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A new generalized class of difference sequence spaces defined by Orlicz function

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ABSTRACT: In this article, our aim is to introduce a new generalized class of vector valued sequence space $F(E_k, \Delta_\nu^m, M, p, q)$ using Orlicz function M, where $(E_k)_{k=1}^{\infty}$ is the class of all seminormed space (E_k, q_k) with $E_{k+1} \subseteq E_k$. It is assumed that F is a normal, AK-sequence space with absolutely monotone paranorm g_F and $p = (p_k)$ is a bounded sequence of positive real numbers. Here it is also proved that the space is a complete paranormed space under the paranorm g along with cetrain inclusion relations.

Key Words: Orlicz function, paranorm, $AK-$space, $BK-$space.

Contents

1. Introduction

In 1981 Kızmaz [10] first introduced the concept of difference sequence spaces. Since then attempts have been made by several authors to introduce and investigate some more new sequence spaces. Among them are Ahmad and Mursaleen [1], Colak [3], Et and Colak [5], Et and Basarir [6], Baliarsingh and Dutta [2], Dutta and Baliarsingh [19], Dutta et al. [20] and many. Then Et and Esi [7]introduced the space:

$$X(\Delta_\nu^m) = \{x = (x_k) : (\Delta_\nu^m x) \in X\},$$

for $X = l_\infty, c$ and c_0, where $\nu = (\nu_k)$ is any fixed sequence of non-zero complex numbers and $(\Delta_\nu^m x_k) = \left(\Delta_\nu^{m-1} x_k - \Delta_\nu^{m-1} x_{k+1}\right)$ with

$$\Delta_\nu^m x_k = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} \nu_{k+i} x_{k+i} \quad \text{for all} \quad k \in \mathbb{N}.$$

These sequence spaces are Banach spaces normed by

$$\|x\|_\Delta = \sum_{i=1}^{m} \nu_i x_i + \|\Delta_\nu^m x\|_\infty.$$

Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x \in S : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right), \quad \text{for some} \quad \rho > 0 \right\},$$

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where S be the space of all complex sequences and the space l_M with norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

becomes a Banach space which is called an Orlicz sequence space. Then Orlicz sequence spaces were extended by Parashar and Choudhary [16], Maddox [14], Tripathy et al.[[24],[25],[26]] and many others.

The vector valued sequence spaces under suitable topology was studied by Leonard [12], Rosier [18], Ratha and Srivastava [17], Srivastava and Ghosh [21], Srivastava and Kumar [22] and many others.

2. Seminormed sequence space $F(E_k, \Delta_\nu^m, M, p, q)$

Now we define class of sequence space as follows:

$$F(E_k, \Delta_\nu^m, M, p, q) = \left\{x = (x_k) \in S(E_k) : x_k \in E_k \text{ for each } k, \left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right) \in F \right. \\ \left. \text{for some } \rho > 0\right\},$$

where (E_k, q_k) is a seminormed space with monotonocally decreasing sequence of set E_k. Define

$$S(E_k) = \{x = (x_k) : x_k \in E_k \text{ for each } k \in \mathbb{N}\}.$$

Here $S(E_k)$ is a linear space under the usual co-ordinatewise operations i.e.,

$$\alpha x = (\alpha_k x_k) \text{ and } x + y = (x_k + y_k),$$

with $\alpha \in \mathbb{C}$. Here F is a normal AK-sequence space with absolutely monotone paranorm g_F and having a Schauder basis (e_k), where $e_k = (0, 0, 0, ..., 1, 0, 0, ...0)$ with 1 in kth place, $p = (p_k)$ is any bounded sequence of strictly positive real numbers and M is any Orlicz function. It can be shown that for suitable choices of E_k, F, m and p_k, the proposed sequence space $F(E_k, M, p, \Delta_\nu^m, q)$ generalizes many of the earlier known scalar as well as vector valued sequence spaces. For example:

(i) For $F = \ell_1, E_k = \mathbb{C}$ for all $k \in \mathbb{N}, m = 0, \nu = (1, 1, 1, \dots)$ and $p_k = 1$ for all $k \in \mathbb{N}$, the space reduces ℓ_M [13] by Lindenstrauss and Tzafriri.

(ii) For $F = \ell_1, E_k = \mathbb{C}$ for all $k \in \mathbb{N}, m = 0$ and $\nu = (1, 1, 1, \dots)$ it reduces to $\ell_M(p)$ [16] by Parashar and Choudhary.

(iii) By choosing $F = c_0$ and $\ell_\infty, E_k = \mathbb{C}$ for all $k \in \mathbb{N}$ and $\nu = (1, 1, 1, \dots)$ it reduces to $c_0(\Delta^m, M, p)$ and $\ell_\infty(\Delta^m, M, p)$ [8] by Gökhan et al.

(iv) By choosing $F = \ell_1, E_k = \mathbb{C}$ for all $k \in \mathbb{N}, m = 1$ and $\nu = (1, 1, 1, \dots)$ it reduces to the space $\ell(\Delta^m, M, p)$ [23] by Subramanian.

Thus, study of the sequence space $F(E_k, \Delta_\nu^m, M, p, q)$ gives an unified approach to many of the earlier known spaces.

3. Definitions and preliminaries

The following definitions and preliminaries will be used in sequel.

Definition 3.1 *[9] A paranorm g on a sequence space X is said to be absolutely monotone if $x = (x_k), y = (y_k) \in X$ and $|x_k| \leq |y_k|$ for each $k \in \mathbb{N} \Rightarrow g(x) \leq g(y)$.*

Definition 3.2 *[9]. A sequence space X is called a $K-$space if the co-ordinate function $P_k : X \to K$ given by $P_k(x) = x_k$ is continuous for each $k \in \mathbb{N}$.*

Definition 3.3 *[27] An FK-space is a Fréchet sequence space with continuous co-ordinates.*

Definition 3.4 *[28] A linear space X is called $BK-$space, if it is equipped with a norm under which it is a Banach space with continuous co-ordinates.*

Definition 3.5 *[27] An $FK-$space X is said to be an $AK-$space if $X \supset \phi$, the set of all finitely non-zero sequences and $\{\delta^n\}$ is a basis for X, i.e., for each x, $x^{[n]} \to x$, where $x^{[n]}$ the n^{th} section of x is $\sum_{k=1}^{n} x_k\delta^k$, otherwise expressed $x = \sum x_k\delta^k$ for all $x \in X$. For example, $l(p), c_0(p), w_0(p)$ are $AK-$spaces.*

Definition 3.6 *[4] A sequence space X is said to be convergence free when, if $x = (x_k)$ in X and if $y_k = 0$ whenever $x_k = 0$, then $y = (y_k)$ is in X.*

Definition 3.7 *[9]. A sequence space X is said to be normal if $(x_k) \in X$ implies $(\alpha_k x_k) \in X$ for all sequence of scalars α_k with $|\alpha_k| \leq 1$ for all $n \in \mathbb{N}$.*

Definition 3.8 *[4] A sequence space X is said to be symmetric if,when $x = (x_k)$ is in X, then $y = (y_k)$ is in X when the co-ordinates of y are those of x, but in a different order.*

Definition 3.9 *[9] An Orlicz function is a function $M : [0, \infty) \longrightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.*

Lemma 3.1 *[11] An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if a constant $K > 0$ such that $M(2u) = KM(u), u \geq 0$.*

The Δ_2-condition is equivalent to the inequality $M(lu) \leq K'lM(u)$, for some $K' > 0$ which holds for all values of u and $l > 1$.

Lemma 3.2 *[15] Let $p = (p_n)$ be a bounded sequence of positive real numbers. Then for any complex numbers a_n and b_n, $|a_n+b_n|^{p_n} \leq D\left(|a_n|^{p_n} + |b_n|^{p_n}\right)$, where $0 < p_n \leq \sup p_n = G$ and $D = \max\{1, 2^{G-1}\}$.*

Lemma 3.3 *[15] Let $0 < p \leq 1$. Then for any complex numbers a and b, $|a + b|^p \leq |a|^p + |b|^p$.*

4. Sequence space $F(E_k, \Delta_\nu^m, M, p, q)$

Theorem 4.1 *$F(E_k, \Delta_\nu^m, M, p, q)$ is a linear space over the complex field $\mathbb{C}$.*

Proof: Let $x = (x_k), y = (y_k) \in F(E_k, M, p, \Delta_\nu^m, q)$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_1}\right)\right]^{p_k}\right) \in F \text{ and } \left(\left[M\left(\frac{q_k(\Delta_\nu^m y_k)}{\rho_2}\right)\right]^{p_k}\right) \in F.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_1)$, where $\alpha, \beta \in \mathbb{C}$. Then by convexity of the Orlicz function M we have

$$\begin{aligned} M\left(\frac{q_k(\Delta_\nu^m(\alpha x_k + \beta y_k))}{\rho_3}\right) &\leq M\left(\frac{|\alpha|q_k(\Delta_\nu^m x_k) + |\beta|q_k(\Delta_\nu^m y_k))}{\rho_3}\right) \\ &\leq \frac{1}{2}\left[M\left(\frac{q_k(\Delta^m x_k)}{\rho_1}\right) + M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_2}\right)\right]. \end{aligned}$$

This implies by using Lemma 3.2,

$$\begin{aligned} \left[M\left(\frac{q_k(\Delta_\nu^m(\alpha x_k + \beta y_k))}{\rho_3}\right)\right]^{p_k} &\leq \left(\frac{1}{2}\right)^{p_k}\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_1}\right) + M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_2}\right)\right]^{p_k} \\ &\leq \left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_1}\right) + M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_2}\right)\right]^{p_k} \\ &\leq D\left\{\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_1}\right)\right]^{p_k} + \left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_2}\right)\right]^{p_k}\right\} \end{aligned}$$

This implies that, $\left(\left[M\left(\frac{q_k(\Delta_\nu^m(\alpha x_k+\beta y_k))}{\rho_3}\right)\right]^{p_k}\right) \in F$, because F is a normal AK-sequence space. i.e., $\alpha x + \beta y \in F(E_k, \Delta_\nu^m, M, p, q)$. Hence $F(E_k, \Delta_\nu^m, M, p, q)$ is a linear space. □

Theorem 4.2 *The function $\mathcal{M}_t : [0,\infty) \to [0,\infty)$ defined by*

$$\mathcal{M}_t(u) = \left[g_F \left(\sum_{k=1}^{t} \left[M\left(\frac{uq_k(\Delta_\nu^m x_k)}{\rho} \right) \right]^{p_k} e_k \right) \right]^{\frac{1}{H}}$$

is a continuous function of u for each positive integer t, where $x = (x_k) \in F(E_k, \Delta_\nu^m, M, p, q)$ and (e_k) is a basis of F with $g_F(e_k) = 1$, for all $k \in \mathbb{N}$ and g_F an absolutely monotone paranorm on F, a normal AK-sequence space.

Proof: We define the function $\mathcal{N}_k : [0,\infty) \to F$ by $\mathcal{N}_k(u) = \left[M\left(\frac{uq_k(\Delta_\nu^m x_k)}{\rho} \right) \right]^{p_k} e_k$. Let $u_i \to 0$ as $i \to \infty$. Then by continuity of Orlicz function M,

$$\mathcal{N}_k(u_i) = \left[M\left(\frac{u_i q_k(\Delta_\nu^m x_k)}{\rho} \right) \right]^{p_k} e_k \to (0,0,0,\dots) \text{as} i \to \infty \text{ for each } k = 1,2,3,\dots,t.$$

Thus for fixed t, $\sum_{k=1}^{t} \mathcal{N}_k(u_i) = \sum_{k=1}^{t} \left[M\left(\frac{u_i q_k(\Delta_\nu^m x_k)}{\rho} \right) \right]^{p_k} e_k \to (0,0,0,\dots)$.
Since paranorm g_F is a continuous function, so it gives $g_F\left(\sum_{k=1}^{t} \mathcal{N}_k(u_i) \right) \to 0$ as $i \to \infty$.
i.e., $\mathcal{M}_n(u_i) \to 0$ as $i \to \infty$.
Hence $\mathcal{M}_n$ is a continuous function of u for each positive integer t. □

Theorem 4.3 *$F(E_k, \Delta_\nu^m, M, p, q)$ is a paranormed space with paranorm*

$$g(x) = \sum_{i=1}^{m} q_i(\nu_i x_i) + \inf \left\{ \rho^{\left(\frac{p_n}{H}\right)} : \left(g_F \left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{H}} \le 1, n \in \mathbb{N} \right\} \tag{4.1}$$

where $x \in F(E_k, \Delta_\nu^m, M, p, q)$ and $H = \max(1, \sup p_k)$.

Proof: It is obvious that, $g(x) \ge 0, g(0) = 0$ and $g(x) = g(-x)$ for any $x \in F(E_k, \Delta_\nu^m, M, p, q)$. Then there exists $\rho_1, \rho_2 > 0$ such that

$$\left(g_F \left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_1} \right) \right]^{p_k} \right)^{\frac{1}{H}} \le 1 \text{ and } \left(g_F \left[M\left(\frac{q_k(\Delta_\nu^m y_k)}{\rho_2} \right) \right]^{p_k} \right)^{\frac{1}{H}} \le 1.$$

Let $\rho_3 = \max(2\rho_1, 2\rho_2)$. Then by convexity of Orlicz function M, it can be shown that

$$\begin{aligned}
\left[M\left(\frac{q_k(\Delta_\nu^m (x_k + y_k)}{\rho_3} \right) \right]^{p_k} &\le \left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_1} \right) + M\left(\frac{q_k(\Delta_\nu^m y_k)}{\rho_2} \right) \right]^{p_k} \\
&\le \left[\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_1} \right) + M\left(\frac{q_k(\Delta_\nu^m y_k)}{\rho_2} \right) \right]^{\frac{p_k}{H}} \right]^H \\
&\le \left[\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_1} \right) \right]^{\frac{p_k}{H}} + \left[M\left(\frac{q_k(\Delta_\nu^m y_k)}{\rho_2} \right) \right]^{\frac{p_k}{H}} \right]^H
\end{aligned}$$

As g_F is absolute monotone paranorm on F and by using Lemma 3.3, it can be shown that

$$\begin{aligned}
&\left(g_F \left[M\left(\frac{q_k(\Delta_\nu^m (x_k + y_k))}{\rho_3} \right) \right]^{p_k} \right)^{\frac{1}{H}} \\
\le\ &\left[g_F \left\{ \left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho} \right) \right]^{p_k} + \left[M\left(\frac{q_k(\Delta_\nu^m y_k)}{\rho} \right) \right]^{p_k} \right\} \right]^{\frac{1}{H}} \\
\le\ &\left[g_F \left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho} \right) \right]^{p_k} \right) \right]^{\frac{1}{H}} + \left[g_F \left(\left[M\left(\frac{q_k(\Delta_\nu^m y_k)}{\rho} \right) \right]^{p_k} \right) \right]^{\frac{1}{H}}.
\end{aligned} \tag{4.2}$$

Since q_k is a seminorm on E_k, therefore

$$\sum_{i=1}^{m} q_i(\nu_i(x_i+y_i)) \leq \sum_{i=1}^{m} q_i(\nu_i x_i) + \sum_{i=1}^{m} q_i(\nu_i x_i). \tag{4.3}$$

From (4.2) and (4.3), $g(x+y) \leq g(x)+g(y)$.
Now to prove that scalar multiplication is continuous under g, i.e., it is required to show that $g(\alpha_l x^l - \alpha x) \to 0$ as $l \to \infty$, where $x^l \to x$ as $l \to \infty$ in $F(E_k, \Delta_\nu^m, M, p, q)$ and $\alpha_l \to \alpha$ as $l \to \infty$ in $\mathbb{C}$.
Consider,

$$\begin{aligned}
g(\alpha_l x^l - \alpha x) &= g(\alpha_l x^l - \alpha_l x + \alpha_l x - \alpha x) \\
&\leq g(\alpha_l(x^l - x)) + g((\alpha_l - \alpha)x) \\
&= \sum_{i=1}^{m} q_i(\nu_i \alpha_l (x_i^l - x_i)) + \sum_{i=1}^{m} q_i(\nu_i(\alpha_l - \alpha)x_i) \\
&\quad + \inf\left\{\rho^{\frac{p_n}{H}} : \left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m \alpha_l(x_k^l - x_k))}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, n \in \mathbb{N}\right\} \\
&\quad + \inf\left\{\rho^{\frac{p_n}{H}} : \left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m (\alpha_l - \alpha)x_k)}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, n \in \mathbb{N}\right\} \\
&\leq |\alpha_l| \sum_{i=1}^{m} q_i(\nu_i(x_i^l - x_i)) + |\alpha_l - \alpha| \sum_{i=1}^{m} q_i(\nu_i x_i) \\
&\quad + \inf\left\{(|\alpha_l| r)^{\frac{p_n}{H}} : \left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m (x_k^l - x_k))}{r}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, n \in \mathbb{N}\right\} \\
&\quad + \inf\left\{\rho^{\frac{p_n}{H}} : \left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m (\alpha_l - \alpha)x_k)}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, n \in \mathbb{N}\right\} \\
&\quad \left[\text{where} \quad r = \frac{\rho}{|\alpha|}\right] \\
&\leq N_1 \sum_{i=1}^{m} q_i(\nu_i(x_i^l - x_i)) + |\alpha_l - \alpha| \sum_{i=1}^{m} q_i(\nu_i x_i) \\
&\quad + N_2 \inf\left\{(|\alpha_l| r)^{\frac{p_n}{H}} : \left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m (x_k^l - x_k))}{r}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, n \in \mathbb{N}\right\} \\
&\quad + \inf\left\{\rho^{\frac{p_n}{H}} : \left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m (\alpha_l - \alpha)x_k)}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, n \in \mathbb{N}\right\} \\
&\quad \left(\text{where} \quad N_1 = \sup_l |\alpha_l|, N_2 = \sup_l [\max(1, |\alpha_l|)], \text{ because } \quad |\alpha|^{\frac{p_n}{H}} \leq \max(1, |\alpha_l|).\right)
\end{aligned}$$

This implies,

$$g(\alpha_l x^l - \alpha x) \leq \max(N_1, N_2) g(x^l - x) + |\alpha_l - \alpha| \sum_{i=1}^{m} q_i(\nu_i x_i) + \Omega \tag{4.4}$$

where $\Omega = \inf\left\{\rho^{\frac{p_n}{H}} : \left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m(\alpha_l - \alpha)x_k)}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, n \in \mathbb{N}\right\}$.
First and second expression of inequality (4.4) tends to 0 as $x^l \to x$ in $F(E_k, \Delta_\nu^m, M, p, q)$ and $\alpha_l \to \alpha$ in $\mathbb{C}$ as $l \to \infty$. Now it remains to show that $\Omega \to 0$ as $l \to \infty$.

Since $\left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right) \in F$ and (e_k) is Schauder basis of F, therefore

$$g_F\left[\left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right) - \sum_{k=1}^{t}\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} e_k\right] \to 0 \text{ as } t \to \infty$$

i.e., $g_F\left(\sum_{k=t+1}^{\infty}\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} e_k\right) \to 0$ as $t \to \infty$.

Therefore for every $\epsilon > 0$, there exists a positive integer t_0 such that

$$g_F\left(\sum_{k=t+1}^{\infty}\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} e_k\right) < \left(\frac{\varepsilon}{2}\right)^H \text{ for all } t \geq t_0. \tag{4.5}$$

Replacing t by t_0 in in the inequality (4.4) we get

$$\left[g_F\left(\sum_{k=t_0+1}^{\infty}\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} e_k\right)\right]^{\frac{1}{H}} < \frac{\varepsilon}{2}. \tag{4.6}$$

As $\alpha_l \to \alpha$ for $l \to \infty$, therefore for $\varepsilon = 1$, there exists a positive integer l_1 such that $|\alpha_l - \alpha| < 1$ for all $l \geq l_1$. Consequently, for all $l \geq l_1$,

$$\sum_{k=t_0+1}^{\infty}\left[M\left(\frac{|\alpha_l - \alpha| q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} e_k \leq \sum_{k=t_0+1}^{\infty}\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} e_k.$$

Since g_F is monotone paranorm, so for all $l \geq l_1$,

$$\left[g_F\left(\sum_{k=t_0+1}^{\infty}\left[M\left(\frac{|\alpha_l - \alpha| q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} e_k\right)\right]^{\frac{1}{H}} \leq \left[g_F\left(\sum_{k=t_0+1}^{\infty}\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} e_k\right)\right]^{\frac{1}{H}}.$$

Using inequality (4.6) we get,

$$\left[g_F\left(\sum_{k=t_0+1}^{\infty}\left[M\left(\frac{|\alpha_l - \alpha| q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} e_k\right)\right]^{\frac{1}{H}} < \frac{\varepsilon}{2} \text{ for all } l \geq l_1. \tag{4.7}$$

Next, we show

$$\left[g_F\left(\sum_{k=1}^{t_0}\left[M\left(\frac{|\alpha_l - \alpha| q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} e_k\right)\right]^{\frac{1}{H}} < \frac{\varepsilon}{2} \text{ for all } l \geq l_2,$$

where l_2 is a positive integer. For this first we claim that the function $\mathcal{M}_{t_0}$ defined by

$$\mathcal{M}_{t_0} = \left[g_F\left(\sum_{k=t_0+1}^{\infty}\left[M\left(\frac{u q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} e_k\right)\right]^{\frac{1}{H}}, u \geq 0$$

is continuous for any fixed integer t_0, which is proved in Theorem 4.2. Hence there exists $\delta \in (0, 1)$ such that $\mathcal{M}_{t_0} < \left(\frac{\varepsilon}{2}\right)^H$, whenever $u < \delta$. Now, since $\alpha_l \to \alpha$ as $l \to \infty$, therefore for $\delta \in (0, 1)$, there exists a positive integer l_2 such that $|\alpha_l - \alpha| < \delta$ for all $l \geq l_2$.

We have $\mathcal{M}_{t_0}(|\alpha_l - \alpha|) < \left(\frac{\varepsilon}{2}\right)^H$ for all $l \geq l_2$.

$$i.e., \left[g_F\left(\sum_{k=1}^{t_0}\left[M\left(\frac{|\alpha_l - \alpha| q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} e_k\right)\right]^{\frac{1}{H}} < \frac{\varepsilon}{2} \text{ for all } l \geq l_2. \tag{4.8}$$

Using (4.7),(4.8) and Lemma 3.3, we have

$$\left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m(\alpha_l-\alpha)x_k)}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \text{ for all } l \geq l_0$$

where $l_0=\max(l_1,l_2)$. Thus, inequality (4.4) gives $g(\alpha_l x^l-\alpha x)\to 0$ as $l\to\infty$.
Hence $F(E,\Delta_\nu^m,M,p,q)$ is a paranormed sequence space. □

Remark 4.1 *$F(E_k,\Delta_\nu^m,M,p,q)$ is not a total paranormed space under the paranorm defined by equation (4.1).*

Proof: We have to show that $g(x)=0$ need not imply $x=0$.
Suppose $g(x)=0$, which implies $\sum_{i=1}^m q_i(\nu_i x_i)=0$ and $\left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}}=0$ together. By $\sum_{i=1}^m q_i(\nu_i x_i)=0$ i.e., $q_i(\nu_i x_i)=0$ for each $i=1,2,3,\ldots,m$. which need not imply that $x_i=0$, since $\nu_i\neq 0$, as q is a seminorm on E. Hence g is not a total paranormed space on $F(E,\Delta_\nu^m,M,p,q)$. □

Theorem 4.4 *$F(E_k,\Delta_\nu^m,M,p,q)$ is a K-space if F is a K-space.*

Proof: We have to show that the co-ordinate function

$$P_k: F(E_k,\Delta_\nu^m,M,p,q)\longrightarrow E_k$$

given by $P_k(x)=x_k$ is continuous for each $k\in\mathbb{N}$, where $x\in F(E_k,\Delta_\nu^m,M,p,q)$. For this let (x^l) be any sequence in $F(E_k,\Delta_\nu^m,M,p,q)$ such that $x^l\to 0$ as $l\to\infty$ in $F(E_k,\Delta_\nu^m,M,p,q)$.

$$\sum_{i=1}^m q_i(\nu_i x_i)+\inf\left\{\rho^{\left(\frac{p_n}{H}\right)}:\left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}}\leq 1, n\in\mathbb{N}\right\}\to 0 \text{ as } l\to\infty.$$

this means that

$$q(\nu_l^l x_i^l)\to 0 \text{ as } l\to\infty \text{ for each } i=1,2,3,\ldots m \tag{4.9}$$

as well as

$$\inf\left\{\rho^{\left(\frac{p_n}{H}\right)}:\left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}}\leq 1, n\in\mathbb{N}\right\}\to 0 \text{ as } l\to\infty.$$

Since F is a K-space, therefore for fixed k

$$\left[M\left(\frac{q_k(\Delta_\nu^m x_k^l)}{\rho}\right)\right]^{p_k}\to 0 \text{ as } l\to\infty.$$

Which implies

$$M\left(\frac{q_k(\Delta_\nu^m x_k^l)}{\rho}\right)\to 0 \text{ as } l\to\infty.$$

Thus for any $\delta>0$, there exists $l_0\in\mathbb{N}$ such that

$$M\left(\frac{q_k(\Delta_\nu^m x_k^l)}{\rho}\right)<\delta \text{ for all } l\geq l_0.$$

Let $\delta=M\left(\frac{\varepsilon}{\rho}\right)$, where $\varepsilon>0$. Then for all $l>l_0$

$$M\left(\frac{q_k(\Delta_\nu^m x_k^l)}{\rho}\right)<M\left(\frac{\varepsilon}{\rho}\right).$$

As M is continuous and non-decreasing

$$q_k(\Delta_\nu^m x_k^l) < \varepsilon.$$

This shows that for fixed k,

$$\Delta_\nu^m x_k^l \to 0 \text{ in } E_k \text{ as } l \to \infty. \tag{4.10}$$

Now from equation(4.9) and the simillar lines used above, it can be shown that

$$x_i^l \to 0 \text{ in } E_i \text{ as } l \to \infty \text{ for each } i = 1, 2, \ldots, m. \tag{4.11}$$

We know that, for each $k \in \mathbb{N}$

$$\Delta_\nu^m x_k = x_k^l \nu_k^l - \binom{m}{1} x_{k+1}^l \nu_{k+1}^l + \binom{m}{2} x_{k+2}^l \nu_{k+2}^l - \cdots + (-1)^m \binom{m}{m} x_{k+m}^l \nu_{k+m}^l \tag{4.12}$$

Put $k = 1$ in equation (4.12), we get

$$\Delta_\nu^m x_1 = x_1^l \nu_1^l - \binom{m}{1} x_2^l \nu_2^l + \binom{m}{2} x_3^l \nu_3^l - \cdots + (-1)^m \binom{m}{m} x_{1+m}^l \nu_{1+m}^l \tag{4.13}$$

Now using equation (4.10) and (4.11) in equation (4.13), we get

$$x_{1+m}^l \nu_{1+m}^l \to 0 \text{ as } n \to \infty.$$

Similarly, one can show that $x_{2+m}^l \nu_{2+m}^l \to 0$ as $n \to \infty$ and so on. Thus, we conclude that $x_k^l \to 0$ in E_k as $l \to \infty$ for each $k \in \mathbb{N}$. Hence co-ordinate wise function P_k is continuous for each $k \in \mathbb{N}$, so $F(E_k, \Delta_\nu^m, M, p, q)$ is a K-space. □

Theorem 4.5 *Let F is a K-space and (E_k, q_k) be complete seminormed space such that $E_{k+1} \subseteq E_k$ for each $k \in \mathbb{N}$. Then $F(E_k, \Delta_\nu^m, M, p, q)$ is a complete paranormed space under the paranorm g defined by equation (4.1) where $x \in F(E_k, \Delta_\nu^m, M, p, q)$.*

Proof: It is shown in Theorem-4.3 that the sequence space $F(E_k, \Delta_\nu^m, M, p, q)$ is a paranormed space under g. For completeness, consider $x^l = \left((x_k^l)_k\right)$ to be any Cauchy sequence in $F(E_k, \Delta_\nu^m, M, p, q)$. Then $g(x^l - x^t) \to 0$ as $l, t \to \infty$, which implies

$$q_i\left(\nu_i^l x_i^l - \nu_i^t x_i^t\right) \to 0 \text{ for each } i \,(1 \le i \le m) \tag{4.14}$$

and

$$\inf\left\{\rho^{(\frac{p_n}{H})} : \left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m(x_k^l - x_k^t))}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \le 1, n \in \mathbb{N}\right\} \to 0 \tag{4.15}$$

as $l, t \to \infty$. From equation (4.9), it is clear that (x_i^l) is Cauchy sequence in E_i for each $1 \le i \le m$. Next, we will show that $(\Delta_\nu^m x_k^l)$ is a Cauchy sequence in E_k for each $k \in \mathbb{N}$. From condition (4.10) there exists a positive integer l_1 such that

$$\left(g_F\left[M\left(\frac{q_k(\Delta_\nu^m(x_k^l - x_k^t))}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \le 1 \quad \text{for all} \quad l, t \ge l_1.$$

Which implies,

$$g_F\left[M\left(\frac{q_k(\Delta_\nu^m(x_k^l - x_k^t))}{\rho}\right)\right]^{p_k} \le 1 \quad \text{for all} \quad l, t \ge l_1.$$

Since $\rho < g(x^l - x^t)$, replacing ρ by $g(x^l - x^t)$ in the above inequality, we get

$$g_F\left[M\left(\frac{q_k(\Delta_\nu^m(x_k^l - x_k^t))}{g(x^l - x^t)}\right)\right]^{p_k} \le 1 \text{ for all } l, t \ge l_1. \tag{4.16}$$

Since (x^l) is a Cauchy sequence, therefore for a given $\varepsilon > 0$, there exists a positive integer l_2 such that $g(x^l - x^t) < \frac{\varepsilon}{r\delta}$, for all $l, t \geq l_2$. For $r > 0$ with $g_F\left[M\left(\frac{r\delta}{2}\right)\right]^{p_k} \geq 1$ we have from (4.16),

$$g_F\left[M\left(\frac{q_k(\Delta_\nu^m(x_k^l - x_k^t))}{g(x^l - x^t)}\right)\right]^{p_k} \leq g_F\left[M\left(\frac{r\delta}{2}\right)\right]^{p_k}.$$

By using Theorem 4.4, as $F(E_k, \Delta_\nu^m, M, p, q)$ is a K-space,

$$M\left(\frac{q_k(\Delta_\nu^m(x_k^l - x_k^t))}{g(x^l - x^t)}\right) \leq M\left(\frac{r\delta}{2}\right)$$

Since M is continuous and non-decreasing,

$$\frac{q_k(\Delta_\nu^m(x_k^l - x_k^t))}{g(x^l - x^t)} \leq \frac{r\delta}{2}$$

$$\begin{aligned} q_k(\Delta_\nu^m(x_k^l - x_k^t)) &\leq \frac{r\delta}{2} g(x^l - x^t) \\ &\leq \frac{r\delta}{2}\frac{\varepsilon}{r\delta} \\ &= \frac{\varepsilon}{2}, \quad \text{for} \quad l, t \geq l_0 = \max(l_1, l_2). \end{aligned}$$

This shows that

$$(\Delta_\nu^m x_k^l) \text{ is a Cauchy sequence in } E_k \text{ for each } k \in \mathbb{N}. \tag{4.17}$$

Now using (4.14) it can be shown that

$$(x_i^l) \text{ is a Cauchy sequence in } E_i \text{ for each } i = 1, 2, 3, \ldots, m. \tag{4.18}$$

Using (4.17) and (4.18), we will show that (x_k^l) is Cauchy sequence in E_k as $l \to \infty$ for each $k \in \mathbb{N}$. We know that, for each $k \in \mathbb{N}$

$$\Delta_\nu^m x_k = x_k^l \nu_k^l - \binom{m}{1} x_{k+1}^l \nu_{k+1}^l + \binom{m}{2} x_{k+2}^l \nu_{k+2}^l - \cdots + (-1)^m \binom{m}{m} x_{k+m}^l \nu_{k+m}^l \tag{4.19}$$

Put $k = 1$ in equation (4.19), we get

$$\Delta_\nu^m x_1 = x_1^l \nu_1^l - \binom{m}{1} x_2^l \nu_2^l + \binom{m}{2} x_3^l \nu_3^l - \cdots + (-1)^m \binom{m}{m} x_{1+m}^l \nu_{1+m}^l \tag{4.20}$$

Now using (4.17) and (4.18) in equation (4.19), we get

$$x_{1+m}^l \nu_{1+m}^l \to 0 \text{ as } n \to \infty.$$

Similarly, one can show that $x_{2+m}^l \nu_{2+m}^l \to 0$ as $n \to \infty$ and so on. Thus, we conclude that $x_k^l \to 0$ in E_k as $l \to \infty$ for each $k \in \mathbb{N}$. Thus $\Delta_\nu^m(x_k^l) = \left(\left(\Delta_\nu^m(x_k^1)\right), \left(\Delta_\nu^m(x_k^2)\right), \ldots\right)$ converges to $\Delta_\nu^m x_k$, for each $k \in \mathbb{N}$ in E_k. But E_k is complete, therefore (x_k^l) is a convergent sequence in E_k for each $k \in \mathbb{N}$. Then for each fixed $i (1 \leq i \leq m)$,

$$q_i(\nu_i^l x_i^l - \nu_i x_i) < \frac{\varepsilon}{2m} \text{ for all } l \geq l_0(i). \tag{4.21}$$

Again (x^l) is a Cauchy sequence in $F(E_k, \Delta_\nu^m, M, p, q)$. So for given $\varepsilon > 0$, there exists a positive integer $l_0(\varepsilon)$ such that $g(x^l - x^t) < \frac{\varepsilon}{2}$ for all $l, t \geq l_0(\varepsilon)$. We can choose $\eta > 0$ such that

$$g(x^l - x^t) < \eta < \frac{\varepsilon}{2} \text{ for all } l, t \geq l_0(\varepsilon). \tag{4.22}$$

Since g_F is absolute monotone paranorm on a normal AK-space F, therefore

$$\begin{aligned} g_F\left(\sum_{k=1}^{n}\left[M\left(\frac{q_k(\Delta_\nu^m(x_k^l-x_k^t))}{\eta}\right)\right]^{p_k}e_k\right) &\leq g_F\left(\left[M\left(\frac{q_k(\Delta_\nu^m(x_k^l-x_k^t))}{\rho}\right)\right]^{p_k}\right) \\ &\leq 1. \end{aligned}$$

Using continuity of Orlicz function and g_F, we get

$$g_F\left(\sum_{k=1}^{n}\left[M\left(\frac{q_k(\Delta_\nu^m(x_k^l-x_k))}{\eta}\right)\right]^{p_k}e_k\right) \leq 1 \quad \text{as} \quad l\to\infty.$$

Letting $n\to\infty$ we get

$$g_F\left(\left[M\left(\frac{q_k(\Delta_\nu^m(x_k^l-x_k))}{\rho}\right)\right]^{p_k}\right) \leq 1 \quad \text{as} \quad l\to\infty.$$

So,

$$\begin{aligned} g(x^l-x) &= \sum_{i=1}^{m} q_i(x_i^l-x_i)+\inf\left\{\xi^{\frac{p_n}{H}} : \left[g_F\left(M\left(\frac{q_k(\Delta_\nu^m(x_k^l-x_k))}{\xi}\right)\right)^{p_k}\right]^{\frac{1}{H}} \leq 1, n\in\mathbb{N}\right\} \\ &< m\left(\frac{\varepsilon}{2m}\right)+\eta<\varepsilon, \quad \text{for all} \quad l\geq \max\left(l_0(1),l_0(2),\ldots,l_0(m),l_0(\varepsilon)\right). \end{aligned}$$

Using conditions (4.17) and (4.18). Thus, (x^l) converges to x in the paranorm of $F(E,\Delta_\nu^m,M,p,q)$. Now, we shall show that $x\in F(E_k,\Delta_\nu^m,M,p,q)$.

Since $(x^l)=(x_k^l)\in F(E_k,\Delta_\nu^m,M,p,q)$, so there exists some $\rho>0$ such that

$$\left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k^l)}{\rho}\right)\right]^{p_k}\right)\in F.$$

Since $q\left(\Delta_\nu^m(x_k^l-x_k)\right)\to 0$ as $l\to\infty$ for each $k\in\mathbb{N}$, therefore we can choose positive number δ_k^l satisfying $0<\delta_k^l<1$ such that

$$\left[M\left(\frac{q_k(\Delta_\nu^m(x_k^l-x_k))}{\rho}\right)\right]^{p_k} < \delta_k^l\left[M\left(\frac{q_k(\Delta_\nu^m x_k^l)}{\rho}\right)\right]^{p_k}.$$

We have

$$\begin{aligned} \left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{2\rho}\right)\right]^{p_k} &= \left[M\left(\frac{q_k(\Delta_\nu^m((x_k^l-x_k)-x_k^l))}{2\rho}\right)\right]^{p_k} \\ &\leq \left[\frac{1}{2}M\left(\frac{q_k(\Delta_\nu^m(x_k^l-x_k))}{2\rho}\right)+\frac{1}{2}M\left(\frac{q_k(\Delta_\nu^m x_k^l)}{\rho}\right)\right]^{p_k} \\ &\leq \left[M\left(\frac{q_k(\Delta_\nu^m(x_k^l-x_k))}{2\rho}\right)+M\left(\frac{q_k(\Delta_\nu^m x_k^l)}{\rho}\right)\right]^{p_k} \\ &\leq D\left\{\left[M\left(\frac{q_k(\Delta_\nu^m(x_k^l-x_k))}{2\rho}\right)\right]^{p_k}+\left[M\left(\frac{q_k(\Delta_\nu^m x_k^l)}{\rho}\right)\right]^{p_k}\right\} \\ &\leq D\left(1+\delta_k^l\right)\left[M\left(\frac{q_k(\Delta_\nu^m x_k^l)}{\rho}\right)\right]^{p_k}. \end{aligned}$$

This implies,

$$\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{2\rho}\right)\right]^{p_k} \leq DM_l\left[M\left(\frac{q_k(\Delta_\nu^m x_k^l)}{\rho}\right)\right]^{p_k}.$$

By using Lemma 3.2, where $D=\max\left(1,2^{G-1}\right), G=\sup_k p_k$ and $M_l=\sup_k\left(1+\delta_k^l\right)$.
Since F is normal sequence space, it follows that

$$\left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{2\rho}\right)\right]^{p_k}\right)\in F \quad i.e.,\quad x\in F(E_k,\Delta_\nu^m,M,p,q).$$

Hence $F(E_k,\Delta_\nu^m,M,p,q)$ is a complete paranormed space under the paranorm g. □

Theorem 4.6 *$F(E_k,\Delta_\nu^m,M,p,q)$ is normal if $m=0$ and $\nu=(1,1,1,\dots)$.*

Proof: For $m=0$ and $\nu=(1,1,1,\dots)$, the sequence space $F(E_k,\Delta_\nu^m,M,p,q)$ reduces to $F(E_k,M,p,q)$, where

$$F(E_k,M,p,q)=\left\{x=(x_k)\in S(E_k): x_k\in E_k,\quad\text{for each}\quad k,\left(\left[M\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}\right)\in F \text{ for some } \rho>0\right\}.$$

For a sequence scalars $\lambda=(\lambda_k)$ such that $|\lambda_k|\le 1$ for all $k\in\mathbb{N}$, we have
$q_k(\lambda_k x_k)=|\lambda_k|q_k(x_k)\le q_k(x_k)$ i.e., $q_k(\lambda_k x_k)\le q_k(x_k)$, therefore

$$\left[M\left(\frac{q_k(\lambda_k x_k)}{\rho}\right)\right]^{p_k}\le\left[M\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}\quad\text{for each}\quad k\in\mathbb{N}.$$

It follows that $\left(\left[M\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}\right)\in F$, as F is a normal sequence space, i.e., $\lambda x\in F(E,M,p,q)$. Hence $F(E,\Delta_\nu^m,M,p,q)$ is normal space. □

Theorem 4.7 *For different Orlicz functions M,M_1 and M_2 the following inclusion holds:*

(i) $F(E_k,\Delta_\nu^m,M_1,p,q)\cap F(E_k,\Delta_\nu^m,M_2,p,q)\subseteq F(E_k,\Delta_\nu^m,M_1+M_2,p,q)$,

(ii) $F(E_k,\Delta_\nu^m,M_2,p,q)\subset F(E_k,\Delta_\nu^m,M_1,p,q)$, *if* $\sup_t\left[\frac{M_1(t)}{M_2(t)}\right]<\infty$ *and*

(iii) $F_1(E_k,\Delta_\nu^m,M,p,q)\subseteq F_2(E_k,\Delta_\nu^m,M,p,q)$, *if* $F_1\subseteq F_2$.

Proof: (i) Let $x\in F(E,\Delta_\nu^m,M_1,p,q)\cap F(E_k,\Delta_\nu^m,M_2,p,q)$. Then there exists $\rho_1>0$ and $\rho_2>0$ such that

$$\left(\left[M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_1}\right)\right]^{p_k}\right)\in F\quad\text{and}\quad\left(\left[M_2\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_2}\right)\right]^{p_k}\right)\in F.$$

Let $\rho=\max(\rho_1,\rho_2)$. Then for each $k\in\mathbb{N}$ we have

$$\begin{aligned}\left[(M_1+M_2)\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} &\le \left[M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)+M_2\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\\ &\le D\left\{\left[M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_1}\right)\right]^{p_k}+\left[M_2\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_2}\right)\right]^{p_k}\right\}\end{aligned}$$

by using Lemma 3.2, where $D=\max\left(1,2^{G-1}\right), G=\sup p_k$. Since F is normal AK-sequence space, it implies that $\left[(M_1+M_2)\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\in F$ and hence $x\in F(E_k,\Delta_\nu^m,M_1+M_2,p,q)$.

(ii)Let $x\in F(E_k,\Delta_\nu^m,M_2,p,q)$. Then there exists $\rho>0$ such that

$$\left(\left[M_2\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right)\in F.$$

Since, $\sup_t \left[\frac{M_1(t)}{M_2(t)}\right] < \infty$, therefore there exists $\eta > 0$ such that $\left(\frac{M_1(t)}{M_2(t)}\right) \leq \eta$, for all $t \geq 0$. Replacing t by $\frac{q(\Delta_\nu^m x_k)}{\rho}$ in the above,

$$M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_1}\right) \leq \eta M_2\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho_1}\right).$$

Thus for each $k \in \mathbb{N}$, we have

$$\left[M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k} \leq \max\left(1, \eta^G\right)\left[M_2\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}, \text{ where } G = \sup_k p_k.$$

Since F is normal AK-sequence space, it implies that

$$\left(\left[M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right) \in F \text{ and hence } x \in F(E, \Delta_\nu^m, M_1, p, q).$$

(iii) Let $x \in F_1(E, \Delta_\nu^m, M, p, q)$. Then there exists $\rho > 0$ such that

$$\left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right) \in F_1.$$

But $F_1 \subseteq F_2$ and consequenly,

$$\left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right) \in F_2.$$

Hence $x \in F_1(E, \Delta_\nu^m, M, p, q)$. □

Theorem 4.8 *Let M and M_1 be two Orlicz functions. If M satisfies the Δ_2-condition, then $F(E_k, \Delta_\nu^m, M_1, p, q) \subseteq F(E_k, \Delta_\nu^m, M \circ M_1, p, q)$.*

Proof: Let $x \in F(E_k, \Delta_\nu^m, M_1, p, q)$. Then there exists $\rho > 0$ such that

$$\left(\left[M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right) \in F.$$

Case-(i): If $M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right) \leq 1$, then using convexity of Orlicz function M,

$$\begin{aligned}\left[M\left(M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right)\right]^{p_k} &\leq \left[M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)M(1)\right]^{p_k} \\ &\leq \max\left(1, [M(1)]^G\right)\left[M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}.\end{aligned}$$

Case-(ii): If $M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right) \leq 1$, then using Δ_2-condition of Orlicz function M, we get

$$\begin{aligned}\left[M\left(M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right)\right]^{p_k} &\leq \left[KM_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)M(1)\right]^{p_k} \\ &\leq \max\left(1, [KM(1)]^G\right)\left[M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\end{aligned}$$

where $G = \sup_k p_k$. But F is a normal sequence space, it implies that $\left[M\left(M_1\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right)\right]^{p_k} \in F$, i.e., $x \in F(E_k, \Delta_\nu^m, M \circ M_1, p, q)$ in both cases.
Hence $F(E_k, \Delta_\nu^m, M_1, p, q) \subseteq F(E_k, \Delta_\nu^m, M \circ M_1, p, q)$. □

5. Sequence space $[F(E_k, \Delta_\nu^m, M, p, q)]$

In this section, we introduce the class $[F(E_k, \Delta_\nu^m, M, p, q)]$ of vector valued sequences as

$$[F(E_k, \Delta_\nu^m, M, p, q)] = \left\{ x = (x_k) \in S(E_k) : x_k \in E_k \text{ for each } k, \left(\left[M \left(\frac{q_k(\Delta_\nu^m x_k)}{\rho} \right) \right]^{p_k} \right) \in F \right.$$
$$\left. \text{for every } \quad \rho > 0 \right\}.$$

Clearly, set $[F(E_k, \Delta_\nu^m, M, p, q)]$ is a subspace of $F(E_k, \Delta_\nu^m, M, p, q)$. The topology on sequence space $[F(E_k, \Delta_\nu^m, M, p, q)]$ is introduced under the paranorm g, which is given by (4.1).

Theorem 5.1 *Sequence space $[F(E_k, \Delta_\nu^m, M, p, q)]$ is a complete paranormed space under paranorm g if (E_k, q_k) is a complete seminormed space.*

Proof: Since $[F(E_k, \Delta_\nu^m, M, p, q)]$ is a subspace of $F(E_k, \Delta_\nu^m, M, p, q)$, so in order to prove $[F(E_k, \Delta_\nu^m, M, p, q)]$ is complete paranormed space, it is sufficient to show that $[F(E_k, \Delta_\nu^m, M, p, q)]$ is closed in $F(E_k, \Delta_\nu^m, M, p, q)$. For this, consider $x \in \overline{F(E_k, \Delta_\nu^m, M, p, q)}$. Then there exists a sequence $(x^l) = ((x_k^l)_k)$ in $F(E_k, \Delta_\nu^m, M, p, q)$ such that $x^l \to x$ under the paranorm g i.e., $g(x^l - x) \to 0$ as $l \to \infty$, where $x = (x_k) \in F(E_k, \Delta_\nu^m, M, p, q)$. Thus for a given ε, there exists a positive integer l_0 such that

$$g(x^l - x) < \frac{\varepsilon}{2} \quad \text{for all } l \geq l_0. \tag{5.1}$$

We have,

$$\begin{aligned}
\left[M \left(\frac{q_k(\Delta_\nu^m x_k)}{\varepsilon} \right) \right]^{p_k} &= \left[M \left(\frac{q_k(\Delta_\nu^m ((x_k^l - x_k) - x_k^l))}{\varepsilon} \right) \right]^{p_k} \\
&\leq \left[\frac{1}{2} M \left(\frac{q_k(\Delta_\nu^m (x_k^l - x_k))}{\varepsilon/2} \right) + \frac{1}{2} M \left(\frac{q_k(\Delta_\nu^m x_k^l)}{\varepsilon/2} \right) \right]^{p_k} \\
&\leq \left[M \left(\frac{q_k(\Delta_\nu^m (x_k^l - x_k))}{\varepsilon/2} \right) + M \left(\frac{q_k(\Delta_\nu^m x_k^l)}{\varepsilon/2} \right) \right]^{p_k} \\
&\leq D \left\{ \left[M \left(\frac{q_k(\Delta_\nu^m (x_k^l - x_k))}{\varepsilon/2} \right) \right]^{p_k} + \left[M \left(\frac{q_k(\Delta_\nu^m x_k^l)}{\varepsilon/2} \right) \right]^{p_k} \right\} \\
&\leq D \left\{ \left[M \left(\frac{q_k(\Delta_\nu^m (x_k^l - x_k))}{g(x^l - x)} \right) \right]^{p_k} + \left[M \left(\frac{q_k(\Delta_\nu^m x_k^l)}{\varepsilon/2} \right) \right]^{p_k} \right\}
\end{aligned}$$

by using Lemma 3.2 and (5.1), where $D = \max\left(1, 2^{G-1}\right)$ and $G = \sup_k p_k$.
Since

$$\left(\left[M \left(\frac{q_k(\Delta_\nu^m (x_k^l - x_k))}{g(x^l - x)} \right) \right]^{p_k} \right) \in F \quad \text{and} \quad \left(\left[M \left(\frac{q_k(\Delta_\nu^m x_k^l)}{\varepsilon/2} \right) \right]^{p_k} \right) \in F.$$

It implies that

$$\left(\left[M \left(\frac{q_k(\Delta_\nu^m x_k)}{\varepsilon} \right) \right]^{p_k} \right) \in F$$

because F is a normal sequence space. Since ε is arbitrary, so $x \in [F(E_k, \Delta_\nu^m, M, p, q)]$. Thus,

$$\overline{F(E_k, \Delta_\nu^m, M, p, q)} \subseteq [F(E_k, \Delta_\nu^m, M, p, q)]$$

□

Theorem 5.2 *$[F(E_k, \Delta_\nu^m, M, p, q)]$ is an AK-space.*

Proof: Let $x \in [F(E_k, \Delta_\nu^m, M, p, q)]$. Then $\left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right) \in F$ for every $\rho > 0$. Since (e_k) is a Schauder basis of F, therefore for a given $\varepsilon > 0$

$$g_F\left[\left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\varepsilon}\right)\right]^{p_k}\right) - \sum_{k=1}^{t}\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\varepsilon}\right)\right]^{p_k} e_k\right] \to 0 \text{ as } t \to \infty$$

i.e., $g_F\left(\sum_{k=t+1}^{\infty}\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\varepsilon}\right)\right]^{p_k} e_k\right) \to 0$ as $t \to \infty$.
So, there exists a positive integer $t_0(\varepsilon)$ such that

$$g_F\left(\sum_{k=t+1}^{\infty}\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\varepsilon}\right)\right]^{p_k} e_k\right) < 1, \text{ for all } t \geq t_0(\varepsilon). \tag{5.2}$$

Using the definition of paranorm g, we get

$$g\left(x - x^{[t]}\right) = \inf\left\{\sigma^{\frac{p_n}{H}} : \left[g_F\left(\sum_{k=t+1}^{\infty}\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\sigma}\right)\right]^{p_k} e_k\right)\right]^{\frac{1}{H}} \leq 1, n \in \mathbb{N}\right\} \tag{5.3}$$

for all $t > m$ and any $\sigma > 0$. From (5.2) and (5.3), we have

$$g\left(x - x^{[t]}\right) < \varepsilon^{\frac{p_n}{H}} < \varepsilon \text{ for all } t \geq t_0(\varepsilon),$$

where $x^{[t]}$ is t^{th} section of x.
Hence $[F(E_k, \Delta_\nu^m, M, p, q)]$ is an AK-space. □

Theorem 5.3 *Let* $x^l = \left((x_k^l)\right)$ *be a sequence of element of* $[F(E_k, \Delta_\nu^m, M, p, q)]$ *and* $x = (x_k) \in [F(E_k, \Delta_\nu^m, M, p, q)]$. *Then* $x^l \to x$ *in* $[F(E_k, \Delta_\nu^m, M, p, q)]$ *if and only if*

(i) $x_k^l \to x_k$ *in* E_k *for each* $k \geq 1$ *and*

(ii) $g(x^l) \to g(x)$ *as* $i \to \infty$.

Proof: Necessity is obvious.
Sufficient part: Suppose (i) and (ii) hold and let t be any arbitrary positive integer, then

$$\begin{aligned} g\left(x^l - x\right) &= g\left(x^l - x^{l[t]} + x^{l[t]} - x^{[t]} + x^{[t]} - x\right) \\ &\leq g\left(x^l - x^{l[t]}\right) + g\left(x^{l[t]} - x^{[t]}\right) + g\left(x^{[t]} - x\right) \end{aligned}$$

where $x^{l[t]}, x^{[t]}$ denotes t^{th} section of x^l and x, respectively. Taking $l \to \infty$ we get,

$$\begin{aligned} \lim_{l\to\infty} \sup g(x^l - x) &\leq \lim_{l\to\infty} \sup g\left(x^l - x^{l[t]}\right) + \lim_{l\to\infty} \sup g\left(x^{l[t]} - x^{[t]}\right) + g\left(x^{[t]} - x\right) \\ &\leq 2g\left(x^{l[t]} - x^{[t]}\right). \end{aligned}$$

For t is arbitrary, letting $t \to \infty$, we get

$$\lim_{l\to\infty} \sup g(x^l - x) = 0 \text{ i.e., } g(x^l - x) \to 0 \text{ as } l \to \infty.$$

□

Theorem 5.4 *If* E_k *is separable for each* $k \in \mathbb{N}$, *then* $[F(E_k, \Delta_\nu^m, M, p, q)]$ *is separable.*

Proof: Let us assume that E_k is separable for each $k \in \mathbb{N}$. In this case, there exists a countable dense subset U_k of E_k. Let Z denotes the set of finite sequences $z = (z_k)$, where $z_k \in U_k$ for each $k \in \mathbb{N}$ and $(z_k) = (z_1, z_2, z_3, \dots,_t, \theta_{t+1}, \theta_{t+2}, \dots)$ for arbitrary $t \in \mathbb{N}$. Clearly, Z is countable subset of $[F(E_k, \Delta_\nu^m, M, p, q)]$. To show that Z is dense in $[F(E_k, \Delta_\nu^m, M, p, q)]$. Let $x = (x_k) \in [F(E_k, \Delta_\nu^m, M, p, q)]$. Since $[F(E_k, \Delta_\nu^m, M, p, q)]$ is an AK-space, so $g\left(x - x^{[t]}\right) \to 0$ as $t \to \infty$, where $x^{[t]} = t^{th}$section of x. So for a given $\varepsilon > 0$, there exists an integer $t_1 > 1$ such that

$$g\left(x - x^{[t]}\right) < \frac{\varepsilon}{2} \text{ for all } \ t \geq t_1.$$

If $t = t_1$ is taken,

$$g\left(x - x^{[t]}\right) < \frac{\varepsilon}{2}.$$

Let us choose

$$y = (y_k) = (y_1, y_2, y_3, \dots, y_n, \theta_{t_1+1}, \theta_{t_2+2}, \dots) \in Z$$

such that

$$g\left(x_k^{t_1} - y_k\right) < \frac{\varepsilon}{2M(1)t_1} \text{ for each } k.$$

Now,

$$g(x-y) \leq g\left(x - x^{[t_1]}\right) + g\left(x^{[t_1]} - y)\right) < \varepsilon.$$

This implies Z is dense in $[F(E_k, \Delta_\nu^m, M, p, q)]$. Hence $[F(E_k, \Delta_\nu^m, M, p, q)]$ is separable. □

Theorem 5.5 *If Orlicz function M satisfies Orlicz M satisfies Δ_2-condition, then* $F(E_k, \Delta_\nu^m, M, p, q) = [F(E_k, \Delta_\nu^m, M, p, q)]$.

Proof: From the definition of introduced space, we can say easily that

$$[F(E_k, \Delta_\nu^m, M, p, q)] \subseteq F(E_k, \Delta_\nu^m, M, p, q). \tag{5.4}$$

For inverse inclusion, let $x \in F(E_k, \Delta_\nu^m, M, p, q)$. Then there exists some $\rho > 0$ such that

$$\left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right) \in F.$$

Again, let $\sigma > 0$ be any arbitrary number. Then two cases arise.
Case-(i):If $\rho \leq \sigma$, then for each $k \in \mathbb{N}$,

$$\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\sigma}\right)\right]^{p_k} \leq \left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}$$

i.e.,$x \in [F(E_k, \Delta_\nu^m, M, p, q)]$.
Case-(ii): If $\rho > \sigma$, then $\frac{\rho}{\sigma} > 1$. From Δ_2-condition of Orlicz function, there exists a constant $k > 0$ such that

$$M\left(\frac{q_k(\Delta_\nu^m x_k)}{\sigma}\right) \leq \left(\frac{k\rho}{\sigma}\right) M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right).$$

Consequently, for each $k \in \mathbb{N}$

$$\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\sigma}\right)\right]^{p_k} \leq \sup_k\left\{\left(\frac{k\rho}{\sigma}\right)^{p_k}\right\}\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}.$$

Since F is normal space, it implies that

$$\left(\left[M\left(\frac{q_k(\Delta_\nu^m x_k)}{\rho}\right)\right]^{p_k}\right) \in F.$$

i.e., $x \in F(E_k, \Delta_\nu^m, M, p, q)$ in both cases. Hence

$$F(E_k, \Delta_\nu^m, M, p, q) \subseteq [F(E_k, \Delta_\nu^m, M, p, q)] \tag{5.5}$$

From (5.4) and (5.5), we get $F(E_k, \Delta_\nu^m, M, p, q) = [F(E_k, \Delta_\nu^m, M, p, q)]$. □

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