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On Rough $\mathcal{I}$-Convergence of Complex Uncertain Sequences

Shyamal Debnath* and Amit Halder

ABSTRACT: In this paper, we introduce the different notion of rough $\mathcal{I}$-convergence of complex uncertain sequence using the concept of $\mathcal{I}$-convergence and rough convergence of complex uncertain sequence namely rough $\mathcal{I}$-convergent almost surely, rough $\mathcal{I}$-convergence in measure, rough $\mathcal{I}$-convergence in distribution, rough $\mathcal{I}$-convergence in mean. Finally, we studied some of their basic properties and examined the relationship among them.

Key Words: Uncertainty theory, complex uncertain variable, rough convergence, $\mathcal{I}$-convergence.

Contents

1. Introduction

In modern probability theory, there are many concepts of convergence for random sequence such as convergence in probability, almost surely convergence, convergence in distribution, convergence in mean, and so on. However, in the real world, due to lack of observed data, when making decisions, people have to consult with domain experts. In this case, information and knowledge can't be described well by random variables. In order to model this type of human uncertainty, Liu suggests to deal with it by uncertainty theory [3,4]. Also Liu [3] proposed some convergence concepts of uncertain sequences consisting of convergence almost surely, convergence in measure, convergence in mean, convergence in distribution, and discussed the relationship among those convergence concepts in detail. Nowadays uncertainty theory has successfully been applied for investigations in different areas by famous researchers like You and Yan [7,8], Ye and Zhu [28] and many others.

In real life, uncertainty not only appears in real quantities but also in complex quantities. In order to model complex uncertain quantities, Peng [30] presented the concepts of complex uncertain variables and complex uncertain distribution. Then Chen et al. [29] introduced the convergence of complex uncertain sequences. After that, many researchers widely studied the complex uncertain sequences and made significant progress, such as Tripathy and Nath [5,20], Roy et al. [26], Saha et al. [27], Kisi [19], Debnath and Das [22,23].

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [11] and Steinhaus [12]. Later it was studied by Fridy [**?**] and many other researchers. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [21] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of a subset of the set of natural numbers. Later, further findings on $\mathcal{I}$-convergence were established by many famous researchers, including [1,16,10,6].

* Corresponding author.
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The idea of rough convergence was first introduced by Phu [13] in finite-dimensional normed spaces. In 2003, Phu [14] extended the results to infinite-dimensional normed spaces. Dündar and Çakan [9] introduced the notion of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence. After that, lots of interesting developments have occurred in this field like [2,17,18,24]. Very recently, Debnath and Das [25] introduced the rough convergence of complex uncertain sequences.

In this paper, using the concepts of $\mathcal{I}$-convergence [21] and the concepts of rough convergence [25] of complex uncertain sequence, we introduce the different notion of rough $\mathcal{I}$-convergence of complex uncertain sequence namely rough $\mathcal{I}$-convergence almost surely, rough $\mathcal{I}$-convergence in measure, rough $\mathcal{I}$-convergence in distribution, rough $\mathcal{I}$-convergence in mean of complex uncertain sequences and investigate some inter-relationship among them.

2. Definitions and Preliminaries

Throughout the article let r be a non-negative real number and $\mathcal{I}$ be a nontrivial admissible ideal.

Definition 2.1 *[21] A non-void class $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if $\mathcal{I}$ is additive (i.e., $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$) and hereditary (i.e., $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$).
An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq 2^{\mathbb{N}}$. A non-trivial ideal $\mathcal{I}$ is said to be admissible if $\mathcal{I}$ contain every finite subset of $\mathbb{N}$.*

Example 2.1 *(i) $\mathcal{I}_f :=$ The set of all finite subsets of $\mathbb{N}$ forms a non-trivial admissible ideal.
(ii) $\mathcal{I}_d :=$ The set of all subsets of $\mathbb{N}$ whose natural density is zero forms a non-trivial admissible ideal.*

Definition 2.2 *[21] A sequence $x = (x_n)$ is said to be $\mathcal{I}$ convergent if there exists $L \in \mathbb{R}$ such that for all $\varepsilon > 0$, the set $\{n \in \mathbb{N} : | x_n - L | \geq \varepsilon\} \in \mathcal{I}$. The usual convergence of sequences is a special case of $\mathcal{I}$-convergence ($\mathcal{I}$=$\mathcal{I}_f$-the ideal of all finite subsets of $\mathbb{N}$). The statistical convergence of sequences is also the special case of $\mathcal{I}$-convergence. In this case $\mathcal{I}$=$\mathcal{I}_d = \{A \subseteq \mathbb{N} : \lim_{n\to\infty} \frac{|A \cap \{1,2,\ldots,n\}|}{n} = 0\}$, where $| A |$ being the cardinality of the set A. For more examples on $\mathcal{I}$-convergence see [21].*

Definition 2.3 *[9] Let r be a non-negative real number. A sequence $x = (x_n)$ is said to be rough $\mathcal{I}$-convergent to x_*, denoted by $x_n \xrightarrow{r-\mathcal{I}} x_*$ provided that*

$$\{n \in \mathbb{N} : | x_n - x_* | \geq r + \varepsilon\} \in \mathcal{I},$$

for every $\varepsilon > 0$. The set

$$\mathcal{I} - LIM^r x := \{x_* \in \mathbb{R} : x_n \xrightarrow{r-\mathcal{I}} x_*\}$$

is called rough $\mathcal{I}$-limit set of the sequence $x = (x_n)$. A sequence $x = (x_n)$ is said to be rough $\mathcal{I}$-convergent if $\mathcal{I} - LIM^r x \neq \phi$. In this case, r is called the roughness degree of the sequence $x = (x_n)$. For r=0, we get the ideal convergence.

Definition 2.4 *[3] Let $\mathfrak{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:
Axiom 1 (Normality): $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2 (Duality): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathfrak{L}$;
Axiom 3 (Subadditivity): For every countable sequence of $\{\Lambda_j\} \in \mathfrak{L}$,*

$$\mathcal{M}\{\bigcup_{j=1}^{\infty} \Lambda_j\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathfrak{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in $\mathfrak{L}$ is called an event. In order to obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu [4] as:

$$\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}.$$

Definition 2.5 *[30] A variable $\zeta = \xi + i\eta$ from an uncertainty space $(\Gamma, \mathfrak{L}, \mathcal{M})$ to the set of complex numbers is a complex uncertain variable if and only if ξ and η are uncertain variables, where ξ and η are real and imaginary part of ζ, respectively.*

Definition 2.6 *[30] Let $\zeta = \xi + i\eta$ be a complex uncertain variable, where ξ and η are real and imaginary part of ζ, respectively. Then the complex uncertainty distribution of ζ is a function from $\mathbb{C}$ to $[0,1]$ defined by*

$$\Phi(z) = \mathcal{M}\{\xi \leq x, \eta \leq y\}$$

for any complex number $z = x + iy$.

Definition 2.7 *[30] Let $\zeta = \xi + i\eta$ be a complex uncertain variable. If the expected value of ξ and η i.e., $E[\xi]$ and $E[\eta]$ exists, then the expected value of ζ is defined by*

$$E[\zeta] = E[\xi] + iE[\eta].$$

Definition 2.8 *[25] A complex uncertain sequence (ζ_n) is said to be rough convergent almost surely to ζ with roughness degree r if for every $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ and for any event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$|| \ \zeta_n(\gamma) - \zeta(\gamma) \ || < r + \varepsilon, \forall n \geq n_0 \ and \ for \ every \ \gamma \in \Lambda.$$

Definition 2.9 *[25] A complex uncertain sequence (ζ_n) is said to be rough convergent in measure to ζ with roughness degree r if for every $\varepsilon, \delta > 0$, $\exists n_0 \in \mathbb{N}$ such that*

$$\mathcal{M}\Big\{\gamma : || \ \zeta_n(\gamma) - \zeta(\gamma) \ || \geq \delta\Big\} < r + \varepsilon, \forall n \geq n_0.$$

Definition 2.10 *[25] Let $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$, respectively. A complex uncertain sequence (ζ_n) is said to be rough convergent in distribution to ζ with roughness degree r if for every $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that*

$$|| \ \Phi_n(z) - \Phi(z) \ || < r + \varepsilon, \forall n \geq n_0$$

and for all z at which $\Phi(z)$ is continuous.

Definition 2.11 *[25] A complex uncertain sequence (ζ_n) is said to be rough convergent in mean to ζ with roughness degree r if for every $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that*

$$E\Big[\ || \ \zeta_n(\gamma) - \zeta(\gamma) \ || \ \Big] < r + \varepsilon, \forall n \geq n_0.$$

Definition 2.12 *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-convergent almost surely ($\mathcal{I}$.a.s.) to ζ if for every $\varepsilon > 0$, there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\Big\{n \in \mathbb{N} : || \ \zeta_n(\gamma) - \zeta(\gamma) \ || \geq \varepsilon\Big\} \in \mathcal{I}, \ for \ every \ \gamma \in \Lambda.$$

Definition 2.13 *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-convergent in measure to ζ if for every $\varepsilon, \delta > 0$*

$$\Big\{n \in \mathbb{N} : \mathcal{M}\Big(\ || \ \zeta_n(\gamma) - \zeta(\gamma) \ || \geq \delta\Big) \geq \varepsilon\Big\} \in \mathcal{I}.$$

Definition 2.14 *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-convergent in mean to ζ if for every $\varepsilon > 0$*

$$\Big\{n \in \mathbb{N} : E\Big[\ || \ \zeta_n(\gamma) - \zeta(\gamma) \ || \ \Big] \geq \varepsilon\Big\} \in \mathcal{I}.$$

Definition 2.15 *Let $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$, respectively. Then the complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-convergent in distribution to ζ if for every $\varepsilon > 0$*

$$\Big\{n \in \mathbb{N} : || \ \Phi_n(z) - \Phi(z) \ || \geq \varepsilon\Big\} \in \mathcal{I}, \ for \ all \ z \ at \ which \ \Phi(z) \ is \ continuous.$$

3. Main Results

Definition 3.1 *A complex uncertain sequence* (ζ_n) *is said to be rough* $\mathcal{I}$*-convergent almost surely to* ζ *with roughness degree* r *if, for every* $\varepsilon > 0$ *and for any event* Λ *with* $\mathcal{M}\{\Lambda\} = 1$ *such that*

$$\Big\{n \in \mathbb{N} : \| \zeta_n(\gamma) - \zeta(\gamma) \| \geq r + \varepsilon\Big\} \in \mathcal{I}, \ for\ every\ \gamma \in \Lambda.$$

If the above condition holds, ζ *is a rough* $\mathcal{I}$*-limit point of* (ζ_n)*, which is usually no more unique (for* $r > 0$*). So we have to consider rough* $\mathcal{I}$*-limit set of* (ζ_n) *defined by*

$$\mathcal{I}.a.s - LIM^r \zeta_n := \big\{\zeta : \zeta_n \xrightarrow{r-\mathcal{I}.a.s} \zeta\big\}.$$

Example 3.1 *Consider the uncertainty space* $(\Gamma, \mathfrak{L}, \mathcal{M})$ *to be* $\{\gamma_1, \gamma_2\}$ *with power set and* $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_1\} = \frac{1}{2}$*. We define a complex uncertain variables as*

$$\zeta_n(\gamma) = \begin{cases} (1 + \frac{1}{n})i, & if\ \gamma = \gamma_1 \\ -(1 + \frac{1}{n})i, & if\ \gamma = \gamma_2 \end{cases} \quad for\ n = 1, 2, 3, ...$$

We also define $\zeta \equiv 0$ *and take* $\mathcal{I}$=$\mathcal{I}_d$*.*
Then we have $\| \zeta_n(\gamma) - \zeta(\gamma) \| = \| \zeta_n(\gamma) \| = (1 + \frac{1}{n})$*, for* $\gamma = \gamma_1, \gamma_2$*.*
Clearly the sequence (ζ_n) *is not* $\mathcal{I}$*-convergent almost surely to* ζ*, but it is rough* $\mathcal{I}$*-convergent almost surely to* ζ *for* $r \geq 1$*.*

Theorem 3.1 *If a complex uncertain sequence* (ζ_n) *is rough* $\mathcal{I}$*-convergent a.s to* ζ*, then*

(*i*) $(\zeta_n - \zeta)$ *is rough* $\mathcal{I}$*-convergent a.s to* 0*.*

(*ii*) $(c\zeta_n)$ *is rough* $\mathcal{I}$*-convergent a.s to* $(c\zeta)$*, where* $c \in \mathbb{C}$*.*

Proof: Proofs are straight forward so omitted. □

Theorem 3.2 *If the complex uncertain sequences* (ζ_n) *and* (ζ_n^*) *are rough* $\mathcal{I}$*-convergent a.s to* ζ *and* ζ^**, respectively, then*

(*i*) $(\zeta_n + \zeta_n^*)$ *is rough* $\mathcal{I}$*-convergent a.s to* $(\zeta + \zeta^*)$*.*

(*ii*) $(\zeta_n - \zeta_n^*)$ *is rough* $\mathcal{I}$*-convergent a.s to* $(\zeta - \zeta^*)$*.*

Proof: (*i*) Let $\varepsilon > 0$, then $A = \Big\{n \in \mathbb{N} : \| \zeta_n(\gamma) - \zeta(\gamma) \| < \frac{r+\varepsilon}{2}\Big\} \in \mathcal{F}(\mathcal{I})$
and $B = \Big\{n \in \mathbb{N} : \| \zeta_n^*(\gamma) - \zeta^*(\gamma) \| < \frac{r+\varepsilon}{2}\Big\} \in \mathcal{F}(\mathcal{I})$.
Since $A \cap B \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$ this implies $A \cap B \neq \phi$. Therefore for all $n \in A \cap B$ we have,

$$\big\|\big(\zeta_n(\gamma) + \zeta_n^*(\gamma)\big) - \big(\zeta(\gamma) + \zeta^*(\gamma)\big)\big\| \leq \|\zeta_n(\gamma) - \zeta(\gamma)\| + \|\zeta_n^*(\gamma) - \zeta^*(\gamma)\| < r + \varepsilon,$$

i.e, $\Big\{n \in \mathbb{N} : \| \big(\zeta_n(\gamma) + \zeta_n^*(\gamma)\big) - \big(\zeta(\gamma) + \zeta^*(\gamma)\big) \| < r + \varepsilon\Big\} \in \mathcal{F}(\mathcal{I})$.
Hence $(\zeta_n + \zeta_n^*)$ is rough $\mathcal{I}$-convergent a.s to $(\zeta + \zeta^*)$.

(*ii*) It is similar to the proof of (*i*) above and therefore omitted. □

Theorem 3.3 *If the complex uncertain sequences* (ζ_n) *and* (ζ_n^*) *are rough* $\mathcal{I}$*-convergent a.s to* ζ *and* ζ^**, respectively, and there exist positive numbers* $p_1, p, q_1,$ *and* q *such that* $p_1 \leq \|\zeta_n\|, \|\zeta\| \leq p$ *and* $q_1 \leq \|\zeta_n^*\|, \|\zeta^*\| \leq q$ *for any* n*, then*

(*i*) $(\zeta_n\zeta_n^*)$ *is rough* $\mathcal{I}$*-convergent a.s to* $(\zeta\zeta^*)$*.*

(ii) $(\frac{\zeta_n}{\zeta_n^*})$ *is rough $\mathcal{I}$-convergent a.s to* $(\frac{\zeta}{\zeta^*})$.

Proof: Let the complex uncertain sequences (ζ_n), (ζ_n^*) are rough $\mathcal{I}$-convergent a.s to ζ and ζ^*, respectively. Then for every $\varepsilon > 0$ and $p, q > 0$ we have,

$$A = \Big\{n \in \mathbb{N} :\| \zeta_n(\gamma) - \zeta(\gamma) \|< \tfrac{r+\varepsilon}{2q}\Big\} \in \mathcal{F}(\mathcal{I})$$

$$\text{and } B = \Big\{n \in \mathbb{N} :\| \zeta_n^*(\gamma) - \zeta^*(\gamma) \|< \tfrac{r+\varepsilon}{2p}\Big\} \in \mathcal{F}(\mathcal{I}).$$

Since $A \cap B \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$ this implies $A \cap B \neq \phi$. Therefore for all $n \in A \cap B$ we have,

$$\begin{aligned}
||\zeta_n(\gamma)\zeta_n^*(\gamma) - \zeta(\gamma)\zeta^*(\gamma)|| &= ||\zeta_n(\gamma)\zeta_n^*(\gamma) - \zeta_n(\gamma)\zeta^*(\gamma) + \zeta_n(\gamma)\zeta^*(\gamma) - \zeta(\gamma)\zeta^*(\gamma)|| \\
&\leq ||\zeta_n(\gamma)\zeta_n^*(\gamma) - \zeta_n(\gamma)\zeta^*(\gamma)|| + ||\zeta_n(\gamma)\zeta^*(\gamma) - \zeta(\gamma)\zeta^*(\gamma)|| \\
&\leq p||\zeta_n^*(\gamma) - \zeta^*(\gamma)|| + q||\zeta_n(\gamma) - \zeta(\gamma)|| \\
&< r + \varepsilon.
\end{aligned}$$

i.e, $\Big\{n \in \mathbb{N} :\| \zeta_n(\gamma)\zeta_n^*(\gamma) - \zeta(\gamma)\zeta^*(\gamma) \|< r + \varepsilon\Big\} \in \mathcal{F}(\mathcal{I})$.
Hence $(\zeta_n\zeta_n^*)$ is rough $\mathcal{I}$-convergent a.s to $(\zeta\zeta^*)$.

(ii) It is similar to the proof of *(i)* above and therefore omitted. □

Theorem 3.4 *The complex uncertain sequence (ζ_n) where $\zeta_n = \xi_n + i\eta_n$ is rough $\mathcal{I}$-convergent almost surely to $\zeta = \xi + i\eta$ if and only if the uncertain sequence (ξ_n) and (η_n) are rough $\mathcal{I}$-convergent almost surely to ξ and η, respectively.*

Proof: Let the uncertain sequence (ξ_n) and (η_n) are rough $\mathcal{I}$-convergent almost surely to ξ and η, respectively. Then from the definition of rough $\mathcal{I}$-convergent almost surely of uncertain sequences, it follows that for any small $\varepsilon > 0$,

$$\Big\{n \in \mathbb{N} :| \xi_n(\gamma) - \xi(\gamma) |\geq \tfrac{r+\varepsilon}{\sqrt{2}}\Big\} \in \mathcal{I}$$

and

$$\Big\{n \in \mathbb{N} :| \eta_n(\gamma) - \eta(\gamma) |\geq \tfrac{r+\varepsilon}{\sqrt{2}}\Big\} \in \mathcal{I}.$$

Note that $\| \zeta_n - \zeta \|= \sqrt{|\xi_n - \xi|^2 + |\eta_n - \eta|^2}$.
Thus we have

$$\Big\{ \| \zeta_n - \zeta \|\geq r + \varepsilon\Big\} \subset \Big\{|\xi_n - \xi| \geq \tfrac{r+\varepsilon}{\sqrt{2}}\Big\} \cup \Big\{|\eta_n - \eta| \geq \tfrac{r+\varepsilon}{\sqrt{2}}\Big\}$$

Therefore

$$\begin{aligned}
&\Big\{n \in \mathbb{N} :\| \zeta_n(\gamma) - \zeta(\gamma) \|\geq r + \varepsilon\Big\} \\
&\qquad \subset \Big\{n \in \mathbb{N} : |\xi_n(\gamma) - \xi(\gamma)| \geq \frac{r+\varepsilon}{\sqrt{2}}\Big\} \cup \Big\{n \in \mathbb{N} : |\eta_n(\gamma) - \eta(\gamma)| \geq \frac{r+\varepsilon}{\sqrt{2}}\Big\} \in \mathcal{I}.
\end{aligned}$$

Hence

$$\Big\{n \in \mathbb{N} :\| \zeta_n(\gamma) - \zeta(\gamma) \|\geq r + \varepsilon\Big\} \in \mathcal{I}.$$

Conversely let, the complex uncertain sequence be (ζ_n) is rough $\mathcal{I}$-convergent almost surely to ζ. Then from the definition of rough $\mathcal{I}$-convergent almost surely of uncertain sequences, it follows that for any small $\varepsilon > 0$,

$$\Big\{n \in \mathbb{N} :\| \zeta_n(\gamma) - \zeta(\gamma) \|\geq r + \varepsilon\Big\} \in \mathcal{I}$$

Note that

$$|\xi_n - \xi| \leq |(\xi_n - \xi) + i(\eta_n - \eta)| = |(\xi_n + i\eta_n) - (\xi + i\eta)| =|| \zeta_n - \zeta ||$$

Thus we have

$$\Big\{n \in \mathbb{N} :| \xi_n - \xi |\geq r + \varepsilon\Big\} \subseteq \Big\{n \in \mathbb{N} :|| \zeta_n - \zeta ||\geq r + \varepsilon\Big\}$$

Therefore

$$\Big\{n \in \mathbb{N} :| \xi_n(\gamma) - \xi(\gamma) |\geq r + \varepsilon\Big\} \subseteq \Big\{n \in \mathbb{N} :|| \zeta_n(\gamma) - \zeta(\gamma) ||\geq r + \varepsilon\Big\} \in \mathcal{I}.$$

Hence

$$\Big\{n \in \mathbb{N} :| \xi_n(\gamma) - \xi(\gamma) |\geq r + \varepsilon\Big\} \in \mathcal{I}.$$

Similarly,

$$\Big\{n \in \mathbb{N} :| \eta_n(\gamma) - \eta(\gamma) |\geq r + \varepsilon\Big\} \in \mathcal{I}.$$

This complete the proof. □

Definition 3.2 *A complex uncertain sequence (ζ_n) is said to be rough $\mathcal{I}$-convergent in measure to ζ with roughness degree r if for every $\varepsilon, \delta > 0$ such that*

$$\Big\{n \in \mathbb{N} : \mathcal{M}\Big(|| \zeta_n(\gamma) - \zeta(\gamma) ||\geq \delta\Big) \geq r + \varepsilon\Big\} \in \mathcal{I}.$$

Example 3.2 *Consider the uncertainty space $(\Gamma, \mathfrak{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ and*

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{n}{(2n+1)}, & if \ \sup\limits_{\gamma_n \in \Lambda} \frac{n}{(2n+1)} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)}, & if \ \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)} < \frac{1}{2} \\ \frac{1}{2}, \ otherwise \end{cases} \quad for \ n = 1, 2, 3, ...$$

and a complex uncertain variables defined by

$$\zeta_n(\gamma) = \begin{cases} ni, \ if \ \gamma = \gamma_n \\ 0, \ \ otherwise \end{cases} \quad for \ n = 1, 2, 3, ...$$

and $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_d$. For $\varepsilon, \delta > 0$ we have,

$$\begin{aligned} &\Big\{n \in \mathbb{N} : \mathcal{M}\Big(|| \zeta_n(\gamma) - \zeta(\gamma) ||\geq \delta\Big) \geq r + \varepsilon\Big\} \\ &\qquad = \Big\{n \in \mathbb{N} : \mathcal{M}\Big(\gamma :|| \zeta_n(\gamma) - \zeta(\gamma) ||\geq \delta\Big) \geq r + \varepsilon\Big\} \\ &\qquad = \Big\{n \in \mathbb{N} : \mathcal{M}(\gamma_n) \geq r + \varepsilon\Big\} \\ &\qquad = \Big\{n \in \mathbb{N} : \frac{n}{2n+1} \geq r + \varepsilon\Big\} \in \mathcal{I}, \ for \ r \geq \frac{1}{2}. \end{aligned}$$

Thus the sequence (ζ_n) is not $\mathcal{I}$-convergent in measure to ζ, but it is rough $\mathcal{I}$-convergent in measure to ζ for $r \geq \frac{1}{2}$.

Theorem 3.5 *The complex uncertain sequence (ζ_n) where $\zeta_n = \xi_n + i\eta_n$ is rough $\mathcal{I}$-convergent in measure to $\zeta = \xi + i\eta$ if and only if the uncertain sequence (ξ_n) and (η_n) are rough $\mathcal{I}$-convergent in measure to ξ and η, respectively.*

Proof: Let the uncertain sequence (ξ_n) and (η_n) are rough $\mathcal{I}$-convergent in measure to ξ and η, respectively. Then from the definition of rough $\mathcal{I}$-convergent in measure of uncertain sequences, it follows that for any small $\varepsilon, \delta > 0$,

$$\Big\{n \in \mathbb{N} : \mathcal{M}(\| \xi_n - \xi \| \geq \frac{\delta}{\sqrt{2}}) \geq \frac{r+\varepsilon}{2}\Big\} \in \mathcal{I}$$
$$and \ \Big\{n \in \mathbb{N} : \mathcal{M}(\| \eta_n - \eta \| \geq \frac{\delta}{\sqrt{2}}) \geq \frac{r+\varepsilon}{2}\Big\} \in \mathcal{I}.$$

Note that $\| \zeta_n - \zeta \| = \sqrt{|\xi_n - \xi|^2 + |\eta_n - \eta|^2}$
Thus we have $\Big\{ \| \zeta_n - \zeta \| \geq \delta\Big\} \subset \Big\{|\xi_n - \xi| \geq \frac{\delta}{\sqrt{2}}\Big\} \cup \Big\{|\eta_n - \eta| \geq \frac{\delta}{\sqrt{2}}\Big\}$.
$\implies \mathcal{M}\Big\{ \| \zeta_n - \zeta \| \geq \delta\Big\} \leq \mathcal{M}\Big\{|\xi_n - \xi| \geq \frac{\delta}{\sqrt{2}}\Big\} + \mathcal{M}\Big\{|\eta_n - \eta| \geq \frac{\delta}{\sqrt{2}}\Big\}$.
Therefore $\Big\{n \in \mathbb{N} : \mathcal{M}\big(\| \zeta_n - \zeta \| \geq \delta\big) \geq r + \varepsilon\Big\}$

$$\subseteq \Big\{n \in \mathbb{N} : \mathcal{M}\big(\| \xi_n - \xi \| \geq \tfrac{\delta}{\sqrt{2}}\big) \geq \tfrac{r+\varepsilon}{2}\Big\}$$
$$\cup \Big\{n \in \mathbb{N} : \mathcal{M}\big(\| \eta_n - \eta \| \geq \tfrac{\delta}{\sqrt{2}}\big) \geq \tfrac{r+\varepsilon}{2}\Big\} \in \mathcal{I}.$$

Hence $\Big\{n \in \mathbb{N} : \mathcal{M}\big(\| \zeta_n - \zeta \| \geq \delta\big) \geq r + \varepsilon\Big\} \in \mathcal{I}$.
Conversely, let the complex uncertain sequence (ζ_n) is rough $\mathcal{I}$-convergent in measure to ζ. Then from the definition of rough $\mathcal{I}$-convergent in measure of complex uncertain sequences, it follows that for any small $\varepsilon, \delta > 0$, $\Big\{n \in \mathbb{N} : \mathcal{M}\big(\| \zeta_n - \zeta \| \geq \delta\big) \geq r + \varepsilon\Big\} \in \mathcal{I}$.
Note that $|\xi_n - \xi| \leq |(\xi_n - \xi) + i(\eta_n - \eta)| = |(\xi_n + i\eta_n) - (\xi + i\eta)| = \| \zeta_n - \zeta \|$
Thus we have $\Big\{ | \xi_n - \xi | \geq \delta\Big\} \subseteq \Big\{ \| \zeta_n - \zeta \| \geq \delta\Big\}$
$\implies \mathcal{M}\Big\{ | \xi_n - \xi | \geq \delta\Big\} \leq \mathcal{M}\Big\{ \| \zeta_n - \zeta \| \geq \delta\Big\}$.
Therefore $\Big\{n \in \mathbb{N} : \mathcal{M}\big(| \xi_n - \xi \| \geq \delta\big) \geq r + \varepsilon\Big\}$

$$\subseteq \Big\{n \in \mathbb{N} : \mathcal{M}\big(\| \zeta_n - \zeta \| \geq \delta\big) \geq r + \varepsilon\Big\} \in \mathcal{I}.$$

Hence $\Big\{n \in \mathbb{N} : \mathcal{M}\big(| \xi_n - \xi | \geq \delta\big) \geq r + \varepsilon\Big\} \in \mathcal{I}$.
Similarly $\Big\{n \in \mathbb{N} : \mathcal{M}\big(| \eta_n - \eta | \geq \delta\big) \geq r + \varepsilon\Big\} \in \mathcal{I}$.
This complete the proof. □

Definition 3.3 *Let $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$, respectively. Then the complex uncertain sequence (ζ_n) is said to be rough $\mathcal{I}$-convergent in distribution to ζ with roughness degree r if for every $\varepsilon > 0$,*

$$\Big\{n \in \mathbb{N} : \| \Phi_n(z) - \Phi(z) \| \geq r + \varepsilon\Big\} \in \mathcal{I},$$

for all z at which $\Phi(z)$ is continuous.

Example 3.3 *Consider the uncertainty space $(\Gamma, \mathfrak{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ and*

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{n}{(2n+1)}, & if \ \sup\limits_{\gamma_n \in \Lambda} \frac{n}{(2n+1)} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)}, & if \ \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)} < \frac{1}{2} \\ \frac{1}{2}, \ otherwise \end{cases} \quad for \ n = 1, 2, 3, ...$$

and a complex uncertain variables be defined by

$$\zeta_n(\gamma) = \begin{cases} n^2 i, \ if \ \gamma = \gamma_n \\ 0, \quad otherwise \end{cases} \qquad for \ n = 1, 2, ...$$

and $\zeta \equiv 0$. *Take* $\mathcal{I}$=$\mathcal{I}_d$. *Then we have the uncertainty distributions of* ζ_n *are*

$$\Phi_n(z) = \Phi_n(x+iy) = \begin{cases} 0, \ if \ x < 0, y < \infty \\ 0, \ if x \geq 0, y < 0 \\ 1 - \frac{n}{2n+1}, \ if \ x \geq 0, 0 \leq y < n^2 \\ 1, \ \ if \ x \geq 0, y \geq n^2 \end{cases}$$

for $n = 1, 2, 3, ...$ *and also the complex uncertainty distribution of* ζ *is*

$$\Phi(z) = \begin{cases} 0, \ if \ x < 0, y < \infty \\ 0, \ if x \geq 0, y < 0 \\ 1, \ \ if \ x \geq 0, y \geq 0. \end{cases}$$

For $\varepsilon > 0$, $r \geq \frac{1}{2}$, *we have*

$$\Big\{ n \in \mathbb{N} : \mid\mid \Phi_n(z) - \Phi(z) \mid\mid \geq r + \varepsilon \Big\} \in \mathcal{I}$$

for all z *at which* $\Phi(z)$ *is continuous.*
Thus the sequence (ζ_n) *is not* $\mathcal{I}$*-convergent in distribution to* ζ, *but it is rough* $\mathcal{I}$*-convergent in distribution to* ζ *for* $r \geq \frac{1}{2}$.

Theorem 3.6 *The complex uncertain sequence* (ζ_n) *where* $\zeta_n = \xi_n + i\eta_n$ *is rough* $\mathcal{I}$*-convergent in distribution to* $\zeta = \xi + i\eta$ *if the uncertain sequence* (ξ_n) *and* (η_n) *are rough* $\mathcal{I}$*-convergent in distribution to* ξ *and* η, *respectively.*

Proof: Let $\Phi(z), \Phi_1(z), \Phi_2(z), ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$ respectively and $\phi(x), \phi_n(x), \phi(y), \phi_n(y)$ be the uncertainty distribution of uncertain variables ξ, ξ_n, η, η_n respectively. Let (ξ_n) and (η_n) be rough $\mathcal{I}$-convergent in distribution to ξ and η, respectively. Then from the definition of rough $\mathcal{I}$-convergent in distribution of uncertain sequences, it follows that for every $\varepsilon > 0$,

$$\Big\{ n \in \mathbb{N} : \mid \phi_n(x) - \phi(x) \mid \geq \tfrac{r+\varepsilon}{2} \Big\} \in \mathcal{I} \text{ , for all } x \text{ at which } \phi(x) \text{ is continuous}$$
$$\text{and } \Big\{ n \in \mathbb{N} : \mid \phi_n(y) - \phi(y) \mid \geq \tfrac{r+\varepsilon}{2} \Big\} \in \mathcal{I} \text{ , for all } y \text{ at which } \phi(y) \text{ is continuous.}$$

$$\begin{aligned} \text{Now } \ ||\Phi_n(z) - \Phi(z)|| &= |\mathcal{M}\{\xi_n \leq x, \eta_n \leq y\} - \mathcal{M}\{\xi \leq x, \eta \leq y\}| \\ &= |\mathcal{M}\{\xi_n \leq x\} \wedge \mathcal{M}\{\eta_n \leq y\} - \mathcal{M}\{\xi \leq x\} \wedge \mathcal{M}\{\eta \leq y\}| \\ &= |\phi_n(x) \wedge \phi_n(y) - \phi(x) \wedge \phi(y)| \\ &= |min\{\phi_n(x), \phi_n(y)\} - min\{\phi(x), \phi(y)\}| \\ &= |\tfrac{\phi_n(x)+\phi_n(y)+|\phi_n(x)-\phi_n(y)|}{2} - \tfrac{\phi(x)+\phi(y)+|\phi(x)-\phi(y)|}{2}|. \end{aligned}$$

Then it can be easily shown that, $||\Phi_n(z) - \Phi(z)|| \leq |\phi_n(x) - \phi(x)| + |\phi_n(y) - \phi(y)|$.

Therefore $\Big\{ n \in \mathbb{N} : ||\Phi_n(z) - \Phi(z)|| \geq r + \varepsilon \Big\}$
$$\subseteq \Big\{ n \in \mathbb{N} : |\phi_n(x) - \phi(x)| \geq \tfrac{r+\varepsilon}{2} \Big\} \cup \Big\{ n \in \mathbb{N} : |\phi_n(y) - \phi(y)| \geq \tfrac{r+\varepsilon}{2} \Big\} \in \mathcal{I}.$$
Hence $\Big\{ n \in \mathbb{N} : ||\Phi_n(z) - \Phi(z)|| \geq r + \varepsilon \Big\} \in \mathcal{I}$ for all z at which $\Phi(z)$ is continuous.
This complete the proof. □

Definition 3.4 *A complex uncertain sequence* (ζ_n) *is said to be rough* $\mathcal{I}$*-convergent in mean to* ζ *with roughness degree* r *if for every* $\varepsilon > 0$ *such that*

$$\Big\{ n \in \mathbb{N} : E\Big[\mid\mid \zeta_n(\gamma) - \zeta(\gamma) \mid\mid \Big] \geq r + \varepsilon \Big\} \in \mathcal{I}.$$

Example 3.4 *Consider the uncertainty space $(\Gamma, \mathfrak{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ and*

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{4}{(3n+1)}, & if \ \sup\limits_{\gamma_n \in \Lambda} \frac{4}{(3n+1)} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{4}{(3n+1)}, & if \ \sup\limits_{\gamma_n \in \Lambda^c} \frac{4}{(3n+1)} < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases} \quad for \ n = 1, 2, 3, ...$$

and a complex uncertain variables be defined by

$$\zeta_n(\gamma) = \begin{cases} ni, & if \ \gamma = \gamma_n \\ 0, & otherwise \end{cases} \quad for \ n = 1, 2, 3, ...$$

and $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_d$. Then for each $n \geq 3$, we have the uncertainty distribution of uncertain variable $||\zeta_n - \zeta|| = ||\zeta_n||$ is

$$\Phi_n(x) = \begin{cases} 0, & if \ x < 0, \\ 1 - \frac{4}{3n+1}, & if \ 0 \leq x < n \\ 1, & if \ x \geq n \end{cases} \quad for \ n = 1, 2, 3, ...$$

Now for each $n \geq 3$, $E[|| \zeta_n(\gamma) - \zeta(\gamma) ||] = \int_0^n (1 - (1 - \frac{4}{3n+1}))dx = \frac{4n}{3n+1}$.
For $\varepsilon > 0$, $r \geq \frac{4}{3}$ we have,

$$\Big\{n \in \mathbb{N} : E\Big[\ || \ \zeta_n(\gamma) - \zeta(\gamma) \ || \ \Big] \geq r + \varepsilon\Big\} \in \mathcal{I}.$$

Thus the sequence (ζ_n) is not $\mathcal{I}$-convergent in mean to ζ, but it is rough $\mathcal{I}$-convergent in mean to ζ for $r \geq \frac{4}{3}$.

Relationship among convergence concepts :

In this section, relationships among the convergence concepts of complex uncertain sequences are studied.

Rough $\mathcal{I}$-Convergence in measure and rough $\mathcal{I}$-convergence in mean

Theorem 3.7 *If a complex uncertain sequence (ζ_n) is rough $\mathcal{I}$-convergent in mean to ζ , then it is rough $\mathcal{I}$-convergent in measure to ζ.*

Proof: Let the complex uncertain sequence (ζ_n) is rough $\mathcal{I}$-convergent in mean to ζ. Then from the definition of rough $\mathcal{I}$-convergent in mean of complex uncertain sequences, it follows that for every $\varepsilon > 0$,

$$\Big\{n \in \mathbb{N} : E\Big[\ || \ \zeta_n(\gamma) - \zeta(\gamma) \ || \ \Big] \geq r + \varepsilon\Big\} \in \mathcal{I}.$$

Using Markov inequality we can see that for given $\delta \geq 1$, we have

$$\mathcal{M}\big(\ || \ \zeta_n(\gamma) - \zeta(\gamma) \ || \geq \delta\big) \leq \frac{E\big[||\zeta_n(\gamma) - \zeta(\gamma)||\big]}{\delta} \leq E\big[\ || \ \zeta_n(\gamma) - \zeta(\gamma) \ || \ \big].$$

Then for $\varepsilon > 0$, $\Big\{n \in \mathbb{N} : \mathcal{M}\big(\ || \ \zeta_n(\gamma) - \zeta(\gamma) \ || \geq \delta\big) \geq r + \varepsilon\Big\}$

$$\subseteq \Big\{n \in \mathbb{N} : E\big[\ || \ \zeta_n(\gamma) - \zeta(\gamma) \ || \ \big] \geq r + \varepsilon\Big\} \in \mathcal{I}.$$

Hence $\Big\{n \in \mathbb{N} : \mathcal{M}\big(\ || \ \zeta_n(\gamma) - \zeta(\gamma) \ || \geq \delta\big) \geq r + \varepsilon\Big\} \in \mathcal{I}.$

Thus (ζ_n) is rough $\mathcal{I}$-convergent in measure to ζ and the theorem is proved. □

Example 3.5 *Rough $\mathcal{I}$-convergence in measure does not imply rough $\mathcal{I}$-convergence in mean.*
From Example 3.2, we have the uncertainty distribution of uncertain variable $||\zeta_n - \zeta|| = ||\zeta_n||$ is

$$\Phi_n(x) = \begin{cases} 0, & if\ x < 0, \\ 1 - \frac{n}{2n+1}, & if\ 0 \le x < n \\ 1, & if\ x \ge n \end{cases} \quad for\ n = 1, 2, 3, ...$$

Now $\mathcal{M}\Big(|| \zeta_n(\gamma) - \zeta(\gamma) || \ge \delta \Big) = \mathcal{M}\Big(\gamma : || \zeta_n(\gamma) - \zeta(\gamma) || \ge \delta \Big) = \mathcal{M}(\gamma_n) = \frac{n}{2n+1}$ *for* $n = 1, 2, 3....$
and $E\big[||\zeta_n - \zeta||\big] = \int_0^n \big(1 - (1 - \frac{n}{2n+1})\big)dx = \frac{n^2}{2n+1}$ *for* $n = 1, 2, 3....$
Then for $r = \frac{1}{2}$ *and* $\varepsilon > 0$, $\Big\{n \in \mathbb{N} : \mathcal{M}\Big(|| \zeta_n(\gamma) - \zeta(\gamma) || \ge \delta \Big) \ge r + \varepsilon \Big\} \in \mathcal{I}$
and $\Big\{n \in \mathbb{N} : E\big[||\zeta_n - \zeta||\big] \ge r + \varepsilon \Big\} \notin \mathcal{I}$.
Thus the sequence (ζ_n) is rough $\mathcal{I}$-convergent in measure for $r = \frac{1}{2}$ but the sequence (ζ_n) is not rough $\mathcal{I}$-convergent in mean.

Rough $\mathcal{I}$-Convergence in measure and rough $\mathcal{I}$-convergence almost surely

Example 3.6 *Rough $\mathcal{I}$-convergence in measure does not imply rough $\mathcal{I}$convergence a.s.*
Consider the uncertainty space $(\Gamma, \mathfrak{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = \frac{1}{2}$. We define a complex uncertain variables as

$$\zeta_n(\gamma) = \begin{cases} (2 + \frac{1}{n})i, & if\ \gamma = \gamma_1 \\ -(2 + \frac{1}{n})i, & if\ \gamma = \gamma_2 \end{cases} \quad for\ n = 1, 2, ...$$

and also we define $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_d$.
Now $|| \zeta_n(\gamma) - \zeta(\gamma) || = || \zeta_n(\gamma) || = 2 + \frac{1}{n}$ *for* $\gamma = \gamma_1, \gamma_2$.
and $\mathcal{M}\Big(|| \zeta_n(\gamma) - \zeta(\gamma) || \ge \delta \Big) = \mathcal{M}\Big(\gamma : || \zeta_n(\gamma) - \zeta(\gamma) || \ge \delta \Big) = 1$.
Then for $\varepsilon > 0$ *and* $r = 1.5$, $\Big\{n \in \mathbb{N} : \mathcal{M}\Big(|| \zeta_n(\gamma) - \zeta(\gamma) || \ge \delta \Big) \ge r + \varepsilon \Big\} \in \mathcal{I}$
and $\{n \in \mathbb{N} : || \zeta_n(\gamma) - \zeta(\gamma) || \ge r + \varepsilon\} \notin \mathcal{I}$.
Thus the sequence (ζ_n) is rough $\mathcal{I}$-convergent in measure but the sequence (ζ_n) is not rough $\mathcal{I}$-convergent almost surely for $r = 1.5$.

Example 3.7 *Rough $\mathcal{I}$-convergence a.s does not imply rough $\mathcal{I}$-convergence in measure.*
Consider the uncertainty space $(\Gamma, \mathfrak{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ and

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{n}{(2n+1)}, & if\ \sup\limits_{\gamma_n \in \Lambda} \frac{n}{(2n+1)} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)}, & if\ \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)} < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases}$$

and a complex uncertain variables defined by

$$\zeta_n(\gamma) = \begin{cases} in, & if\ \gamma = \gamma_n \\ 0, & otherwise \end{cases} \quad for\ n = 1, 2, 3, ...$$

and $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_d$.
Now $|| \zeta_n(\gamma) - \zeta(\gamma) || = || \zeta_n(\gamma) || = \begin{cases} n, & if\ \gamma = \gamma_n \\ 0, & otherwise \end{cases}$ *for n=1,2,3,...*
and $\mathcal{M}\Big(|| \zeta_n(\gamma) - \zeta(\gamma) || \ge \delta \Big) = \mathcal{M}\Big(\gamma : || \zeta_n(\gamma) - \zeta(\gamma) || \ge \delta \Big) = \mathcal{M}(\gamma_n) = \frac{n}{2n+1}$ *for* $n = 1, 2, 3....$

Then for $r = 0.1$ *and* $\varepsilon > 0$, $\Big\{n \in \mathbb{N} :\| \zeta_n(\gamma) - \zeta(\gamma) \|\geq r + \varepsilon\Big\} \in \mathcal{I}$
and $\Big\{n \in \mathbb{N} : \mathcal{M}\Big(\| \zeta_n(\gamma) - \zeta(\gamma) \|\geq \delta\Big) \geq r + \varepsilon\Big\} \notin \mathcal{I}$.
Thus the sequence (ζ_n) *is rough* $\mathcal{I}$*-convergent almost surely but the sequence* (ζ_n) *is not rough* $\mathcal{I}$*-convergent in measure for* $r = 0.1$.

Rough $\mathcal{I}$-Convergence in distribution and rough $\mathcal{I}$-convergence almost surely

Example 3.8 *Rough* $\mathcal{I}$*-convergence in distribution does not imply rough* $\mathcal{I}$*-convergence a.s.*
Consider the uncertainty space $(\Gamma, \mathfrak{L}, \mathcal{M})$ *to be* $\{\gamma_1, \gamma_2\}$ *with power set and* $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = \frac{1}{2}$. *We define a complex uncertain variable as*

$$\zeta(\gamma) = \begin{cases} i, & if\ \gamma = \gamma_1 \\ -i, & if\ \gamma = \gamma_2. \end{cases}$$

We also define $\zeta_n = -\zeta$ *for* $n = 1, 2, \ldots.$ *Take* $\mathcal{I}$=$\mathcal{I}_d$.
Then we have the complex uncertainty distributions of ζ_n *and* ζ *are same, which are defined by*

$$\Phi_n(z) = \Phi_n(x + iy) = \begin{cases} 0, & if\ x < 0, y < \infty \\ 0, & if x \geq 0, y < -1 \\ \frac{1}{2}, & if\ x \geq 0, -1 \leq y < 1 \\ 1, & if\ x \geq 0, y \geq 1 \end{cases} \qquad for\ n = 1, 2, 3, \ldots.$$

Now $\| \Phi_n(z) - \Phi(z) \|= 0$ *for* $-\infty < x < \infty, -\infty < y < \infty$.
and $\| \zeta_n(\gamma) - \zeta(\gamma) \|=\| 2\zeta_n(\gamma) \|= 2$ *for* $\gamma = \gamma_1, \gamma_2$.
Then for $r = 1.5$ *and* $\varepsilon > 0$, $\Big\{n \in \mathbb{N} :\| \Phi_n(z) - \Phi(z) \|\geq r + \varepsilon\Big\} \in \mathcal{I}$
and $\{n \in \mathbb{N} :\| \zeta_n(\gamma) - \zeta(\gamma) \|\geq r + \varepsilon\} \notin \mathcal{I}$.
Thus the sequence (ζ_n) *is rough* $\mathcal{I}$*-convergent in distribution but it is not rough* $\mathcal{I}$*-convergent almost surely for* $r = 1.5$.
Also, we see that the sequence (ζ_n) *is rough* $\mathcal{I}$*-convergent in distribution as well as rough* $\mathcal{I}$*-convergent almost surely for* $r \geq 2$.

Example 3.9 *Rough* $\mathcal{I}$*-convergence a.s does not imply rough* $\mathcal{I}$*-convergence in distribution.*
From Example 3.2, the complex uncertainty distributions of ζ_n *are*

$$\Phi_n(z) = \Phi_n(x + iy) = \begin{cases} 0, & if\ x < 0, y < \infty \\ 0, & if x \geq 0, y < 0 \\ 1 - \frac{n}{2n+1}, & if\ x \geq 0, 0 \leq y < n \\ 1, & if\ x \geq 0, y \geq n \end{cases}$$

for $n = 1, 2, 3, \ldots$ *and also the complex uncertainty distribution of uncertain variable* ζ *is*

$$\Phi(z) = \begin{cases} 0, & if\ x < 0, y < \infty \\ 0, & if x \geq 0, y < 0 \\ 1, & if\ x \geq 0, y \geq 0. \end{cases}$$

Now $\| \Phi_n(z) - \Phi(z) \|= \begin{cases} \frac{n}{2n+1}, & if\ x \geq 0, 0 \leq y < n \\ 0, & otherwise \end{cases} \qquad for\ n = 1, 2, 3, \ldots.$

Also $\| \zeta_n(\gamma) - \zeta(\gamma) \|=\| \zeta_n(\gamma) \|= \begin{cases} n, & if\ \gamma = \gamma_n \\ 0, & otherwise \end{cases} \qquad$ *for n=1,2,3,....*
Then for $r = 0.1$ *and* $\varepsilon > 0$, $\{n \in \mathbb{N} :\| \zeta_n(\gamma) - \zeta(\gamma) \|\geq r + \varepsilon\} \in \mathcal{I}$

and $\Big\{n \in \mathbb{N} :|| \Phi_n(z) - \Phi(z) ||\geq r+\varepsilon\Big\} \notin \mathcal{I}.$

Thus the sequence (ζ_n) *is rough* $\mathcal{I}$*-convergent almost surely but the sequence* (ζ_n) *is not rough* $\mathcal{I}$*-convergent in distribution for* $r = 0.1$.

Also, we see that the sequence (ζ_n) *is rough* $\mathcal{I}$*-convergent in distribution as well as rough* $\mathcal{I}$*-convergent almost surely for* $r \geq \frac{1}{2}$.

Rough $\mathcal{I}$***-Convergence in mean and rough*** $\mathcal{I}$***-convergence almost surely***

Example 3.10 *Rough* $\mathcal{I}$*-convergence a.s does not imply rough* $\mathcal{I}$*-convergence in mean.*
Take an uncertainty space $(\Gamma, \mathfrak{L}, \mathcal{M})$ *to be* $\{\gamma_1, \gamma_2, ...\}$ *with power set and*

$$\mathcal{M}\{\Lambda\} = \sum_{\gamma_n \in \Lambda} \frac{1}{2^n}.$$

The complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} i2^n, & if\ \gamma = \gamma_n \\ 0, & otherwise \end{cases} \quad for\ n = 1, 2, 3, ...$$

Take $\zeta \equiv 0$ *and* $\mathcal{I}$=$\mathcal{I}_d$.
We have the uncertain distribution of uncertain variables $||\zeta_n - \zeta|| = ||\zeta_n||$ *is*

$$\Phi_n(x) = \begin{cases} 0, & if\ x < 0 \\ 1 - \frac{1}{2^n}, & if\ 0 \leq x < 2^n \\ 1, & if\ x \geq 2^n \end{cases} \quad for\ n = 1, 2, 3, ...$$

Then we have $E\big[||\zeta_n - \zeta||\big] = \int_0^\infty \big(1 - \Phi_n(x) + \Phi_n(-x)\big)dx = \int_0^{2^n} \big(1 - (1 - \frac{1}{2^n})\big)dx = 1$

and $|| \zeta_n(\gamma) - \zeta(\gamma) ||=|| \zeta_n(\gamma) ||= \begin{cases} 2^n, & if\ \gamma = \gamma_n \\ 0, & otherwise \end{cases}$ *for n=1,2,3,...*

Then for $r = 0.5$ *and* $\varepsilon > 0$, $\{n \in \mathbb{N} :|| \zeta_n(\gamma) - \zeta(\gamma) ||\geq r + \varepsilon\} \in \mathcal{I}$

and $\Big\{n \in \mathbb{N} : E\big[||\zeta_n - \zeta||\big] \geq r + \varepsilon\Big\} \notin \mathcal{I}.$

Thus the sequence (ζ_n) *is rough* $\mathcal{I}$*-convergent almost surely but it is not rough* $\mathcal{I}$*-convergent in mean for* $r = 0.5$.

Also, we see that the sequence (ζ_n) *is rough* $\mathcal{I}$*-convergent in mean as well as rough* $\mathcal{I}$*-convergent almost surely for* $r \geq 1$.

Rough $\mathcal{I}$***-Convergence in measure and rough*** $\mathcal{I}$***-convergence in distribution***

Example 3.11 *Rough* $\mathcal{I}$*-convergence in distribution does not imply rough* $\mathcal{I}$*-convergence in measure.*
It follows from Example 3.8

4. Conclusion

In this paper, we introduced a new concept of sequence convergence(rough $\mathcal{I}$-convergence) of complex uncertain sequences for the first time. Then in the setting of uncertainty theory, we discussed the relationships among these different convergence concepts, namely rough $\mathcal{I}$-convergence almost surely, rough $\mathcal{I}$-convergence in measure, rough $\mathcal{I}$-convergence in distribution, rough $\mathcal{I}$-convergence in mean. In the future, we will investigate the relationship among the above defined convergent concepts with rough $\mathcal{I}$-convergence in metric of complex uncertain sequence. Furthermore, we will try to apply these generalized convergence concepts of complex uncertain sequences into real problems in engineering and mathematical finance.

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Shyamal Debnath,
Department of Mathematics,
Tripura University (A Central University),
Suryamaninagar-799022, Agartala, India..
E-mail address: shyamalnitamath@gmail.com

and

Amit Halder,
Department of Mathematics,
Tripura University (A Central University),
Suryamaninagar-799022, Agartala, India..
E-mail address: mit2905halder@gmail.com