have the following theorem wich can be used to derive an inversion formula for the Jacobi-Dunkl-Wigner transform.

Theorem 2.1 *Let $f, g \in L^1_{\alpha,\beta}(\mathbb{R}) \cap L^2_{\alpha,\beta}(\mathbb{R})$ such that $c_g := \int_\mathbb{R} g(y)d\mu_{\alpha,\beta}(y) \neq 0$ then we have*

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = \frac{1}{c_g}\int_\mathbb{R} \mathcal{W}(f,g)(x,\lambda)d\mu_{\alpha,\beta}(x). \tag{2.26}$$

Proof: From the relation (2.25) we deduce that the right quantity in (2.26) is well defined furthermore, for $\lambda \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} \mathcal{W}(f,g)(x,\lambda)d\mu_{\alpha,\beta}(x) = \int_{\mathbb{R}}[\int_{\mathbb{R}} f(y)\mathcal{T}^x_{\alpha,\beta}(g)(-y)\psi^{\alpha,\beta}_{\lambda}(y)d\mu_{\alpha,\beta}(y)]\mathrm{d}\mu_{\alpha}(x),$$

by using the relations (2.11),(2.12) we find that

$$\int_{\mathbb{R}} \mathcal{W}(f,g)(x,\lambda)d\mu_{\alpha,\beta}(x) = c_g\mathcal{F}_{\alpha,\beta}(f)(\lambda),$$

which give the result. □

Corollary 2.1 *With the hypothesis of the theorem 2.1, if furthermore we have $\mathcal{F}_{\alpha,\beta}(f) \in L^1_{\alpha,\beta}(\mathbb{R})$ then the following inversion formula for the Jacobi-Dunkl-Wigner transform $\mathcal{W}$ holds:*

$$f(x) = \frac{1}{c_g}\int_{\mathbb{R}} \psi^{\alpha,\beta}_{-\lambda}(x)[\int_{\mathbb{R}} \mathcal{W}(f,g)(y,\lambda)d\mu_{\alpha,\beta}(y)]d\sigma(\lambda).$$

Proof: By using the relation (2.26) we have

$$\int_{\mathbb{R}} \psi^{\alpha,\beta}_{-\lambda}(x)[\int_{\mathbb{R}} \mathcal{W}(f,g)(y,\lambda)d\mu_{\alpha,\beta}(y)]d\sigma(\lambda) = c_g\int_{\mathbb{R}} \mathcal{F}_{\alpha,\beta}(f)(\lambda)\psi^{\alpha,\beta}_{-\lambda}(x)d\sigma(\lambda),$$

by using inversion formula for the Jacobi-Dunkl transform (2.8) we find the result. □

3. Localization operators associated with the Jacobi-Dunkl-Wigner transform

3.1. Introduction

In this section we will define and give sufficient conditions for the boundedness, compactness and Schatten class properties of localization operators $\mathscr{L}_{u,v}(\sigma)$ associated with the Jacobi-Dunkl-Wigner transform in terms of properties of the symbol σ and the functions u and v.

Definition 3.1 *Let u and v be measurable functions on $\mathbb{R}$, σ be a measurable function on the set $\mathbb{R}^2$, we define the localization operator $\mathscr{L}_{u,v}(\sigma)$ asoociated with the Jacobi-Dunkl-Wigner transform by*

$$\mathscr{L}_{u,v}(\sigma)(f)(y) := \int_{\mathbb{R}^2} \sigma(x,\lambda)\mathcal{W}(f,u)(x,\lambda)\psi^{\alpha,\beta}_{\lambda}(y)\overline{\mathcal{T}^x_{\alpha,\beta}(v)(y)}\mathrm{d}\theta_{\alpha,\beta}(x,\lambda). \tag{3.1}$$

Remark 3.1 In accordance with the different choices of the symbol σ and the different continuities required, we need to impose different conditions on u,v, and then we obtain an operator on $L^p_{\alpha,\beta}(\mathbb{R})$ for all $1 \le p \le +\infty$.
It is more convenient to interpret the definition of $\mathscr{L}_{u,v}(\sigma)$ in a weak sense, that is for all $f \in L^p_{\alpha,\beta}(\mathbb{R}), g \in L^q_{\alpha,\beta}(\mathbb{R})$ we have

$$\langle \mathscr{L}_{u,v}(\sigma)(f) \mid g\rangle_{\mu_{\alpha,\beta}} = \int_{\mathbb{R}^2} \sigma(x,\lambda)\mathcal{W}(f,u)(x,\lambda)\overline{\mathcal{W}(g,v)(x,\lambda)}\mathrm{d}\theta_{\alpha,\beta}(x,\lambda). \tag{3.2}$$

we have the following result

Proposition 3.1 *Let $1 \le p \le +\infty$, the adjoint of the linear operator*

$$\mathscr{L}_{u,v}(\sigma) : L^p_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^p_{\alpha,\beta}(\mathbb{R})$$

is the operator

$$\mathscr{L}^*_{u,v}(\sigma) : L^{p'}_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^{p'}_{\alpha,\beta}(\mathbb{R})$$

where

$$\mathscr{L}^*_{u,v}(\sigma) = \mathscr{L}_{v,u}(\bar{\sigma}). \tag{3.3}$$

Proof: Let $f \in L^p_{\alpha,\beta}(\mathbb{R}), g \in L^q_{\alpha,\beta}(\mathbb{R})$ by using the relation (3.2) we have

$$\begin{aligned}\langle \mathscr{L}_{u,v}(\sigma)(f) \mid g\rangle_{\mu_{\alpha,\beta}} &= \int_{\mathbb{R}^2} \sigma(x,\lambda)\mathcal{W}(f,u)(x,\lambda)\overline{\mathcal{W}(g,v)(x,\lambda)}\mathrm{d}\theta_{\alpha,\beta}(x,\lambda) \\ &= \overline{\int_{\mathbb{R}^2} \overline{\sigma(x,\lambda)}\mathcal{W}(g,v)(x,\lambda)\overline{\mathcal{W}(f,u)(x,\lambda)}d\theta_{\alpha,\beta}(x,\lambda)} \\ &= \overline{\langle \mathscr{L}_{v,u}(\bar{\sigma})(g) \mid f\rangle_{\mu_{\alpha,\beta}}} = \langle f \mid \mathscr{L}_{v,u}(\bar{\sigma})(g)\rangle_{\mu_{\alpha,\beta}},\end{aligned}$$

we get

$$\mathscr{L}^*_{u,v}(\sigma) = \mathscr{L}_{v,u}(\bar{\sigma}).$$

□

In the sequel of this section, u and v will be any functions in $L^2_{\alpha,\beta}(\mathbb{R})$ such that $\|u\|_{2,\mu_\alpha} = \|v\|_{2,\mu_\alpha} = 1$.We note that this hypothesis is not essential and the result still true up some constant depending on $\|u\|_{2,\mu_\alpha}$ and $\|v\|_{2,\mu_\alpha}$.

3.2. Boundedness for $\mathscr{L}_{u,v}(\sigma)$ in S_∞

The main purpose of this subsection is to prove that the linear operator

$$\mathscr{L}_{u,v}(\sigma) : L^2_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^2_{\alpha,\beta}(\mathbb{R})$$

is bounded for all symbol $\sigma \in L^p_\theta(\mathbb{R}^2)$ with $1 \leq p + \infty$.We consider first the problem for $\sigma \in L^1_\theta(\mathbb{R}^2)$, next $\sigma \in L^\infty_\theta(\mathbb{R}^2)$ and we conclude by using interpolation theory.

Proposition 3.2 *Let $\sigma \in L^2_\theta(\mathbb{R}^2)$ then the localization operator $\mathscr{L}_{u,v}(\sigma)$ is in S_∞ and we have*

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{1,\theta_\alpha}. \tag{3.4}$$

Proof: Let $f, g \in L^2_{\alpha,\beta}(\mathbb{R})$ by using the relation (3.2) we have

$$\begin{aligned}&\mid \langle \mathscr{L}_{u,v}(\sigma)(f) \mid g\rangle_{\mu_{\alpha,\beta}} \leq \int_{\mathbb{R}^2} |\sigma(x,\lambda)||\mathcal{W}(f,u)(x,\lambda)|\mathcal{W}(g,v)(x,\lambda) \mid \mathrm{d}\theta_{\alpha,\beta}(x,\lambda) \\ &\leq \|\mathcal{W}(f,u)\|_{\infty,\theta}\|\mathcal{W}(g,v)\|_{\infty,\theta}\|\sigma\|_{1,\theta},\end{aligned}$$

by using the relation (2.24) we get

$$\left|\langle \mathscr{L}_{u,v}(\sigma)(f) \mid g\rangle_{\mu_{\alpha,\beta}}\right| \leq \|f\|_{2,\alpha,\beta}\|g\|_{2,\alpha,\beta}\|\sigma\|_{1,\theta_\alpha},$$

by (2.16) we find that

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{1,\theta}$$

□

Proposition 3.3 *Let $\sigma \in L^\infty_\theta(\mathbb{R}^2)$, the localization operators $\mathscr{L}_{u,v}(\sigma)$ is in S_∞ and we have*

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{\infty,\theta}. \tag{3.5}$$

Proof: Let $f, g \in L^2_{\alpha,\beta}(\mathbb{R})$ by using the relation (3.2) we have

$$\mid \langle \mathscr{L}_{u,v}(\sigma)(f) \mid g\rangle_{\mu_{\alpha,\beta}} \mid \leq \int_{\mathbb{R}^2} |\sigma(x,\lambda)||\mathcal{W}(f,u)(x,\lambda)|\mathcal{W}(g,v)(x,\lambda)|\mathrm{d}\theta_{\alpha,\beta}(x,\lambda),$$

by Hölder's inequality we find that

$$\mid \langle \mathscr{L}_{u,v}(\sigma)(f) \mid g\rangle_{\mu_{\alpha,\beta}} \mid \leq \|\sigma\|_{\infty,\theta}\|\mathcal{W}(f,u)\|_{2,\theta}\|\mathcal{W}(g,v)\|_{2,\theta},$$

by using the relation (2.23) we get

$$|\left\langle \mathscr{L}_{u,v}(\sigma)(f) \mid g\right\rangle_{\mu_{\alpha,\beta}}| \leq \|\sigma\|_{\infty,\theta}\|f\|_{2,\alpha,\beta}\|g\|_{2,\alpha,\beta},$$

thus

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{\infty,\theta}.$$

□

We can now associate a localization operator $\mathscr{L}_{u,v}(\sigma)$ to every symbol σ in $L^p_\theta(\mathbb{R}^2)$ for all $1 \leq p \leq +\infty$, and prove that $\mathscr{L}_{u,v}(\sigma)$ belongs to S_∞.

Theorem 3.1 *Let $\sigma \in L^p_\theta(\mathbb{R}^2)$,$1 \leq p \leq +\infty$ then there exists a unique bounded linear operator*

$$\mathscr{L}_{u,v}(\sigma) : L^2_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^2_{\alpha,\beta}(\mathbb{R})$$

such that

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{p,\theta}. \tag{3.6}$$

Proof: Let $\sigma \in L^p_\theta\left(\mathbb{R}^2\right), 1 \leq p \leq +\infty$ and $f \in L^2_{\alpha,\beta}(\mathbb{R})$ we consider the following operator

$$T : L^1_\theta\left(\mathbb{R}^2\right) \cap L^\infty_\theta\left(\mathbb{R}^2\right) \longrightarrow L^2_{\alpha,\beta}(\mathbb{R}),$$

given by

$$T(\sigma) = \mathscr{L}_{u,v}(\sigma)(f),$$

then by using the relations (3.4) and (3.5) we have

$$\|T(\sigma)\|_{2,\alpha,\beta} \leq \|f\|_{2,\alpha,\beta}\|\sigma\|_{1,\theta} \tag{3.7}$$

and

$$\|T(\sigma)\|_{2,\alpha,\beta} \leq \|f\|_{2,\alpha,\beta}\|\sigma\|_{\infty,\theta_\alpha}, \tag{3.8}$$

by using the relations (3.7),(3.8) and the Riesz-Thorin interpolation Theorem see [18, 21], the operator T may be uniquely extended to a linear operator on $L^2_{\alpha,\beta}(\mathbb{R})$ for all $1 \leq p \leq +\infty$ and we have

$$\|T(\sigma)\|_{2,\alpha,\beta} = \left\|\mathscr{L}_{u,v}(\sigma)(f)\right\|_{2,\alpha,\beta} \leq \|f\|_{2,\alpha,\beta}\|\sigma\|_{p,\theta}, \tag{3.9}$$

since (3.9) true for all $f \in L^2_\alpha(\mathbb{R})$ which gives the disered result.

□

3.3. $L^p_{\alpha,\beta}$-Boundedness of localization operator $\mathscr{L}_{u,v}(\sigma)$

Using Schur's technique [8] our main purpose of this subsection is to prove that the linear operator

$$\mathscr{L}_{u,v}(\sigma) : L^p_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^p_{\alpha,\beta}(\mathbb{R}),$$

is bounded for all $1 \leq p \leq +\infty$, we have the following theorem.

Theorem 3.2 *Let $\sigma \in L^1_\theta(\mathbb{R}^2)$ and $u, v \in L^1_{\alpha,\beta}(\mathbb{R}) \cap L^\infty_{\alpha,\beta}(\mathbb{R})$ then the localization operator $\mathscr{L}_{u,v}(\sigma)$ extend to a unique bounded linear operator from $L^p_{\alpha,\beta}(\mathbb{R})$ into itself for all $1 \leq p \leq +\infty$, furthermore we have*

$$\|\mathscr{L}_{u,v}(\sigma)\|_{B(L^p_\alpha(\mathbb{R}))} \leq max(\|u\|_{1,\alpha,\beta}\|v\|_{\infty,\alpha,\beta}, \|u\|_{\infty,\alpha,\beta}\|v\|_{1,\alpha,\beta})\|\sigma\|_{1,\theta_\alpha}.$$

Proof: Let F be the function defined on $\mathbb{R}^2$ by

$$F(y, s) = \int_{\mathbb{R}^2} \sigma(x, \lambda)\psi^{\alpha,\beta}_\lambda(y)\overline{\mathcal{T}^x_{\alpha,\beta}(v)(y)}\psi^{\alpha,\beta}_\lambda(s))\mathcal{T}^x_{\alpha,\beta}(u)(s)\mathrm{d}\theta_{\alpha,\beta}(x, \lambda),$$

by using Fubini's theorem we find that

$$\mathscr{L}_{u,v}(\sigma)(f)(y) = \int_{\mathbb{R}} F(y,s)f(s)d\mu_{\alpha,\beta}(s),$$

furthermore by using the relation (2.11) and Fubini's theorem we find that

$$\int_{\mathbb{R}} |F(y,s)|d\mu_{\alpha,\beta}(y) \leq \|u\|_{\infty,\alpha,\beta}\|v\|_{1,\alpha,\beta}\|\sigma\|_{1,\theta} \tag{3.10}$$

and

$$\int_{\mathbb{R}} |F(y,s)|d\mu_{\alpha,\beta}(s) \leq \|u\|_{1,\mu_\alpha}\|v\|_{\infty,\mu_\alpha}\|\sigma\|_{1,\theta} \tag{3.11}$$

by using (3.10),(3.11) and Schur's lemma [8] we can conclude that the linear operator

$$\mathscr{L}_{u,v}(\sigma) : L^p_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^p_{\alpha,\beta}(\mathbb{R}),$$

is bounded for all $1 \leq p \leq +\infty$ and we have

$$\|\mathscr{L}_{u,v}(\sigma)\|_{B(L^p_\alpha(\mathbb{R}))} \leq max(\|u\|_{1,\alpha,\beta}\|v\|_{\infty,\alpha,\beta}, \|u\|_{\infty,\alpha,\beta}\|v\|_{1,\alpha,\beta})\|\sigma\|_{1,\theta}.$$

□

3.4. Trace of the localization operators $\mathscr{L}_{u,v}(\sigma)$

The main result of this subsection is to prove that the localization operator

$$\mathscr{L}_{u,v}(\sigma) : L^2_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^2_{\alpha,\beta}(\mathbb{R}),$$

is in the Schatten-Von Neumann class S^p for all $1 \leq p \leq +\infty$, firstly we have the following result

Theorem 3.3 *Let $\sigma \in L^1_\alpha\left(\mathbb{R}^2\right)$ then the localization operator*

$$\mathscr{L}_{u,v}(\sigma) : L^2_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^2_{\alpha,\beta}(\mathbb{R})$$

is an Hilbert-Schmidt operator in particular it is compact and we have

$$\|\mathscr{L}_{u,v}(\sigma)\|_{HS} \leq 1 + \|\sigma\|^2_{1,\theta}.$$

Proof: Let $(\phi_k)_k$ be an orthonormal basis of $L^2_{\alpha,\beta}(\mathbb{R})$, by using Fubini's theorem and the relation (3.2) we get

$$\begin{aligned} \||\, \mathscr{L}_{u,v}(\sigma)\,(\phi_k)\|^2_{2,\alpha,\beta} &= \langle \mathscr{L}_{u,v}(\sigma)\,(\phi_k) \mid \mathscr{L}_{u,v}(\sigma)\,(\phi_k)\rangle_{\mu_{\alpha,\beta}} \\ &= \int_{\mathbb{R}^2} \sigma(x,\lambda)\mathcal{W}\left(\phi_k,u\right)(x,\lambda)\overline{\mathcal{W}\left(\mathscr{L}_{u,v}(\sigma)\,(\phi_k)\,,v\right)(x,\lambda)}\mathrm{d}\theta_{\alpha,\beta}(x,\lambda), \end{aligned}$$

by (2.22) we find that

$$\begin{aligned} \||\, \mathscr{L}_{u,v}(\sigma)\,(\phi_k)\|^2_{2,\alpha,\beta} = \int_{\mathbb{R}^2} & \sigma(x,\lambda)\mathcal{F}_\alpha\left(\phi_k\widetilde{\mathcal{T}^x_{\alpha,\beta}(u)}\right)(\lambda) \\ & \overline{\mathcal{F}_\alpha\left(\mathscr{L}_{u,v}(\sigma)\,(\phi_k)\,\widetilde{\mathcal{T}^x_{\alpha,\beta}(v)}\right)}(\lambda)\mathrm{d}\theta_{\alpha,\beta}(x,\lambda), \end{aligned}$$

but

$$\mathcal{F}_\alpha\left(\phi_k\widetilde{\mathcal{T}^x_{\alpha,\beta}(u)}\right)(\lambda) = \left\langle \psi^{\alpha,\beta}_\lambda(.)\widetilde{\mathcal{T}^x_{\alpha,\beta}(u)} \mid \phi_k\right\rangle_{\mu_{\alpha,\beta}},$$

by the relation (3.3) we get

$$\mathcal{F}_\alpha\left(\mathscr{L}_{u,v}(\sigma)\,(\phi_k)\,\mathcal{T}^x_{\alpha,\beta}(v)\right)(\lambda) = \left\langle \phi_k \mid \mathscr{L}_{v,u}(\bar{\sigma})\left(\overline{\widetilde{\mathcal{T}^x_{\alpha,\beta}(v)}}\right)\right\rangle_{\mu_{\alpha,\beta}},$$

So we find that

$$\left|\left|\left|\mathscr{L}_{u,v}(\sigma)(\phi_k)\right|\right|\right|^2_{2,\alpha,\beta} \leq \int_{\mathbb{R}^2}\left|\sigma(x,\lambda)\right|\left|\left\langle \psi^{\alpha,\beta}_\lambda(.)\widetilde{\mathcal{T}_{\alpha,\beta}(u)} \mid \phi_k\right\rangle_{\mu_{\alpha,\beta}}\right|$$
$$\left|\left\langle \phi_k \left| \mathscr{L}_{v,u}(\bar{\sigma})\left(\widetilde{\mathcal{T}^x_{\alpha,\beta}(v)}\psi^{\alpha,\beta}_\lambda(\cdot)\right\rangle_{\mu_{\alpha,\beta}}\right.\right| \mathrm{d}\theta_{\alpha,\beta}(x,\lambda)$$
$$\leq \frac{1}{2}\int_{\mathbb{R}^2}|\sigma(x,\lambda)|\left[\left|\left\langle \psi^{\alpha,\beta}_\lambda(.)\widetilde{\mathcal{T}^x_{\alpha,\beta}(u)} \mid \phi_k\right\rangle_{\mu_{\alpha,\beta}}\right|^2 +\right.$$
$$\left.\left|\left\langle \mathscr{L}_{v,u}(\bar{\sigma})\left(\left(\widetilde{\mathcal{T}^x_{\alpha,\beta}(v)}\left(\psi^{\alpha,\beta}_\lambda(.)\right)\right) |\phi_k\rangle_{\mu_{\alpha,\beta}}\right.\right|^2\right] \mathrm{d}\theta_{\alpha,\beta}(x,\lambda),$$

by using Fubini's theorem we find that

$$\|\mathscr{L}_{u,v}(\sigma)\|^2_{HS} \leq \frac{1}{2}\left[\int_{\mathbb{R}^2}|\sigma(x,\lambda)|\left[\sum_{k=1}^{+\infty}\left|\left\langle \psi^{\alpha,\beta}_\lambda(.)\widetilde{\mathcal{T}^x_{\alpha,\beta}(u)} \mid \phi_k\right\rangle_{\mu_{\alpha,\beta}}\right|^2 + \sum_{k=1}^{+\infty}\left|\left\langle \mathscr{L}_{v,u}(\bar{\sigma})\left(\widetilde{\mathcal{T}^x_{\alpha,\beta}(v)}\psi^{\alpha,\beta}_\lambda(.)\,|\phi_k\rangle_{\mu_{\alpha,\beta}}\right.\right|^2\right]\mathrm{d}\theta_{\alpha,\beta}(x,\lambda)\right].$$

By using Parseval's identity, the relations (2.5),(2.13),(3.4) and the fact that $\|u\|_{2,\alpha,\beta} = \|v\|_{2,\alpha,\beta} = 1$ we find that

$$\|\mathscr{L}_{u,v}(\sigma)\|^2_{HS} \leq \frac{1}{2}\|\sigma\|_{1,\theta}\left(1+\|\sigma\|^2_{1,\theta}\right) \leq \left(1+\|\sigma\|^2_{1,\theta}\right)^2 < \infty$$

which proves that $\mathscr{L}_{u,v}(\sigma)$ is an Hilbert-Schmidt operator so compact and we have

$$\|\mathscr{L}_{u,v}(\sigma)\|_{HS} \leq 1+\|\sigma\|^2_{1,\theta}.$$

□

In the following we prove that the localization operator

$$\mathscr{L}_{u,v}(\sigma): L^2_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^2_{\alpha,\beta}(\mathbb{R})$$

is compact for all $\sigma \in L^p_{\alpha,\beta}\left(\mathbb{R}^2\right)$.

Proposition 3.4 *Let $\sigma \in L^p_\theta\left(\mathbb{R}^2\right), 1 \leq p < +\infty$ then the localization operator*

$$\mathscr{L}_{u,v}(\sigma): L^2_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^2_{\alpha,\beta}(\mathbb{R})$$

is compact.

Proof: Let $\sigma \in L^1_\theta\left(\mathbb{R}^2\right)$ with $1 \leq p < +\infty$ and let $(\sigma_n)_n$ be a sequence of functions in $L^1_\theta\left(\mathbb{R}^2\right) \cap L^\infty_\theta\left(\mathbb{R}^2\right)$ such that $\sigma_n \longrightarrow \sigma$ in $L^p_\theta\left(\mathbb{R}^2\right)$ as $n \longrightarrow \infty$ then by using the relation (3.6) we find that

$$\|\mathscr{L}_{u,v}(\sigma_n) - \mathscr{L}_{u,v}(\sigma)\|_{S_\infty} \leq \|\sigma_n - \sigma\|_{p,\theta},$$

hence $\mathscr{L}_{u,v}(\sigma_n) \longrightarrow \mathscr{L}_{u,v}(\sigma)$ in S_∞ as $n \longrightarrow \infty$ on the other hand by theorem 3.3 we have $\mathscr{L}_{u,v}(\sigma_n)$ is in S_2 hence compact, it follows that $\mathscr{L}_{u,v}(\sigma)$ is compact. □

In the next theorem we obtain a $L^1_{\alpha,\beta}$-compactness result for the localization operator $\mathscr{L}_{u,v}(\sigma)$.

Theorem 3.4 *Let $\sigma \in L^1_\theta\left(\mathbb{R}^2\right)$, u and v in $L^2_{\alpha,\beta}(\mathbb{R}) \cap L^2_{\alpha,\beta}(\mathbb{R})$ then the localization operator*

$$\mathscr{L}_{u,v}(\sigma): L^1_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^1_{\alpha,\beta}(\mathbb{R})$$

is compact.

Proof: By using theorem 3.2 the linear operator

$$\mathscr{L}_{u,v}(\sigma) : L^1_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^1_{\alpha,\beta}(\mathbb{R})$$

is well defined, let $(f_n) \subset L^1_\alpha(\mathbb{R})$ such that $f_n \longrightarrow 0$ weakly in $L^1_{\alpha,\beta}(\mathbb{R})$ as $n \longrightarrow \infty$, it is enough to prove that $\lim_{n\to+\infty} \|\mathscr{L}_{u,v}(\sigma)(f_n)\|_{1,\alpha,\beta} = 0$. By using the relation (3.1) we have

$$\|\mathscr{L}_{u,v}(\sigma)(f_n)\|_{1,\alpha,\beta}$$

$$\leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}^2} \left| \sigma(x,\lambda) \|\mathcal{W}(f_n,u)(x,\lambda)\| \mathcal{T}^x_{\alpha,\beta}(v)(y) \right| \mathrm{d}\theta_{\alpha,\beta}(x,\lambda) \right] d\mu_\alpha(y). \tag{3.12}$$

Using the fact that $f_n \longrightarrow 0$ weakly in $L^2_{\alpha,\beta}(\mathbb{R})$ as $n \longrightarrow \infty$, we deduce that

$$\lim_{n\to+\infty} \left| \mathcal{W}(f_n,u)(x,\lambda) \| \mathcal{T}^x_{\alpha,\beta}(v)(y) \right| = 0 \tag{3.13}$$

for all $x, y, \lambda \in \mathbb{R}$, on the other hand as $f_n \longrightarrow 0$ weakly in $L^1_{\alpha,\beta}(\mathbb{R})$ as $n \longrightarrow \infty$, there exists a positive conctant c such that $\|f_n\|_{1,\alpha,\beta} \leq c$, so we find that

$$| \mathcal{W}(f_n,u)\left((x,\lambda)\right. \left\| \mathcal{T}^x_{\alpha,\beta}(v)(y) \right| \leq c|\sigma(x,\lambda)\| \, u\|_{\infty,\alpha,\beta}|v(y)|, \tag{3.14}$$

by using Fubuni's theorem we get

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}^2} \left| \sigma(x,\lambda) \|\mathcal{W}(f_n,u)(x,\lambda)\| \mathcal{T}^x_{\alpha,\beta}(v)(y) \right| \mathrm{d}\theta_{\alpha,\beta}(x,\lambda) \right] d\mu_\alpha(y) \leq c\|\sigma\|_{1,\theta}\|u\|_{\infty,\alpha,\beta}\|v\|_{1,\alpha,\beta} < \infty. \tag{3.15}$$

Thus from the relations (3.12),(3.13),(3.14),(3.15) and the Lebesgue dominated convergence theorem we deduce that $\lim_{n\to+\infty} \|\mathscr{L}_{u,v}(\sigma)(f_n)\|_{1,\alpha,\beta} = 0$ and the proof is complete. □

In the following we show that the localization operator $\mathscr{L}_{u,v}$ is in the trace class S^1.

Theorem 3.5 *Let $\sigma \in L^1_\theta(\mathbb{R}^2)$ then the localization operator*

$$\mathscr{L}_{u,v}(\sigma) : L^2_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^2_{\alpha,\beta}(\mathbb{R})$$

is in the trace class operators S_1 and we have

$$\|\tilde{\sigma}\|_{1,\theta} \leq \|\mathscr{L}_{u,v}(\sigma)\|_{S_1} \leq \|\sigma\|_{1,\theta} \tag{3.16}$$

where $\tilde{\sigma}$ is given by

$$\tilde{\sigma}(x,\lambda) = \left\langle \mathscr{L}_{u,v}(\sigma)\left(\psi^{\alpha,\beta}_\lambda(.)\mathcal{T}^x_{\alpha,\beta}(u)\right) \mid \psi^{\alpha,\beta}_\lambda(.)\mathcal{T}^x_{\alpha,\beta}(v) \right\rangle_{\mu_{\alpha,\beta}}.$$

Proof: Let $\sigma \in L^1_\theta(\mathbb{R}^2)$ by using theorem 3.4 we have $\mathscr{L}(\sigma)$ is a compact operator,using [21], there exists an orthonormal basis ϕ_j for $j = 1, 2, \ldots$ for the orthogonal complement of the kernel of the operator $\mathscr{L}_{u,v}(\sigma)$ consisting of eigenvectors of $|\mathscr{L}_{u,v}(\sigma)|$ and $h_j j = 1, 2, \ldots$, an orthonormal set in $L^2_{\alpha,\beta}(\mathbb{R})$ such that the localization operators $\mathscr{L}_{u,v}(\sigma)$ can be diagonalized as

$$\mathscr{L}_{u,v}(\sigma)(f) = \sum_{j=1}^{+\infty} s_j \langle f \mid \phi_j \rangle_{\mu_{\alpha,\beta}} h_j, \tag{3.17}$$

where s_j for $j = 1, 2, \ldots$ are the positive singular values of $\mathscr{L}_{u,v}(\sigma)$ corresponding to ϕ_j, then we get :

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S^1} = \sum_{j=1}^{+\infty} s_j = \sum_{j=1}^{+\infty} \langle \mathscr{L}_{u,v}(\sigma)(\phi_j) \mid h_j \rangle_{\mu_{\alpha,\beta}},$$

by using the relations (3.1) and (3.2) we find that

$$\langle \mathscr{L}_{u,v}(\sigma)(\phi_j) \mid h_j \rangle_{\mu_{\alpha,\beta}} = \int_{\mathbb{R}^2} \left| \sigma(x,\lambda) \right| \left| \left\langle \psi_\lambda^{\alpha,\beta}(.)\mathcal{T}_{\alpha,\beta}^x(u) \mid \phi_j \right\rangle_{\mu_{\alpha,\beta}} \right| \left| \left\langle \psi_\lambda^{\alpha,\beta}(.)\widetilde{\mathcal{T}_{\alpha,\beta}^x(v)} \mid h_j \right\rangle_{\mu_{\alpha,\beta}} \right| \mathrm{d}\theta_{\alpha,\beta}(x,\lambda),$$

So we find that

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S^1} \leq \frac{1}{2} \int_{\mathbb{R}^2} |\sigma(x,\lambda)| \left[\sum_{j=1}^{+\infty} \left| \left\langle \psi_\lambda^{\alpha,\beta}(.)\mathcal{T}_{\alpha,\beta}^x(u) \mid \phi_j \right\rangle_{\mu_{\alpha,\beta}} \right|^2 | + \sum_{j=1}^{+\infty} \left| \left\langle \psi_\lambda^{\alpha,\beta}(.)\mathcal{T}_{\alpha,\beta}^x(v) \mid h_j \right\rangle_{\mu_{\alpha,\beta}} \right|^2 \right] \mathrm{d}\theta_{\alpha,\beta}(x,\lambda)$$

by using parseval's identity we get

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S^1} \leq \frac{1}{2} \int_{\mathbb{R}^2} |\sigma(x,\lambda)| \left[\|\psi_\lambda^{\alpha,\beta}(.)\right) \mathcal{T}_{\alpha,\beta}^x(u) \|_{2,\alpha,\beta}^2 + \| \psi_\lambda^{\alpha,\beta}(.)\mathcal{T}_{\alpha,\beta}^x(v)\|_{2,\alpha,\beta}^2 \right] \mathrm{d}\theta_{\alpha,\beta}(x,\lambda).$$

By using the relation (2.5),(2.13) and the fact that $\|u\|_{2,\alpha,\beta} = \|v\|_{2,\alpha,\beta} = 1$ we get

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_1} \leq \|\sigma\|_{1,\theta}.$$

Now we prove that $\mathscr{L}_{u,v}(\sigma)$ satisfies the first member of (3.16), it is easy to see that $\tilde{\sigma} \in L_\theta^1\left(\mathbb{R}^2\right)$ and by using the relation (3.17) and Fubini's theorem we find that

$$\int_{\mathbb{R}^2} |\tilde{\sigma}(x,\lambda)| \mathrm{d}\theta_{\alpha,\beta}(x,\lambda) \leq \frac{1}{2} \sum_{j=1}^{+\infty} s_j \left[\int_{\mathbb{R}^2} \left(\left| \left\langle \psi_\lambda^{\alpha,\beta}(.)\mathcal{T}_{\alpha,\beta}^x(u) \mid \phi_j \right\rangle_{\mu_{\alpha,\beta}} \right|^2 + \left| \left\langle h_j \mid \psi_\lambda^{\alpha,\beta}(.)\mathcal{T}_{\alpha,\beta}^x(v) \right\rangle_{\mu_{\alpha,\beta}} \right|^2 \right) \mathrm{d}\theta_{\alpha,\beta}(x,\lambda) \right]$$

$$= \frac{1}{2} \sum_{j=1}^{+\infty} s_j \left[\int_{\mathbb{R}^2} |\mathcal{W}(\phi_j, u)(x,\lambda)|^2 + |\mathcal{W}(h_j, v)(x,\lambda)|^2 \right] \mathrm{d}\theta_{\alpha,\beta}(x,\lambda),$$

by using the relation (2.23) and the fact that $\|u\|_{2,\alpha,\beta} = \|v\|_{2,\alpha,\beta} = 1$ we get

$$\int_{\mathbb{R}^2} |\tilde{\sigma}(x,\lambda)| d\theta_{\alpha,\beta}(x,\lambda) \leq \frac{1}{2} \sum_{j=1}^{+\infty} s_j \left(\|u\|_{2,\alpha,\beta}^2 + \|v\|_{2,\alpha,\beta}^2 \right) = \sum_{j=1}^{+\infty} s_j = \|\mathscr{L}_{u,v}(\sigma)\|_{S_1},$$

the proof is complete. □

In the following we give a trace formula for the localization operators $\mathscr{L}_{u,v}(\sigma)$.

Theorem 3.6 *Let $\sigma \in L_\theta^1\left(\mathbb{R}^2\right)$ we have the following trace formula*

$$\mathrm{Tr}\left(\mathcal{L}_{u,v}(\sigma)\right) = \int_{\mathbb{R}^2} \sigma(x,\lambda) \left\langle \psi_\lambda^{\alpha,\beta}(.)\widetilde{\mathcal{T}_{\alpha,\beta}^x(u)} \mid \psi_\lambda^{\alpha,\beta}(.)\widetilde{\mathcal{T}_{\alpha,\beta}^x(v)} \right\rangle_{\mu_{\alpha,\beta}} \mathrm{d}\theta_{\alpha,\beta}(x,\lambda)$$

Proof: Let $\{\phi_j, j = 1, 2, \ldots\}$ be an orthonormal basis for $L_{\alpha,\beta}^2(\mathbb{R})$. From Theorem 3.5, the localization operator $\mathscr{L}_{u,v}(\sigma)$ belongs to S_1, then by the definition of the trace given by the relation (2.17), Fubini's theorem and Parseval's identity, we get

$$\mathrm{Tr}\left(\mathscr{L}_{u,v}(\sigma)\right) = \sum_{j=1}^{\infty} \left\langle \mathscr{L}_{u,v}(\sigma)(\phi_j), \phi_j \right\rangle_{\mu_{\alpha,\beta}}$$

$$= \int_{\mathbb{R}^2} \sigma(x,\lambda) \sum_{j=1}^{\infty} \left\langle \phi_j, \psi_\lambda^{\alpha,\beta}(.), \overline{\widetilde{\mathcal{T}_{\alpha,\beta}^x(u)}.} \right\rangle_{\mu_{\alpha,\beta}} \overline{\left\langle \widetilde{\mathcal{T}_{\alpha,\beta}^x(v)}\, \psi_\lambda^{\alpha,\beta}(.), \phi_j \right\rangle_{\mu_{\alpha,\beta}}} \mathrm{d}\theta_{\alpha,\beta}(x,\lambda)$$

$$= \int_{\mathbb{R}^2} \sigma(x,\lambda) \left\langle \psi_\lambda^{\alpha,\beta}(.)\widetilde{\mathcal{T}_{\alpha,\beta}^x(u)} \mid \psi_\lambda^{\alpha,\beta}(.)\widetilde{\mathcal{T}_{\alpha,\beta}^x(v)} \right\rangle_{\mu_{\alpha,\beta}} \mathrm{d}\theta_{\alpha,\beta}(x,\lambda),$$

and the proof is complete. □

Corollary 3.1 *If $u = v$ and if σ is a real valued, and nonegative function in $L^1_\theta\left(\mathbb{R}^2\right)$ then the localization operator*

$$\mathscr{L}_u(\sigma) : L^2_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^2_{\alpha,\beta}(\mathbb{R})$$

is a positive operator and by using the relations (2.18) and (3.18) we find that

$$\|\mathscr{L}_u(\sigma)\|_{S_1} = \int_{\mathbb{R}^2} \sigma(x,\lambda) \left\| \psi^{\alpha,\beta}_\lambda(.)\widetilde{\mathcal{T}^x_{\alpha,\beta}(u)} \right\|^2_{2,\alpha,\beta} \, \mathrm{d}\theta_{\alpha,\beta}(x,\lambda)$$

here $\mathscr{L}_u(\sigma)$ denote the operator $\mathscr{L}_{u,u}$.

In the following we give the main result of this section.

Corollary 3.2 *Let σ in $L^p_\theta\left(\mathbb{R}^2\right), 1 \leq p \leq +\infty$ then, the localization operator*

$$\mathscr{L}_{u,v}(\sigma) : L^2_{\alpha,\beta}(\mathbb{R}) \longrightarrow L^2_{\alpha,\beta}(\mathbb{R})$$

is in S^p and we have

$$\|\mathscr{L}_{u,v}(\sigma)\|_{S_p} \leq \|\sigma\|_{p,\theta}.$$

Proof: The result follows from (3.5) and (3.16) and by interpolation theory see [21]. □

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Not applicable.

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