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Time fractional order spatio-temporal SIR model with therapy: Global analysis

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ABSTRACT: In this work, we will study a time fractional order spatio-temporal SIR model with therapy and vaccination. The model is described by a system of reaction-diffusion equations incorporating a fractional derivative. The therapy will be added to the model in order to describe the effect of treatment on the population dynamics. The existence, boundedness and uniqueness of the solution are proved. The global stability of the equilibria is established. Numerical simulations are curried out in order to show the equilibria stability and the effect of therapy.

Key Words: Global stability, fractional derivative, reaction-diffusion systems, spatio-temporal SIR epidemic model, numerical simulation.

Contents

1. Introduction

Infectious diseases have had a devastating impact on the human population through epidemics and pandemics that have broken out over time. According to the World Health Organization, these diseases have been responsible for more than 17 million deaths every year [36,20,26].

Over the years, mathematical modeling of infectious diseases has become an increasingly crucial tool to prevent, predict and control these diseases [37,28,27]. Since 1760, when Daniel Bernoulli developed the first model of smallpox disease, many mathematical models have been used to study disease transmission dynamics, to predict the extent of disease spread, to evaluate prevention and control strategies, and to assist in public health decision-making [40,42,34,7].

The essence of mathematical modeling lies in the formulation of a set of mathematical equations that reproduce reality [38]. Mathematical models have evolved over time from small sets of ordinary differential equations to sophisticated compartmentalized models with multiple equations [19,25]. These models make it possible to describe the complex interactions between the different compartments of the population (for example, healthy, infected, recovered individuals, etc.) and to take into account factors such as the transmission of the disease, the duration of incubation , period of infectiousness, effectiveness of public health interventions, etc.

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One of the simplest, yet most powerful, disease models is the standard Susceptible-Infected-Recovered (SIR) model, which was first introduced by Kermack and McKendrick in 1927 [29,30,31]. In a standard SIR model, the host population is divided into three compartments: susceptible individuals (Susceptibles), infected individuals (Infected) and recovered individuals (Recovered), designated respectively by S(t), I(t) and R (t). These quantities reflect the number of individuals in each compartment at different points in time [10,43].

The standard SIR model without birth and death is represented by a set of ordinary differential equations [23]

$$\begin{aligned} \frac{dS(t)}{dt} &= -\beta S(t)I(t), \\ \frac{dI(t)}{dt} &= \beta S(t)I(t) - rI(t), \\ \frac{dR(t)}{dt} &= rI(t), \end{aligned} \tag{1.1}$$

with β also known as "transmission rate", is the average number of susceptible individuals infected by one infectious individual per contact per unit of time, while r, also known as "recovery rate", is the average number of infected individuals recovered per unit of time.

Over the decades, the standard SIR model has been extended in various forms by adding different compartments to address the spatio-temporal and social aspects of disease dynamics or to study the impact of intervention strategies on the disease transmission dynamics in different communities [33,15]. For instance, contaminated environment [21,22], delays [45], multiple infection strains [1], and multiple infection pathways [2].

Most classical models only consider the temporal variable t. However, the propagation of infection is not solely dependent on time, as spatial factors x can also play a significant role [12,24]. Consequently, many researchers have explored infection propagation in relation to both time and space [8,11,16]. For instance, the numerical study of an SIR epidemic model with diffusion is conducted in [11]. An asymptotic study of an SIR reaction-diffusion model with a linear source is carried out in [16].

In some cases, the state of a system is dependent on its history, and therefore, fractional derivatives can be used in modeling. Fractional derivatives possess a unique characteristic known as the memory effect, which can be useful for several problems. Furthermore, the use of fractional derivatives can expand the region of stability of dynamical systems. Consequently, models that incorporate fractional derivatives have been successfully used to describe various phenomena in different fields, using differential equations of fractional order [14,17,13], The global analysis of a time fractional order spatio-temporal $SEIR$ model with a bilinear incidence rate is studied in a recent work [4], The dynamical stability of a spatio-temporal SIR model with a fractional order derivative and a saturated incidence function is studied and analyzed in [5],

More recently, the global analysis of a spatio-temporal SIR model with a fractional derivative of time and a bilinear incidence rate function $f(I)S = \beta IS$ has been investigated in a recent study [41]. The authors of this study proposed the following model:

$$\begin{cases} {}_0^C D_t^\alpha S(x,t) = \lambda_S \Delta S(x,t) + \Lambda - \beta S(x,t)I(x,t) - (\mu + u)\, S(x,t), \\ {}_0^C D_t^\alpha I(x,t) = \lambda_I \Delta I(x,t) + \beta S(x,t)I(x,t) - (\mu + r)I(x,t), \\ {}_0^C D_t^\alpha R(x,t) = \lambda_R \Delta R(x,t) + uS(x,t) + rI(x,t) - \mu R(x,t), \end{cases} \tag{1.2}$$

with ${}_0^C D_t^\alpha$ is the time fractional derivative of order α in the sense of Caputo with $0 < \alpha \leq 1$.
The birth rate of the susceptibles is denoted Λ, β is the infection rate, u is the term vaccination, r is the recovery rate of the infected individuals and μ is the natural mortality. The positive constants λ_S, λ_I and λ_R are the diffusion coefficients of the subsiptible, the infected and teh recovred, respectevly.

This study extends the SIR model (1.2) by adding a rate of therapy, which allows to simulate different treatment scenarios and analyze the effectiveness of therapeutic interventions in the fight against epidemics. Several studies show that adding the therapy coefficient can reduce the spread of the disease

and reduce the duration of the epidemic. Analysis of the effect of therapy on viral load reduction, duration of infection and likelihood of transmission can help to better understand the impact of therapeutic interventions in the fight against epidemics. This extension can be useful for epidemic modelling and public health decision-making.
Then, we propose a fractional SIR diffusion model with therapy:

$$\begin{cases} {}_0^C D_t^\alpha S(x,t) = \lambda_S \Delta S(x,t) + \Lambda - (1-\nu)\beta S(x,t) I(x,t) - (\mu + u)\, S(x,t), \\ {}_0^C D_t^\alpha I(x,t) = \lambda_I \Delta I(x,t) + (1-\nu)\beta S(x,t) I(x,t) - (\mu + r) I(x,t), \\ {}_0^C D_t^\alpha R(x,t) = \lambda_R \Delta R(x,t) + uS(x,t) + rI(x,t) - \mu R(x,t), \end{cases} \tag{1.3}$$

where the constant ν represent the efficiency of drug therapy in blocking new infection.
The transfer diagram of the infection dynamics is given in Fig. 1.

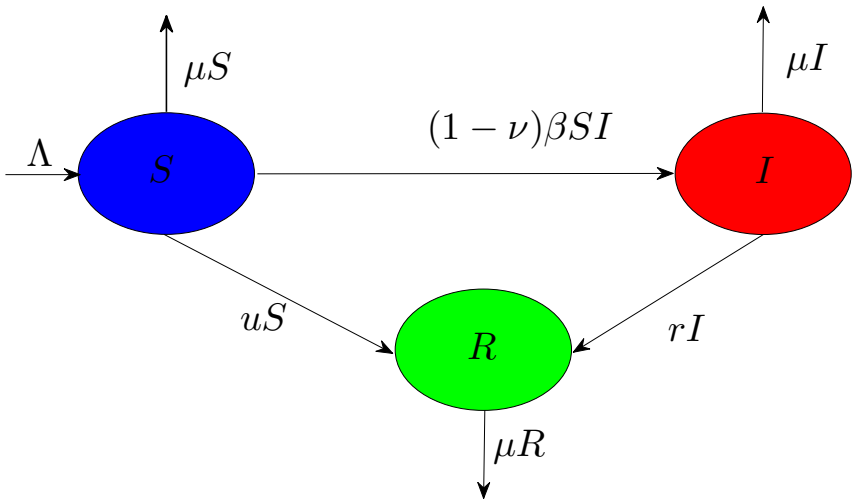

Figure 1: The transfer diagram of the spatio-temporal model SIR.

For biological reasons, we must choose the initial conditions as positive functions

$$S(x,0) = S_0, I(x,0) = I_0 \quad \text{and} \quad R(x,0) = R_0, \quad x \in \Omega \tag{1.4}$$

with Ω is is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. The normal derivatives of the classe S, I and R at the boundary of Ω are zero, which means biologically that the population remain inside the boundary.

$$\frac{\partial S(x,t)}{\partial v} = \frac{\partial I(x,t)}{\partial v} = \frac{\partial R(x,t)}{\partial v} = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{1.5}$$

with $\frac{\partial}{\partial v}$ denotes the outward normal derivative on $\partial\Omega$.

The present work is organized as follows. The next section is devoted to some preliminaries. Section 3 gives the mathematical results concerning the well-posedness of our model. The global analysis of the equlibria is given in Section 4. The numerical tests are given in Section 5. The last section concludes the work.

2. Preliminaries

In this part we will present some basic notions, first the function of Mittag-Leffler is given by:

Definition 2.1 *Mittag-Leffler function The Mittag-Leffler function, $E_\alpha(z)$, is given in the format:*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \quad \alpha > 0, \quad z \in \mathbb{C}, \tag{2.1}$$

$$\text{with} \quad \Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt \quad \text{is the Gamma function.} \tag{2.2}$$

Definition 2.2 *Riemann-Liouville fractional integral [18,32]:*
let f be a function such that $f \in L^1(\mathbb{R}^+)$, the fractional Riemann-Liouville integral. with $\alpha > 0$ of f is:

$$I^\alpha f(t) = \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} f(s) ds. \tag{2.3}$$

Definition 2.3 *The derivative of fractional order in the sense of Caputo [32]*
Let $\alpha > 0$, and let $n \in \mathbb{N}$ check $n-1 < \alpha \leq n$. The Caputo fractional derivative of order α applied to the function $f \in C^n([0,+\infty), \mathbb{R})$ is given by

$$\overset{C}{_0}D_t^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \tag{2.4}$$

with $D = \frac{d}{dt}$.
For $n = 1$. we have $0 < \alpha < 1$, so we have

$$\overset{C}{_0}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds. \tag{2.5}$$

LaSalle's principle of invariance is a widely used tool to study the asymptotic stability of solutions of differential equations, see [3,35].

Theorem 2.1 *(LaSalle's principle of invariance.)*
Let x^* be an equilibrium point of a Cauchy problem
The equilibrium x^* is asymptotically stable if there exists a continuous function $\mathcal{V}$ defined on a neighborhood $U \subset \mathbb{R}^n$ of x^* with values in $\mathbb{R}$, differentiable on $U \setminus \{x^*\}$ such that:
i) $\mathcal{V}(x^*) = 0$ and $\mathcal{V}(x) > 0$ if $x \neq x^*$.
ii) $\dot{\mathcal{V}} \leq 0$ on $U \setminus \{x^*\}$.
ii) The set $S = \{x \in U / \dot{\mathcal{V}}(x) = 0\}$ does not contain any trajectory of the system other than $x(t) = x^*$.

3. Existence result and the equilibria

3.1. Existence of solution

Let $\mathbb{X} = C(\bar{\Omega}, \mathbb{R})$ where $\mathbb{X}^3$ is a Banach space with the usual norms
we set $L = (S, I, R)$, $L_0 = (S_0, I_0, R_0)$, $\lambda = (\lambda_S, \lambda_I, \lambda_R)$ and let A be the linear diffusion operator

$$A : \mathcal{D}(A) \subset \mathbb{X}^3 \to \mathbb{X}^3 \tag{3.1}$$

$$AL = \lambda \Delta L = (\lambda_S \Delta S, \lambda_I \Delta I, \lambda_R \Delta R), \quad \forall L \in \mathcal{D}(A),$$

wher :

$$\mathcal{D}(A) = \left\{ J \in \mathbb{X}^3 : \Delta J \in \mathbb{X}^4, \frac{\partial J}{\partial v} = 0_{\mathbb{R}^3} \text{ for all } x \in \partial\Omega \right\}. \tag{3.2}$$

Consider the function f defined by $f : \mathbb{X}^3 \times [0,T] \mapsto \mathbb{X}^3$, with

$$f(t, L(t)) = f(L(t)) = (f_1(L(t)), f_2(L(t)), f_3(L(t))), \tag{3.3}$$

with

$$\begin{cases} f_1(L(t)) = \Lambda - (1-\nu)\beta S(x,t)I(x,t) - (\mu+u)S(x,t), \\ f_2(L(t)) = (1-\nu)\beta S(x,t)I(x,t) - (\mu+r)I(x,t), \\ f_3(L(t)) = uS(x,t) + rI(x,t) - \mu R(x,t), \end{cases} \tag{3.4}$$

we can rewrite the model (1.3)-(1.4) in the following expression :

$$\begin{cases} {}_0^C D_t^\alpha L = AL + f(L(t)), \\ L(0) = L_0, \end{cases} \tag{3.5}$$

with: $J = (S, I, R)$ and $L_0 = (S_0, I_0, R_0)$.

The proposition (3.3) in [6] allows to show the existence, uniqueness and non-negativity of the solutions of the problem (3.5).

Proposition 3.1 *Let $0 < \alpha \leq 1$, for $J^0 \in \mathcal{D}(A)$, the problem (3.5) has a unique positive solution $J \in C\left([0,T]; X^3\right)$ with*

$$J(t) = \int_0^\infty \Phi_\alpha(\theta) Q\left(t^\alpha \theta\right) J^0 d\theta + F(t), \tag{3.6}$$

with

$$F(t) = \alpha \int_0^t \int_0^\infty \theta (t-\tau)^{\alpha-1} \Phi_\alpha(\theta) Q\left((t-\tau)^\alpha \theta\right) f(\tau) d\theta d\tau, \tag{3.7}$$

and $\Phi_\alpha(\theta)$ is a probability density function defined on $(0, \infty)$.

Proof: It is clear that f is continuous and Lipschitzian, and A a linear operator defined on a set $\mathcal{D}(A) \subset X^3$ dances without itself so according to [6] there is a unique positive solution to the problem (3.5). □

3.2. The problem equilibria

Noting in the model 1.3 that the two first equations does not depend on the class R and then are uncoupled with the last equation. Hence our attention will concentrated on the analysis of the following reduced system

$$\begin{cases} {}_0^C D_t^\alpha S(x,t) = \lambda_S \Delta S(x,t) + \Lambda - (1-\nu)\beta S(x,t)I(x,t) - (\mu+u)\, S(x,t), \\ {}_0^C D_t^\alpha I(x,t) = \lambda_I \Delta I(x,t) + (1-\nu)\beta S(x,t)I(x,t) - (\mu+r)I(x,t). \end{cases} \tag{3.8}$$

The system 3.8 has an infection-free equilibrium

$$\mathcal{E}_f = (S_f, 0) = \left(\frac{\Lambda}{\mu+u}, 0\right). \tag{3.9}$$

By a simple calculation, the basic reproduction number of 3.8 is given by:

$$\mathcal{R}_0 = \frac{(1-\nu)\beta\Lambda}{(\mu+r)(\mu+u)}. \tag{3.10}$$

If $R_0 > 1$, there exists one endemic equilibrium state denoted

$$E^* = (S^*, I^*) = \left(\frac{\mu+r}{(1-\nu)\beta\Lambda}, \frac{\mu+u}{(1-\nu)\beta}(\mathcal{R}_0 - 1)\right). \tag{3.11}$$

4. Global analysis of the model

This section aims to study the global behavior of $\mathcal{E}_f$ and $\mathcal{E}^*$ using Lyapunov functions. We first have the following lemma (Lemma 3.1 in [44]):

Lemma 4.1 *Let Ψ be a positive function defined by $\Psi(y) = y - \ln(y) - 1, y > 0$, and $y(t) \in \mathbb{R}_+^*$ a continuous differentiable function. for all $\alpha \in [0,1]$ and $t \geq t_0$*

$$ {}_{t_0}^{C}D_t^{\alpha}\left[y^*\Psi\left(\frac{y(t)}{y^*}\right)\right] \leq \left(1-\frac{y^*}{y(t)}\right){}_{t_0}^{C}D_t^{\alpha}y(t), y \in \mathbb{R}_+^*. \tag{4.1}$$

First, we discuss the global stability of the infection-free equilibrium $\mathcal{E}_f$ and the immune-free infection equilibrium $\mathcal{E}^*$.

Theorem 4.1 *If $\mathcal{R}_0 \leq 1$ then the infection-free equilibrium $\mathcal{E}_f$ is globally asymptotically stable.*

Proof: Putting:

$$\mathbb{F}(t) = \int_\Omega \left[S_f\Psi\left(\frac{S(x,t)}{S_f}\right) + I(x,t)\right]dx \tag{4.2}$$

Calculating the fractional derivative of $\mathbb{F}$ in Caputo's sense, and according the Lemma (4.1) we have

$$\begin{aligned}
{}_0^{C}D_t^{\alpha}\mathbb{F}(t) &\leq \int_\Omega \left(\left(1-\frac{S_f}{S(x,t)}\right){}_0^{C}D_t^{\alpha}S(x,t) + {}_0^{C}D_t^{\alpha}I(x,t)\right)dx \\
&\leq \int_\Omega \left(\left(1-\frac{S_f}{S(x,t)}\right)(\Lambda - (1-\nu)\beta S(x,t)I(x,t) - (\mu+u)S(x,t))\right. \\
&\quad \left. + (1-\nu)\beta S(x,t)I(x,t) - (\mu+r)I(x,t)\right)dx \\
&\quad + \int_\Omega \left(\lambda_S\Delta S(x,t) - \lambda_S\frac{S_f}{S(x,t)}\Delta S(x,t) + \lambda_I\Delta I(x,t)\right)dx.
\end{aligned} \tag{4.3}$$

We have $\Lambda = (\mu+u)S_f$, then

$$\begin{aligned}
{}_0^{C}D_t^{\alpha}\mathbb{F}(t) &\leq \int_\Omega \left(\left(1-\frac{S_f}{S(x,t)}\right)((\mu+u)S_f - (\mu+u)S(x,t)) + \frac{(1-\nu)\beta\Lambda}{\mu+u}I(x,t) - (\mu+r)I(x,t)\right)dx \\
&\quad + \int_\Omega \left(\lambda_S\Delta S(x,t) - \lambda_S\frac{S_f}{S(x,t)}\Delta S(x,t) + \lambda_I\Delta I(x,t)\right)dx \\
&\leq (\mu+r)\int_\Omega(\mathcal{R}_0-1)I(x,t)dx + (\mu+u)S_f\int_\Omega\left(2-\frac{S(x,t)}{S_f}-\frac{S(x,t)}{S_f}\right)dx \\
&\quad + \int_\Omega \left(\lambda_S\Delta S(x,t) - \lambda_S\frac{S_f}{S(x,t)}\Delta S(x,t) + \lambda_I\Delta I(x,t)\right)dx.
\end{aligned} \tag{4.4}$$

Applying the result of that the arithmetic mean is bigger than or equal to the geometric mean, we get:

$$2-\frac{S(x,t)}{S_f}-\frac{S(x,t)}{S_f} \leq 0. \tag{4.5}$$

According Green's formula, we get

$$\begin{aligned}
{}_0^{C}D_t^{\alpha}\mathbb{F}(t) &\leq (\mu+r)\int_\Omega(\mathcal{R}_0-1)I(x,t)dx + (\mu+u)S_f\int_\Omega\left(2-\frac{S_f}{S(x,t)}-\frac{S(x,t)}{S_f}\right)dx \\
&\quad - \lambda_S S_f\int_\Omega\left[\frac{|\nabla S(x,t)|}{S(x,t)}\right]^2 dx.
\end{aligned} \tag{4.6}$$

For $\mathcal{R}_0 \leq 1$, we deduce that ${}_0^{C}D_t^{\alpha}\mathbb{F}(t)$. In addition

$$\left\{(S,I)\in\mathbb{R}_+^2 : {}_0^C D_t^\alpha \mathbb{F}(x,t)=0\right\}=\{\mathcal{E}_f\}. \tag{4.7}$$

According to the principle of LaSalle invariance if $\mathcal{R}_0\leq 0$ then $\mathcal{E}_f$ is globally asymptotically stable. □

Theorem 4.2 *If $\mathcal{R}_0\geq 1$ then the equilibrium point with infection $\mathcal{E}^*$ is globally asymptotically stable.*

Proof:

Let

$$\mathbb{F}^*(t)=F^*(S(x,t),I(x,t))=\int_\Omega\left[S^*\Psi\left(\frac{S(x,t)}{S^*}\right)+I^*\Psi\left(\frac{I(x,t)}{I^*}\right)\right]dx. \tag{4.8}$$

According the Lemma 4.1, we have

$$\begin{aligned}
{}_0^C D_t^\alpha \mathbb{F}^*(t) &\leq \int_\Omega\left(\left(1-\frac{S^*}{S(x,t)}\right){}_0^C D_t^\alpha S(x,t)+\left(1-\frac{I^*}{S(x,t)}\right){}_0^C D_t^\alpha I(x,t)\right)dx,\\
&\leq \int_\Omega\left[\left(1-\frac{S^*}{S(x,t)}\right)\left[\Lambda-(1-\nu)\beta S(x,t)I(x,t)-(\mu+u)S(x,t)\right]\right.\\
&\quad\left.+\left(1-\frac{I^*}{I(x,t)}\right)\left[(1-\nu)\beta S(x,t)I(x,t)-(\mu+r)I(x,t)\right]\right]dx\\
&\quad+\int_\Omega\left(\lambda_S\Delta S(x,t)-\lambda_S\frac{S_f}{S(x,t)}\Delta S(x,t)+\lambda_I\Delta I(x,t)-\lambda_I\frac{I^*}{I(x,t)}\Delta I(x,t)\right)dx.
\end{aligned} \tag{4.9}$$

We have $\Lambda=(\mu+u)S^*+(\mu+r)I^*$ and $\mu+r=(1-\nu)\beta S^*$, applying then Green's formula, we obtain:

$$\begin{aligned}
{}_0^C D_t^\alpha \mathbb{F}^*(t) &\leq \int_\Omega\left\{(\mu+u)S^*+(\mu+r)I^*-(1-\nu)S(x,t)I(x,t)-(\mu+u)S(x,t)-(\mu+u)S^*\frac{S^*}{S(x,t)}\right.\\
&\quad-(\mu+r)I^*\frac{S^*}{S(x,t)}+(1-\nu)\beta S^*I(x,t)+(\mu+u)S^*+(1-\nu)S(x,t)I(x,t)-(\mu+r)I(x,t)\\
&\quad\left.-(1-\nu)\beta S(x,t)I^*+(\mu+r)I^*-\lambda_S S^*\left[\frac{|\nabla S(x,t)|}{S(x,t)}\right]^2-\lambda_I I^*\left[\frac{|\nabla I(x,t)|}{I(x,t)}\right]^2\right\}dx,\\
&\leq \int_\Omega\left\{(\mu+u)S^*\left[2-\frac{S^*}{S(x,t)}-\frac{S(x,t)}{S^*}\right]+(\mu+r)I^*\left[2-\frac{S^*}{S(x,t)}-\frac{S(x,t)}{S^*}\right]\right\}dx\\
&\quad-\int_\Omega\left\{\lambda_S S^*\left[\frac{|\nabla S(x,t)|}{S(x,t)}\right]^2+\lambda_I I^*\left[\frac{|\nabla I(x,t)|}{I(x,t)}\right]^2\right\}dx.
\end{aligned} \tag{4.10}$$

Then ${}_0^C D_t^\alpha \mathbb{F}^*(t)\leq 0$.

Furthermore, the largest invariant set that verifies

$$\left\{(S,I)\in\mathbb{R}_+^2 : {}_0^C D_t^\alpha \mathbb{F}^*(x,t)=0\right\}=\{\mathcal{E}^*\}. \tag{4.11}$$

Using LaSalle's we achieve the desired result. □

5. Results an discussion

5.1. The equilibria stability

In this subsection, we present and discuss the results of numerical simulations to validate the theoretical results of the previous section. We will use the finite difference numerical method with the Euler scheme for the approximation of the diffusion expression. The Euler's fractional method as mentioned in explains [9,39] for the fractional derivative of order α in Caputo's sense.

We have used the one-dimensional interval $0 \leq x \leq X$ and the time $0 \leq t \leq T$ with $X = 4$, and $T = 35$. The initial conditions were chosen as constants. We used an explicit numerical method with the space step $h_x = 0.176$ and the time step $h_t = 0.02$. The program was implemented with Matlab.

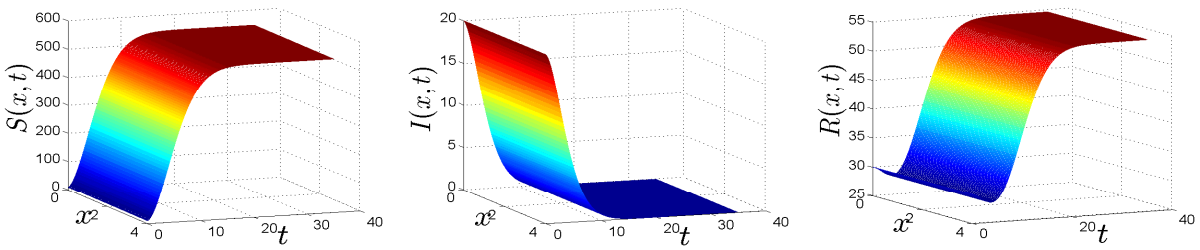

Figure 2: The dynamics of classes S, I and R showing the stability of the free equilibrium $\mathcal{E}_{\mathrm{f}}$.

Parameters	α	Λ	β	ν	μ	u	r	l_S	l_I	l_R	T	X
values for FIGURE 2	0.7	06	0.0001	0.9	0.01	0.001	0.01	0.2	0.2	0.2	40	4
values for FIGURE 3	0.7	60	0.01	0.5	0.1	0.001	0.001	0.2	0.2	0.2	5	4

Table 1: The values of parameters for the two first numerical results

Figure 2 shows the dynamics of classes S, I and R for the parameters in Table 5.1. With these parameters, we have the basic reproduction number is $\mathcal{R}_0 = 0.272 \leq 1$ and we observe the convergence toward the free equilibrium $\mathcal{E}_{\mathrm{f}} = (545, 0, 54.54)$. According to Theorem 4.1 we have $\mathcal{E}_{\mathrm{f}}$ is globally asymptotically stable, which means that the spatio-temporal dynamics converges towards $\mathcal{E}_{\mathrm{f}}$, Biologically speaking, the disease does not exist

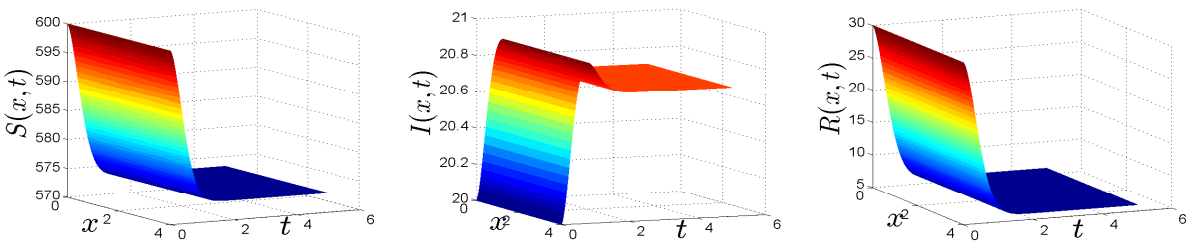

Figure 3: The dynamics of classes S, I and R showing the stability of the endemic equilibrium $\mathcal{E}^*$.

Figure 3 shows the dynamics of classes S, I and R for the following parameters in Table 5.1. With these parameters, we have $\mathcal{R}_0 = 11.76 > 1$ and we observe the convergence toward the endemic equilibrium $\mathcal{E}^* = (573, 20, 6)$. According to Theorem 4.2, we therefore have $\mathcal{E}^*$ is globally asymptotically stable, which means biologically that the endemic persists.

5.2. Effect of therapy

In this party we proposed the values of parameters given in Table 5.2.

Figures 4 and 5 show the dynamics of classes S, I and R for the parameters in Table 5.2. It is clearly observed that when the therapy efficiency is very high the number of the infected individuals is reduced considerably. This shows the importance of the therapy in blocking the infection.

Parameters	α	Λ	β	ν	μ	u	r	l_S	l_I	l_R	T
values for FIGURE 4	0.7	6	0.0001	–	0.01	0.001	0.01	0.2	0.2	0.2	15
values for FIGURE 5	0.7	60	0.01	–	0.1	0.001	0.001	0.2	0.2	0.2	5

Table 2: The values of parameters for the two last numerical results.

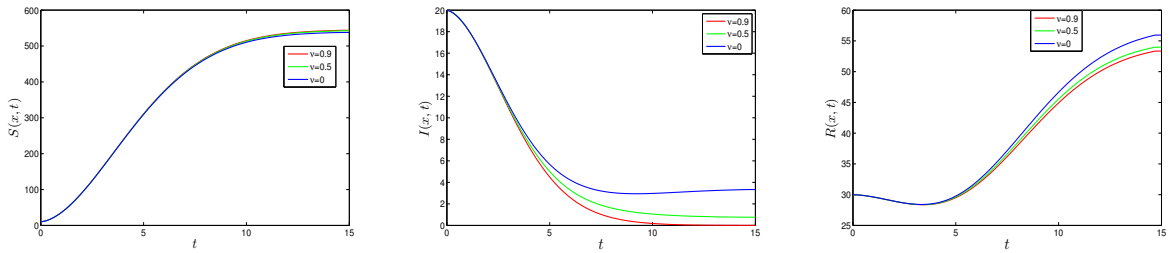

Figure 4: The dynamics of classes S, I and R showing the stability of the free equilibrium $\mathcal{E}_{\mathrm{f}}$ for different values of ν .

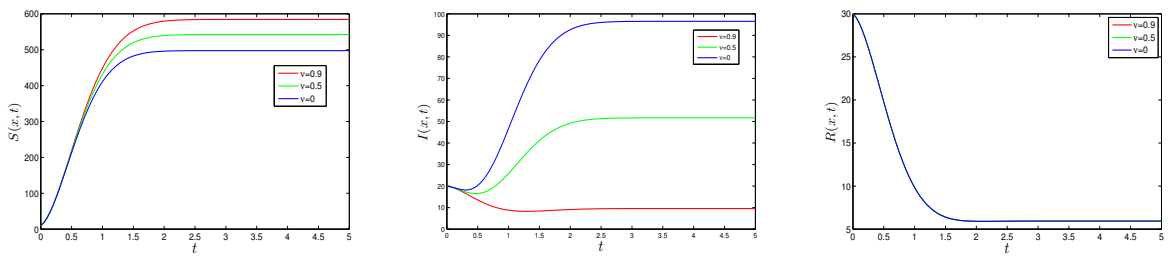

Figure 5: The dynamics of classes S, I and R showing the stability of the endemic equilibrium $\mathcal{E}^*$ for different values of ν.

5.3. Effect of fractional derivative

In this subsection, we examine how fractional derivation order affects disease dynamics. The figures 6 and 7 illustrate the dynamics of the classes S, I and R for different values of α and for the parameters given in the fourth and fifth rows of the table 5.3. We note that, according to these numerical results, the order of fractional derivation α does not affect the stability of the equilibria. However, for small values of α, which describe the behavior of long-term memory, the solutions converge faster to the endemic equilibrium state. We conclude from this interesting experiment that the order of the fractional differentiation only affects the speed of convergence to equilibrium states. However, this fractional order does not affect the final value of the steady state.

Parameters	α	Λ	β	ν	μ	u	r	l_S	l_I	l_R	T
values for FIGURE 6	–	6	0.0001	0.5	0.01	0.001	0.01	0.2	0.2	0.2	15
values for FIGURE 7	–	60	0.01	0.5	0.1	0.001	0.001	0.2	0.2	0.2	5

Table 3: The values of parameters for the two last numerical results.

6. Conclusion

In this work, we have studied a time fractional order spatio-temporal SIR model with therapy. The model is described by a system of reaction-diffusion equations incorporating a fractional derivative. We have studied the effect of therapy on the population dynamics. The existence, boundedness and uniqueness of the solution are proved. The global stability of the equilibria is established. Numerical

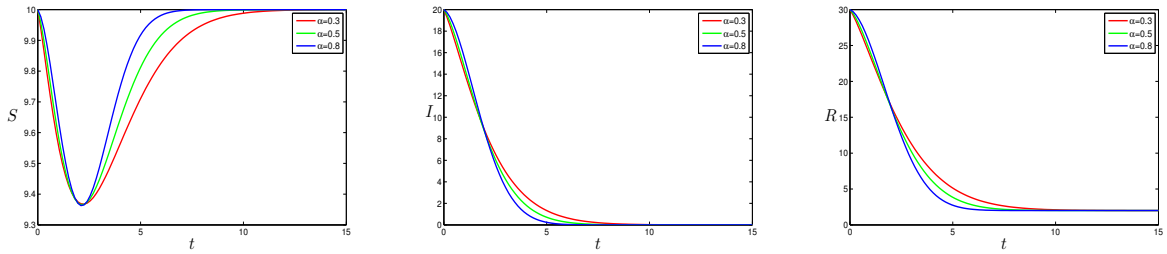

Figure 6: The dynamics of classes S, I and R showing the stability of the free equilibrium $\mathcal{E}_{\mathrm{f}}$ for different values of α .

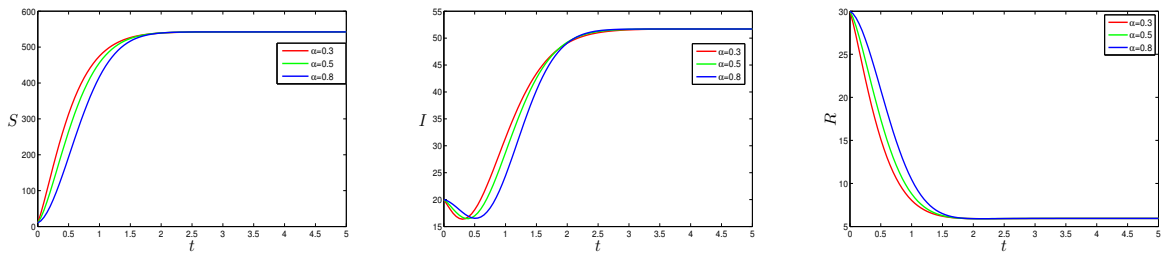

Figure 7: The dynamics of classes S, I and R showing the stability of the endemic equilibrium $\mathcal{E}^*$ for different values of α.

simulations are curried out in order to show the equilibria stability and the effect of therapy. It was observed the therapy plays an essential role in blocking new infections.

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