Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–6.
ISSN-0037-8712
doi:10.5269/bspm.68311

Vertex Stress Polynomial of a Graph

Prajna S. Rai, R. Rajendra* and P. Siva Kota Reddy

ABSTRACT: The notion of stress of a vertex in a graph was introduced by Alfonso Shimbel in 1953. The stress of a vertex in a graph is the number of shortest paths passing through that vertex. In this paper, we introduce the concept of vertex stress polynomial of a graph and obtain some results including a characterization of graphs with non-zero constant vertex stress polynomial.

Key Words: Graph, Geodesic, Graph Polynomial, Stress of a vertex, Stress regular graph.

Contents

1. Introduction

For definitions of common terms and concepts in graph theory, we use the Harary's textbook [4]. Throughout this paper, by a graph $G = (V, E)$ we mean a finite, simple undirected graph. A shortest path between two vertices u and w in G is called u-w geodesic. In 1953, Alfonso Shimbel [15] introduced the notion of vertex stress for graphs as a centrality measure. Stress of a vertex v in a graph G is the number of shortest paths (geodesics) passing through v. This concept has many applications including the study of biological and social networks. Many stress related concepts in graphs and topological indices have been defined and studied by several authors [2,7,5,6,9,10,11,12,13,14,8].

In [2], K. Bhargava et al. have given a characterization of graphs with all vertices of zero stress except for one in their Theorem 4.1. viz., *a connected graph G with at least 3 vertices has all vertices of zero stress except for one if and only if G is a graph with a unique cut-vertex such that all its blocks are complete subgraphs of G.*

A k-regular graph with ν vertices is said to be strongly regular if there exist integers λ and μ such that any two adjacent vertices have λ common neighbors and any two non-adjacent vertices have μ common neighbors [3]. In this case we write $G = \mathrm{srg}(\nu, k, \lambda, \mu)$.

First and second stress indices of graphs have been introduced by R. Rajendra et al. [10]. The First stress index $S_1(G)$ and the second stress index $S_2(G)$ of a simple graph G are defined respectively, as

$$S_1(G) = \sum_{v\in V(G)} \mathrm{str}(v)^2 \tag{1.1}$$

and

$$S_2(G) = \sum_{uv\in E(G)} \mathrm{str}(u)\mathrm{str}(v). \tag{1.2}$$

* Corresponding author
2010 *Mathematics Subject Classification:* 05Cxx, 05C31.
Submitted May 21, 2023. Published June 21, 2023

The concept of stress-sum index of graphs has been introduced by R. Rajendra et al. [9]. The stress-sum index $SS(G)$ of a simple graph G is defined by

$$SS(G) = \sum_{uv \in E(G)} \text{str}(u) + \text{str}(v). \tag{1.3}$$

The concept of vertex degree polynomial of a graph has been introduced by H. Ahmed et al. [1]. The vertex degree polynomial of a graph $G = (V, E)$ is defined as

$$VD(G, x) = \sum_{uv \in E(G)} d(u)x^{d(v)}, \tag{1.4}$$

where the summation is around both the possibilities uv and vu in $E(G)$.

The aim of this paper is to introduce vertex stress polynomial of a graph. The definition of vertex stress polynomial of a graph is given in section 2 followed by an example. In section 3, we obtain some results related to vertex stress polynomial of graphs. Mainly, we characterize the graphs with non-zero constant vertex stress polynomial.

2. Definition and Example

Definition 2.1 *The vertex stress polynomial of a graph $G = (V, E)$ is defined as*

$$VS(G, x) = \sum_{uv \in E(G)} str(u)x^{str(v)}, \tag{2.1}$$

where the summation is around both the possibilities uv and vu in $E(G)$.

Example 2.1 Consider the path P_3.

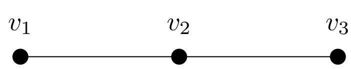

Figure 1: The path P_3

We have, $\text{str}(v_1) = 0$, $\text{str}(v_2) = 1$ and $\text{str}(v_3) = 0$. The vertex stress polynomial of P_3 is

$$\begin{aligned} VS(P_3, x) &= \sum_{uv \in E(G)} \text{str}(u)x^{\text{str}(v)} \\ &= \text{str}(v_1)\text{x}^{strv_2} + str(v_2)\text{x}^{strv_1} + strv_3 x^{strv_2} + strv_2\text{x}^{strv_3} \\ &= 0x^1 + 1x^0 + 1x^0 + 0x^1 \\ &= 2, \text{a constant polynomial.} \end{aligned}$$

3. Results

In this section, we prove some results involving vertex stress polynomial. At the end, we present a characterization of graphs with vertex stress polynomial a non-zero constant.

Proposition 3.1 *If G_1 and G_2 are any two graphs such that $G_1 \cong G_2$, then $VS(G_1, x) = VS(G_2, x)$.*

Proof: Obvious. □

Remark 3.1 The converse of the Proposition 3.1 is not true. There are non-isomorphic graphs having the same vertex stress polynomial. For instance, $VS(K_n, x) = VS(K_1, x) = 0$, for all $n \geq 1$, but $K_n \ncong K_1$ for $n > 1$.

Proposition 3.2 *Let $G_1, G_2, \ldots, G_m$ be components of a disconnected graph H. Then vertex stress polynomial of H is given as*

$$VS(H,x) = VS(G_1,x) + VS(G_2,x) + \cdots + VS(G_n,x).$$

Proof: We have $H = \bigcup_{i=1}^{m} G_i$. Note that $uv \in E(H)$ if and only if uv belongs to the same component. Hence

$$\begin{aligned} VS(H,x) &= VS(\cup_{i=1}^{m} G_i, x) \\ &= \sum_{u_{1_i}v_{1_i} \in E(G_1)} \text{str}(u_{1_i}) x^{str(v_{1_i})} + \cdots + \sum_{u_{m_i}v_{m_i} \in E(G_m)} str(u_{m_i}) x^{\text{str}(v_{m_i})} \\ &= VS(G_1,x) + \cdots + VS(G_m,x). \end{aligned}$$ □

Proposition 3.3 *Let G be a graph with vertex stress polynomial $VS(G,x)$. Then,*

$$\frac{d}{dx} VS(G,x)\bigg|_{x=1} = 2S_2(G)$$

and

$$VS(G,x)|_{x=1} = 2SS(G),$$

where $S_2(G)$ is the second-stress index and $SS(G)$ stress-sum index of G.

Proof: Follows from the Eqs. (2.1), (1.2) and (1.3). □

Proposition 3.4

1. *For any complete graph K_n, $VS(K_n,x) = 0$.*
2. *For any positive integers n,k such that $k \leq n-1$, $VS(G,x) = nk^2x^k$ if and only if G is k-stress regular graph with n vertices.*
3. *For the path P_n,*

$$VS(P_n,x) = \sum_{i=1}^{n-2} i(n-i-1)[x^{(n-i-2)(i+1)} + x^{(n-i)(i-1)}].$$

4. *For the complete bipartite graph $K_{r,s}$,*

$$VS(K_{r,s},x) = rs\left[\frac{r(r-1)}{2} x^{\frac{s(s-1)}{2}} + \frac{s(s-1)}{2} x^{\frac{r(r-1)}{2}}\right].$$

5. *Let $Wd(n,m)$ denote the windmill graph constructed for $n \geq 2$ and $m \geq 2$ by joining m copies of the complete graph K_n at a shared universal vertex v. Then*

$$VS(Wd(n,m),x) = \frac{m^2(m-1)(n-1)^3}{2}.$$

Proof:

1. In a complete graph, every vertex has zero stress. Hence the result follows.

2. Let G be a k-stress regular graph with n vertices. Then $str(v) = k$, for all $v \in V(G)$. By the Definition 2.1, we have

$$\begin{aligned} VS(G,x) &= \sum_{uv \in E(G)} kx^k \\ &= 2ekx^k, \text{ where } e \text{ is the number of edges} \\ &= (\text{sum of degrees of all vertices}) \cdot kx^k \\ &\qquad\qquad (\because \text{By hand shaking lemma}) \\ &= nk^2x^k. \end{aligned}$$

On the other hand, if $VS(G,x) = nk^2x^k$, then we need to prove that $str(v) = k$, for all $v \in V(G)$. Suppose that $VS(G,x) = nk^2x^k$. Then by hand shaking lemma, we have

$$VS(G,x) = 2ekx^k \tag{3.1}$$

So, for any edge uv it follows that $str(u) = 0$ or k and $str(v) = 0$ or k. Hence $VS(G,x)$ will be of the form

$$\sum str(u)x^k + \sum str(v)x^0 + \sum kx^k + \sum 0x^0 \tag{3.2}$$

If $k = 0$, then there is nothing to prove. If $k \neq 0$ and if there is a vertex u with $str(u) = k$, then from Eqs. (3.1) and (3.2) and the fact G is connected it follows that $str(v) = k$, for all $v \in V$.

3. Consider the path P_n shown in Figure 2.

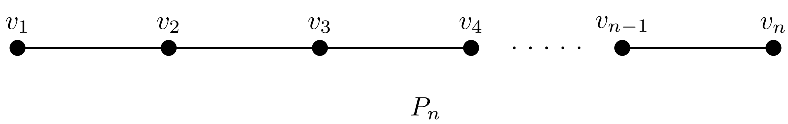

Figure 2: The path P_n on n vertices.

For the vertex v_i, we have

$$\text{str}(v_i) = (i-1)(n-i), \qquad 1 \le i \le n.$$

Then, by the Definition 2.1, we have

$$\begin{aligned} VS(P_n,x) &= \sum_{i=1}^{n-1} \left[\text{str}(v_i)x^{\text{str}(v_{i+1})} + \text{str}(v_{i+1})x^{\text{str}(v_i)}\right] \\ &= \sum_{i=1}^{n-1} (n-i)(i-1)x^{(n-i-1)i} + \sum_{i=1}^{n-1} i(n-i-1)x^{(n-1)(i-1)} \\ &= \sum_{i=0}^{n-2} i(n-i-1)x^{(n-i-2)(i+1)} + \sum_{i=1}^{n-1} i(n-i-1)x^{(n-i)(i-1)} \\ &= \sum_{i=1}^{n-2} i(n-i-1)x^{(n-i-2)(i+1)} + \sum_{i=1}^{n-2} i(n-i-1)x^{(n-i)(i-1)} \\ &= \sum_{i=1}^{n-2} i(n-i-1)[x^{(n-i-2)(i+1)} + x^{(n-i)(i-1)}]. \end{aligned}$$

4. If A and B are the partite sets in a complete bipartite graph $K_{r,s}$ with $|A| = r$ and $|B| = s$, then

$$\mathrm{str}(v) = \begin{cases} \dfrac{s(s-1)}{2}, & \text{if } v \in A; \\ \dfrac{r(r-1)}{2}, & \text{if } v \in B. \end{cases}$$

Then, by Definition 2.1, we have

$$VS(G, x) = rs[(r(r-1)/2)x^{s(s-1)/2} + (s(s-1)/2)x^{r(r-1)/2}].$$

5. In $W_d(n, m)$ all vertices have stress equal to zero except the universal vertex v. We have $str(v) = m(m-1)(n-1)^2/2$ (see [2, Proposition 3.1]). Then, by the Definition 2.1, we have

$$VS(W_d, x) = deg(v)str(v) = \frac{m^2(m-1)(n-1)^3}{2}.$$

□

Corollary 3.1 *For a strongly regular graph $G = srg(v, k, \lambda, \mu)$,*

$$VS(G, x) = 2e\frac{k(k-1-\lambda)}{2}x^{k(k-1-\lambda)/2}.$$

Proof: A strongly regular graph $G = \mathrm{srg}(v, k, \lambda, \mu)$ is stress regular [2, Corollary 5.4] and for any vertex v in G, we have $\mathrm{str}(v) = k(k-1-\lambda)/2$. Hence from Proposition 3.4 (ii), it follows that

$$VS(G, x) = 2e\frac{k(k-1-\lambda)}{2}x^{k(k-1-\lambda)/2}.$$ □

Theorem 3.1 *Let G be a connected graph with at least 3 vertices. Then the vertex stress polynomial of G is a non-zero constant if and only if G is a graph with unique cut-vertex such that all its blocks are complete subgraphs of G.*

Proof: If $VS(G, x)$ is a non-zero constant, then given any $uv \in E(G)$, either $str(u) = 0$ or $str(v) = 0$. So, there exists a vertex v with $strv \neq 0$. Let

$$W = \{v \in V(G) | str(v) \neq 0\}.$$

Note that all the vertices in W are mutually non-adjacent. Otherwise there exists $u \in W$ such that $str(u) \neq 0$ which implies $VS(G, x)$ is not a constant. We claim that W is a singleton set. If W is not a singleton set, then there exist $u, v \in W$, such that $str(u) \neq 0$, $str(v) \neq 0$ and u, v are non-adjacent. Since G is connected, there exists a path between u and v, and hence there exists a geodesic between u and v of length ≥ 2. This implies that there exists a vertex w adjacent to u (also there exist one for v) with stress $\neq 0$. Thus we have $w, u \in W$ with w and u are adjacent, which is a contradiction to the fact that all the vertices in W are mutually non-adjacent. Therefore, W is a singleton set. Applying [2, Theorem 4.1] we see that G is a graph with a unique cut-vertex such that all its blocks are complete subgraph of G.

The converse part follows directly from the Definition 2.1. □

Acknowledgments

The authors would like to thank the referees for their invaluable comments and suggestions which led to the improvement of the manuscript.

References

1. H. Ahmed, A. Alwardi and R. Salestina M, *On the vertex degree polynomial of graphs*, TWMS J. App. and Eng. Math., 13(1), 232–245, (2023).
2. K. Bhargava, N. N. Dattatreya and R. Rajendra, *On stress of a vertex in a graph*, *Preprint*, (2020).
3. R. C. Bose, *Strongly regular graphs, partial geometries and partially balanced designs*, Pacific J. Math., 13, 389–419, (1963).
4. F. Harary, *Graph Theory*, Addison Wesley, Reading, Mass, (1972).
5. John Kok, *Vertex stress related parameters for certain Kneser graphs*, Acta Univ. Sapientiae, Inform., 13(2), 324–334, (2021).
6. John Kok, *Vertex stress in generalized Johnson graphs of diameter 2*, Open J. Discrete Appl. Math., 5(2), 6–11, (2022).
7. K. B. Mahesh, R. Rajendra and P. Siva Kota Reddy, *Square Root Stress Sum Index for Graphs*, Proyecciones, 40(4), 927–937, (2021).
8. Raksha Poojary, K. Arathi Bhat, S. Arumugam and Manjunatha Prasad Karantha, *The stress of a graph*, Commun. Comb. Optim., 8(1), 53–65, (2023).
9. R. Rajendra, P. Siva Kota Reddy and C. N. Harshavardhana, *Stress-Sum index of graphs*, Sci. Magna, 15(1), 94–103, (2020).
10. R. Rajendra, P. Siva Kota Reddy and Ismail Naci CANGÜL, *Stress Indices of Graphs*, Advn. Stud. Contemp. Math., 31, 163–173, (2021).
11. R. Rajendra, P. Siva Kota Reddy and C. N. Harshavardhana, *Tosha Index for Graphs*, Proceedings of the Jangjeon Math. Soc., 24(1), 141–147, (2021).
12. R. Rajendra, P. Siva Kota Reddy, K.B. Mahesh and C. N. Harshavardhana, *Richness of a Vertex in a Graph*, South East Asian J. Math. Math. Sci., 18(2), 149–160, (2022).
13. R. Rajendra, P. Siva Kota Reddy and C. N. Harshavardhana, *Stress-Difference Index for Graphs*, *Bol. Soc. Parana. Mat. (3)*, accepted for publication.
14. R. Rajendra, P. Siva Kota Reddy, C. N. Harshavardhana and Khaled A. A. Alloush, *Squares Stress Sum Index for Graphs*, Proceedings of the Jangjeon Math. Soc., accepted for publication.
15. A. Shimbel, *Structural Parameters of Communication Networks*, Bull. Math. Biol., 15, 501-507, (1953).

Prajna S Rai,
Department of Mathematics
Sri Jayachamarajendra College of Engineering
JSS Science and Technology University
Mysuru–570 006, INDIA
E-mail address: `prajna.seetharam@gmail.com`

and

R. Rajendra,
Department of Mathematics
Field Marshal K.M. Cariappa College
(A Constituent College of Mangalore University)
Madikeri-571 201, INDIA
E-mail address: `rrajendrar@gmail.com`

and

P. Siva Kota Reddy,
Department of Mathematics
Sri Jayachamarajendra College of Engineering
JSS Science and Technology University
Mysuru–570 006, INDIA
E-mail address: `pskreddy@jssstuniv.in`