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Commutativity Theorems in Prime Rings With Involution Involving Symmetric and Skew-Symmetric Elements

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ABSTRACT: In this paper we study commutativity of prime ring R with involution $*$ which admits a generalized derivation satisfying certain algebraic identities for symmetric and skew-symmetric elements. We have also examined specific conclusions related to this topic in our research.

Key Words: Prime ring, involution, commutativity, generalized derivation, symmetric elements, skew-symmetric elements.

Contents

1. Introduction

Throughout this article, R will denote an associative ring with center $Z(R)$. For any $x, y \in R$, we will write the commutator $xy - yx$ as $[x, y]$, while the anti-commutator $xy + yx$ will be denoted as $x \circ y$. R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. As it is known, an additive mapping $d : R \longrightarrow R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$, is called a derivation. On the other hand, if d is a derivation, an additive mapping $F : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$ is called a generalized derivation of R with an associated derivation d. The literature of studying the relationship between the commutativity of a ring R and some special types of maps includes several papers ([1], [2], [3], [4]). One of the most important result in this direction is Posner's theroem [8], where it has been shown that a prime ring becomes commutative, if it admits a nonzero centralizing derivation.
An additive map $* : R \longrightarrow R$ is called an involution if $*$ is an anti-automorphism of order 2; that is $(x^*)^* = x$ for all $x \in R$. An element x in a ring with involution $(R, *)$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian elements of R will be denote by $H(R)$, while $S(R)$ will represent the set of skew-hermitian elements. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. Recently in [6] Nejjar et al. proved that a prime ring R is commutative if R is equipped with a derivation d satisfying $[d(x), d(x^*)] + [x, x^*] \in Z(R)$ for all $x \in R$ or $d(x) \circ d(x^*) + x \circ x^* \in Z(R)$ for all $x \in R$.
In our study, we build on the work of previous authors who have explored the commutativity of prime and semi-prime rings using various types of additive mappings. However, our focus is on rings that have an involution and we consider differential identities on appropriate subsets of the ring involving generalized derivations.

2. Main Theorems

In [5], M. A. Idrissi, and L. Oukhtite studied the commutativity problem of a 2-torsion free prime ring R with involution $*$ of the second kind and proved that if $F[x, x^*] \in Z(R)$, then R is commutative, where F is a nonzero generalized derivation associated with a derivation d. Motivated by the previous result we find the following theorem.

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Theorem 2.1 *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If (F, d) is a nonzero generalized derivation of R, then the following assertions are equivalent:*
(1) $F[h, h'] \in Z(R)$ for all $h, h' \in H(R)$;
(2) $F[k, k'] \in Z(R)$ for all $k, k' \in S(R)$;
(3) R is commutative.

Proof: For the nontrivial implication suppose that

$$F[h, h'] \in Z(R) \quad \text{for all} \quad h, h' \in H(R). \tag{2.1}$$

Hence

$$F[x + x^*, y + y^*] \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{2.2}$$

Developing (2.2), we find that

$$F[x, y] + F[x, y^*] + F[x^*, y] + F[x^*, y^*] \in Z(R). \tag{2.3}$$

Replacing y by yh, where $h \in Z(R) \cap H(R)$, and using the last equation, we obtain

$$([x, y] + [x, y^*] + [x^*, y] + [x^*, y^*])d(h) \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{2.4}$$

Since R is prime, then $d(h) = 0$ or $[x, y] + [x, y^*] + [x^*, y] + [x^*, y^*] \in Z(R)$.
If $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, we have

$$d(z) = 0 \quad \text{for all} \quad z \in Z(R).$$

Substituting ys for y in (2.3) where $s \in S(R) \cap Z(R)\backslash\{0\}$, we have

$$F[x, y] - F[x, y^*] + F[x^*, y] - F[x^*, y^*] \in Z(R). \tag{2.5}$$

Invoking (2.3), (2.5) yields

$$F[x, y] + F[x^*, y] \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{2.6}$$

Replacing y by x^*, we arrive at

$$F[x, x^*] \in Z(R) \quad \text{for all} \quad x \in R \tag{2.7}$$

([5], Theorem 1) implies that R is commutative.
If

$$[x, y] + [x, y^*] + [x^*, y] + [x^*, y^*] \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{2.8}$$

Replacing y by ys in (2.8), where $s \in S(R) \cap Z(R)\backslash\{0\}$, we get

$$[x, y] - [x, y^*] + [x^*, y] - [x^*, y^*] \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{2.9}$$

Taking now $y = x^*$ in (2.9), we find $[x, x^*] \in Z(R)$ for all $x \in R$.
Hence ([6], Lemma 2.1) implies that R is commutative.
(2) $\Rightarrow$ (3) Assume that

$$F[k, k'] \in Z(R) \quad \text{for all} \quad k, k' \in S(R). \tag{2.10}$$

Then

$$F[x - x^*, y - y^*] \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{2.11}$$

We easily get

$$F[x, y] - F[x, y^*] - F[x^*, y] + F[x^*, y^*] \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{2.12}$$

Substituting yh for y where $h \in Z(R) \cap H(R)$ and using the last equation, we obtain

$$([x, y] - [x, y^*] - [x^*, y] + [x^*, y^*])d(h) \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{2.13}$$

In view of primness, it follows that $d(h) = 0$ or $[x, y] - [x, y^*] - [x^*, y] + [x^*, y^*] \in Z(R)$.
If $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then

$$d(z) = 0 \quad \text{for all} \ \ z \in Z(R).$$

Taking ys instead of y in (2.12) where $s \in S(R) \cap Z(R) \backslash \{0\}$, one can obviously see that

$$F[x, y] + F[x, y^*] - F[x^*, y] - F[x^*, y^*] \in Z(R) \quad \text{for all} \ \ x, y \in R. \tag{2.14}$$

Putting $y = x^*$ in (2.14), we get

$$F[x, x^*] \in Z(R) \quad \text{for all} \ \ x \in R. \tag{2.15}$$

Applying ([5], Theorem 1) we conclude that R is commutative.
If

$$[x, y] - [x, y^*] - [x^*, y] + [x^*, y^*] \in Z(R) \quad \text{for all} \ \ x, y \in R. \tag{2.16}$$

The equation (2.16) is similar to the equation (2.8), using the same steps, we find $[x, x^*] \in Z(R)$ for all $x \in R$. Thereby ([6], Lemma 2.1) assures that R is commutative. □

Using the fact that $F + id$ and $F - id$ are generalized derivations, we find the following propositions.

Proposition 2.1 *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If (F, d) is a generalized derivation of R, then the following assertions are equivalent:*
(1) $F[h, h'] + [h, h'] \in Z(R)$ for all $h, h' \in H(R)$;
(2) $F[h, h'] - [h, h'] \in Z(R)$ for all $h, h' \in H(R)$;
(3) R is commutative.

Proposition 2.2 *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If (F, d) is a generalized derivation of R, then the following assertions are equivalent:*
(1) $F[k, k'] + [k, k'] \in Z(R)$ for all $k, k' \in S(R)$;
(2) $F[k, k'] - [k, k'] \in Z(R)$ for all $k, k' \in S(R)$;
(3) R is commutative.

Theorem 2.2 *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If (F, d) is a nonzero generalized derivation of R, then R is commutative if and only if $F[h, k] \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$.*

Proof: For the nontrivial implication assume that

$$F[h, k] \in Z(R) \quad \text{for all} \ \ h \in H(R) \text{ and } k \in S(R). \tag{2.17}$$

In such a way that

$$F[x + x^*, y - y^*] \in Z(R) \quad \text{for all} \ \ x, y \in R. \tag{2.18}$$

Developing (2.18) which gives

$$F[x, y] - F[x, y^*] + F[x^*, y] - F[x^*, y^*] \in Z(R). \tag{2.19}$$

Setting $y = yh$ where $h \in Z(R) \cap H(R)$ and using the last equation, we obtain

$$([x, y] - [x, y^*] + [x^*, y] - [x^*, y^*])d(h) \in Z(R) \quad \text{for all} \ \ x, y \in R. \tag{2.20}$$

By primness we conclude that for each $h \in R$ $d(h) = 0$ or $[x, y] - [x, y^*] + [x^*, y] - [x^*, y^*] \in Z(R)$.
If $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, yields that

$$d(z) = 0 \quad \text{for all} \ \ z \in Z(R).$$

Replacing y by ys in (2.19) where $s \in S(R) \cap Z(R) \backslash \{0\}$, we obtain

$$F[x,y] + F[x,y^*] + F[x^*,y] + F[x^*,y^*] \in Z(R). \tag{2.21}$$

From (2.19) and (2.21) one obtains

$$F[x,y] + F[x^*,y] \in Z(R) \quad \text{for all} \quad x,y \in R. \tag{2.22}$$

Substituting x^* for y, we get

$$F[x,x^*] \in Z(R) \quad \text{for all} \quad x \in R. \tag{2.23}$$

So R is commutative by ([5], Theorem 1).
If

$$[x,y] - [x,y^*] + [x^*,y] - [x^*,y^*] \in Z(R) \quad \text{for all} \quad x,y \in R. \tag{2.24}$$

The above equation is similar to equation (2.8), so using the same reasoning, we arrive at $[x,x^*] \in Z(R)$ for all $x \in R$. So the commutativity of R follows from ([6], Lemma 2.1). □

Using the fact that $F + id$ and $F - id$ are generalized derivations, we find the following proposition.

Proposition 2.3 *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If (F,d) is a generalized derivation of R, then the following assertions are equivalent:*
(1) $F[h,k] + [h,k] \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$;
(2) $F[h,k] - [h,k] \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$;
(3) R is commutative.

In [5], M. A. Idrissi and L. Oukhtite, studied the commutativity problem of a 2-torsion free prime ring R with involution $*$ of the second kind and proved that if $F(x \circ x^*) \in Z(R)$ for all $x \in R$, then R is commutative, where F is a nonzero generalized derivation associated with a derivation d. Based on the previous result as motivation, we get the following result for symmetric and skew-symmetric elements of R.

Theorem 2.3 *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If (F,d) is a nonzero generalized derivation of R, then R is commutative if and only if $F(h \circ h') \in Z(R)$ for all $h,h' \in H(R)$.*

Proof: We are given that

$$F(h \circ h') \in Z(R) \quad \text{for all} \quad h,h' \in H(R). \tag{2.25}$$

Hence

$$F((x+x^*) \circ (y+y^*)) \in Z(R) \quad \text{for all} \quad x,y \in R. \tag{2.26}$$

In consequence of which,

$$F(x \circ y) + F(x \circ y^*) + F(x^* \circ y) + F(x^* \circ y^*) \in Z(R). \tag{2.27}$$

Replacing y by yh, with $h \in Z(R) \cap H(R)$, the last equation leads to

$$(x \circ y + x \circ y^* + x^* \circ y + x^* \circ y^*)d(h) \in Z(R) \quad \text{for all} \quad x,y \in R. \tag{2.28}$$

In view of the primness, the last equation assures that $d(h) = 0$ or $x \circ y + x \circ y^* + x^* \circ y + x^* \circ y^* \in Z(R)$. If $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then

$$d(z) = 0 \quad \text{for all} \quad z \in Z(R).$$

Replacing y by ys in (2.27), where s is a nonzero element in $S(R) \cap Z(R)$, we get

$$F(x \circ y) - F(x \circ y^*) + F(x^* \circ y) - F(x^* \circ y^*) \in Z(R). \tag{2.29}$$

Using (2.29) together with (2.27), we obtain

$$F(x \circ y) + F(x^* \circ y) \in Z(R) \quad \text{for all } \ x, y \in R. \tag{2.30}$$

Putting $x = h$, where $h \in H(R) \cap Z(R) \backslash \{0\}$, we have

$$F(y)h \in Z(R) \quad \text{for all } \ y \in R.$$

Hence

$$F(y) \in Z(R) \quad \text{for all } \ y \in R. \tag{2.31}$$

If $d = 0$, then for $y = xy$ equation (2.31) reduces to

$$F(x)y \in Z(R) \quad \text{for all } \ x, y \in R.$$

The primness and the fact that F is a nonzero generalized derivation implies that R is commutative. If $d \neq 0$, the commutativity of R follows from ([7], Theorem 3).
Now suppose that

$$x \circ y + x \circ y^* + x^* \circ y + x^* \circ y^* \in Z(R) \quad \text{for all } \ x, y \in R. \tag{2.32}$$

Replacing y by ys where $s \in S(R) \cap Z(R) \backslash \{0\}$ and using the last equation, we get

$$x \circ y - x \circ y^* + x^* \circ y - x^* \circ y^* \in Z(R) \quad \text{for all } \ x, y \in R. \tag{2.33}$$

Combining (2.32) and (2.33), we arrive at

$$(x + x^*) \circ y \in Z(R) \quad \text{for all } \ x, y \in R. \tag{2.34}$$

Taking $x = h \in H(R) \cap Z(R) \backslash \{0\}$, we have $hy \in Z(R)$ for all $y \in R$ and $h \in H(R) \cap Z(R) \backslash \{0\}$, and thus $y \in Z(R)$ for all $y \in R$. So that R is commutative. □

Theorem 2.4 *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If (F, d) is a nonzero generalized derivation of R, then R is commutative if and only if $F(k \circ k') \in Z(R)$ for all $k, k' \in S(R)$.*

Proof: We are given that

$$F(k \circ k') \in Z(R) \quad \text{for all } \ k, k' \in S(R). \tag{2.35}$$

Hence

$$F((x - x^*) \circ (y - y^*)) \in Z(R) \quad \text{for all } \ x, y \in R. \tag{2.36}$$

Developing (2.36), we find that

$$F(x \circ y) - F(x \circ y^*) - F(x^* \circ y) + F(x^* \circ y^*) \in Z(R). \tag{2.37}$$

Putting $y = yh$ where $h \in Z(R) \cap H(R)$ and using the last equation, we obtain

$$(x \circ y - x \circ y^* - x^* \circ y + x^* \circ y^*)d(h) \in Z(R) \quad \text{for all } \ x, y \in R. \tag{2.38}$$

In light of primness, it follows that $d(h) = 0$ or $x \circ y - x \circ y^* - x^* \circ y + x^* \circ y^* \in Z(R)$.
If $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, hence

$$d(z) = 0 \quad \text{for all } \ z \in Z(R).$$

Putting ys for y in (2.37), where $s \in S(R) \cap Z(R) \backslash \{0\}$, we conclude that

$$F(x \circ y) + F(x \circ y^*) - F(x^* \circ y) - F(x^* \circ y^*) \in Z(R). \tag{2.39}$$

Adding Eqs (2.39) and (2.37), we obtain

$$F(x \circ y) - F(x^* \circ y) \in Z(R) \quad \text{for all} \;\; x, y \in R. \tag{2.40}$$

Taking $x = s$ in equation (2.40), where s is a nonzero element in $S(R) \cap Z(R)$, one can see that

$$F(y)s \in Z(R) \quad \text{for all} \;\; y \in R$$

which gives

$$F(y) \in Z(R) \quad \text{for all} \;\; y \in R. \tag{2.41}$$

Since (2.41) is the same as (2.31), we conclude that R is commutative.
Now suppose that

$$x \circ y - x \circ y^* - x^* \circ y + x^* \circ y^* \in Z(R) \quad \text{for all} \;\; x, y \in R. \tag{2.42}$$

The last equation is similar to equation (2.32), using the same steps, we get $y \in Z(R)$ for all $y \in R$. Which assures the commutativity of R. □

Using the fact that $F + id$ and $F - id$ are generalized derivations, we find the following propositions.

Proposition 2.4 *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If (F, d) is a generalized derivation of R, then the following assertions are equivalent:*
(1) $F(h \circ h') + h \circ h' \in Z(R)$ for all $h, h' \in H(R)$;
(2) $F(h \circ h') - h \circ h' \in Z(R)$ for all $h, h' \in H(R)$;
(3) R is commutative.

Proposition 2.5 *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If (F, d) is a generalized derivation of R, then the following assertions are equivalent:*
(1) $F(k \circ k') + k \circ k' \in Z(R)$ for all $k, k' \in H(R)$;
(2) $F(k \circ k') - k \circ k' \in Z(R)$ for all $k, k' \in H(R)$;
(3) R is commutative.

Theorem 2.5 *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If (F, d) is a nonzero generalized derivation of R, then R is commutative if and only if $F(h \circ k) \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$.*

Proof: We need to consider only the nontrivial implication.
We are given that

$$F(h \circ k) \in Z(R) \quad \text{for all} \;\; h \in H(R) \quad \text{and} \;\; k \in S(R). \tag{2.43}$$

Hence

$$F((x + x^*) \circ (y - y^*)) \in Z(R) \quad \text{for all} \;\; x, y \in R. \tag{2.44}$$

Developing (2.44), we find that

$$F(x \circ y) - F(x \circ y^*) + F(x^* \circ y) - F(x^* \circ y^*) \in Z(R). \tag{2.45}$$

Replacing y by yh where $h \in Z(R) \cap H(R)$ and using the last equation, we obtain

$$(x \circ y - x \circ y^* + x^* \circ y - x^* \circ y^*)d(h) \in Z(R) \quad \text{for all} \;\; x, y \in R. \tag{2.46}$$

Using the primness hypothesis, it follows that $d(h) = 0$ or $x \circ y - x \circ y^* + x^* \circ y - x^* \circ y^* \in Z(R)$.
If $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then

$$d(z) = 0 \quad \text{for all} \;\; z \in Z(R).$$

Replacing y by ys in (2.45), where $s \in S(R) \cap Z(R) \backslash \{0\}$, we get

$$F(x \circ y) + F(x \circ y^*) + F(x^* \circ y) + F(x^* \circ y^*) \in Z(R). \tag{2.47}$$

Combining equations (2.47) and (2.45), we obviously get

$$F(x \circ y) + F(x^* \circ y) \in Z(R) \quad \text{for all} \ \ x, y \in R. \tag{2.48}$$

The equation (2.48) is equivalent to equation (2.30), then using the same techniques as used above we get the required result.
Now if

$$x \circ y - x \circ y^* + x^* \circ y - x^* \circ y^* \in Z(R) \quad \text{for all} \ \ x, y \in R. \tag{2.49}$$

We notice that the above equation is similar to equation (2.32), repeating the same changes leads to $y \in Z(R)$ for all $y \in R$. We conclude that R is commutative and this completes our proof. □

Using the fact that $F + id$ and $F - id$ are generalized derivations, we find the following proposition.

Proposition 2.6 *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. If (F, d) is a generalized derivation of R, then the following assertions are equivalent:*
(1) $F(h \circ k) + h \circ k \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$;
(2) $F(h \circ k) - h \circ k \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$;
(3) R is commutative.

3. Examples

In this section we discuss some examples showing that our results does not hold in certain cases. We begin by the following examples proving that the condition " $*$ " is of the second kind is necessary.

Example 3.1 *Let us consider* $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Z \right\}$, *and* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. *We have* $Z(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in Z \right\}$, *so it is straightforward to check that R is a prime ring and $*$ is an involution of the first kind. Moreover, we set* $d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$, *and* $F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$. *We have for all $h, h' \in H(R)$ and $k, k' \in S(R)$, $F([h, h'])$, $F([h, k])$, $F([k, k'])$, $F(h \circ h')$, $F(h \circ k)$, $F(k \circ k')$ are in $Z(R)$, but R is not commutative.*

Example 3.2 *Let R be the ring of real quaternion. The involution $*$ defined as $(\alpha + \beta i + \gamma j + \delta k)^* = \alpha - \beta i + \gamma j + \delta k$, then " $*$ " is an involution of the first kind and for all $k, k' \in S(R)$, we have $[k, k'] = 0$ and $k \circ k' \in Z(R)$. Let F be a generalized inner derivation induced by some skew-symmetric elements $a, b \in R$ associated by the inner derivation induced by b, then F satisfies the conditions of Theorem 2 and Theorem 5. But again R is not commutative.*

In the next example, we prove that the "primness hypothesis" of R in our work is not superfluous.

Example 3.3 *Let R and F be as in Example 3.1, and $\mathbb{C}$ be the field of complex numbers. If we set $R_1 = R \times \mathbb{C}$, then R_1 is a semi-prime ring provided with the involution of the second kind $\tau : R_1 \to R_1$ where $\tau(r, s) = (r^*, \bar{s})$ for all $(r, s) \in R \times \mathbb{C}$. Consider the generalized derivation $F_1 : R_1 \to R_1$ defined as $F_1(x, s) = (F(x), 0)$, associated with the derivation $D : R_1 \to R_1$ defined as $D(x, s) = (d(x), 0)$. Furthermore F_1 satisfies the conditions of Theorem 1, but R_1 is not commutative.*

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