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Existence results for a ψ-Hilfer-type fractional Langevin differential inclusion

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ABSTRACT: In this paper we deals with the existence of solution for a new kind of Langevin inclusion involving ψ-Hilfer fractional derivative. The suggested study is based on some basic definitions of fractional calculus and multivalued analysis. The existence result is obtained by making use of the nonlinear alternative of Leray-Schauder type. In the end, we are giving an example to illustrate our results.

Key Words: ψ-Hilfer fractional derivative, Langevin fractional differential inclusions, Langevin equations, ψ-Caputo fractional derivative, Fixed point theorems.

Contents

1. Introduction

Over the past few years, fractional differential equations have attracted the interest of many mathematicians due to their ability to describe several complex problems in different scientific and engineering fields such as physics, biology, chemistry, and control theory, for more details, see [13,12,3,19,20]. Properties of the fractional derivatives make the fractional order models more useful and practical than the classical integral-order models [1,15].

The Langevin equation, introduced by Paul Langevin in 1908, of the form $m\frac{d^2x}{dt^2} = -\lambda\frac{dx}{dt} + \eta(t)$, where, $\frac{dx}{dt}$ is the velocity of the particle, and m is its mass and a noise term $\eta(t)$ representing the effect of the collisions with the molecules of the fluid, for more details see [4,5,7,21,22,23]. For the removal of noise, mathematicians used fractional order differential equations. Thus it is very important to study Langevin inclusion with ψ-Hilfer fractional derivatives.

There are a diverse definitions of fractional integrals and derivatives, the famous definitions are the Riemann-Liouville and the Caputo fractional derivatives. Hilfer [8] introduce the generalization of these derivatives under the name of Hilfer fractional derivative of order α and parameter $\beta \in [0,1]$.

In [14], Rizwan discussed the existence and uniqueness of solutions for a non-local boundary value problem of nonlinear fractional Langevin equation with non-instantaneous impulses by using the Generalized Diaz-Margolis's fixed point theorem. In [2], Alsaedi et al investigated the existence of solutions for Langevin fractional differential inclusions involving two fractional orders with four-point multiterm fractional integral boundary conditions by making use of the nonlinear alternative of Leray-Schauder type.

Motivated by the mentioned works, in this paper, we combine their ideas investigate the existence results of sequential ψ-Hilfer fractional Langevin inclusion with nonlocal boundary conditions :

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$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1;\psi}({}^{H}D^{\alpha_2,\beta_2;\psi}+\lambda)x(t)\in F(t,x(t)), \quad a\leq t\leq b \\ x(a)=0 \quad , \qquad x(b)=\sum_{i=1}^{n}\mu_i(I^{\nu_i;\psi}(x))(\eta_i). \end{cases} \tag{1.1}$$

Where ${}^{H}D^{\alpha_j,\beta_j;\psi}$, $j=1,2$ is the ψ-Hilfer fractional derivative of order α_j, $0<\alpha_j<1$ and parameter β_j, $0\leq\beta_j\leq 1$, $j=1,2$, $\lambda\in\mathbb{R}$, $a\geq 0$, $I^{\nu_i;\psi}$ is the ψ-Riemann-Liouville fractional integral of order $\nu_i>0$, $\mu_i\in\mathbb{R}^*$, $a<\eta_1<...<\eta_n<b$, $i=1,....,n$ and $F:[a,b]\times\mathbb{R}\longrightarrow\mathcal{P}(\mathbb{R})$ is a multivalued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$).

The novelty of this work is to study a new and a challenging case of fractional derivative that called the ψ-Hilfer derivative [18], this brand of fractional derivative generalize the well-known fractional derivatives (Riemman-Liouville, Caputo, ψ-Riemman-Liouville, Hilfer-Hadamard, Katugampola derivetive), for different values of function ψ and parameter β such as

- ⋆ If $\psi(x)=x$ and $\beta=1$, then the problem (1.1) reduces to Caputo-type.
- ⋆ If $\psi(x)=x$ and $\beta=0$, then the problem (1.1) reduces to Riemman-Liouville-type.
- ⋆ If $\beta=0$, then the problem (1.1) reduces to the (1.1)-Riemman-Liouville-type.
- ⋆ If $\psi(x)=x$, then the problem (1.1) reduces to Hilfer-type.
- ⋆ If $\psi(x)=log(x)$, then the problem (1.1) reduces to Hilfer-Hadamard-type.
- ⋆ If $\psi(x)=x^\rho$, then the problem (1.1) reduces to Katugampola-type.

This work is organized as follows: In section 2, we recall some basic concepts of fractional calculus, and multi-valued analysis. In section 3, we prove the first existence result by using Laray-Schauder non linear alternative for multi-valued map. In section 4, we give an example to support our study.

2. Preliminaries

2.1. Fractional Calculus

In this section, we introduce some definitions, lemmas and useful notations that will be used throughout the paper.

Let $\mathcal{C}=C([a,b],\mathbb{R})$ denote the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm defined by $\|f\|=sup_{t\in[a,b]}\{|f(t)|\}$. We denote by $AC^n([a,b],\mathbb{R})$ the n-times absolutely continuous functions given by

$$AC^n([a,b],\mathbb{R})=\left\{f:[a,b]\longrightarrow\mathbb{R};f^{(n-1)}\in AC([a,b],\mathbb{R})\right\}. \tag{2.1}$$

Definition 2.1 *[10]*

Let (a,b), $-\infty\leq a<b\leq+\infty$, be a finite or infinite interval of the half-axis $(0,+\infty)$ and $\alpha>0$. In addition, let $\psi(t)$ be a positive increasing function on $(a,b]$, which has a continuous derivative $\psi^{'}(t)$ on (a,b). The ψ-Riemann-Liouville fractional integral of a function f with respect to another function ψ on $[a,b]$ is defined by

$$I^{\alpha;\psi}_{a^+}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(t)(\psi(t)-\psi(s))^{\alpha-1}f(s)ds, \tag{2.2}$$

where $\Gamma(.)$ represents the Gamma function.

Definition 2.2 *[10]*

Let $\psi^{'}(t) \neq 0$ and $\alpha > 0$, $n \in \mathbb{N}$. The Riemann-Liouville derivative of a function f with respect to another function ψ of order α, correspondent to the Riemann–Liouville is defined by

$$D_{a^+}^{\alpha;\psi} f(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi} f(t), \tag{2.3}$$

$$= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi^{'}(t)(\psi(t)-\psi(s))^{n-\alpha-1} f(s)ds, \tag{2.4}$$

where $n-1 < \alpha < n, n = [\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3 *[10]*
Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, $[a,b]$ is the interval such that $-\infty \leq a < b \leq +\infty$ and $f, \psi \in C^n([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi^{'}(t) \neq 0$ for all $t \in [a,b]$. The ψ-Hilfer fractional derivative of a function f of order α and type $0 \leq \beta \leq 1$, is defined by

$$^{H}D_{a^+}^{\alpha;\psi} f(t) = I_{a^+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(t), \tag{2.5}$$
$$= I_{a^+}^{\gamma-\alpha;\psi} D_{a^+}^{\gamma;\psi} f(t),$$

where $n-1 < \alpha < n, n = [\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α, with $\gamma = \alpha + \beta(n-\alpha)$.

Lemma 2.1 *[10] Let $\alpha, \beta > 0$. Then we have the following semigroup property given by*

$$I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} f(t) = I_{a^+}^{\alpha+\beta;\psi} f(t), t > a. \tag{2.6}$$

Lemma 2.2 *[16,10] Let $a \geq 0$, $\upsilon > 0$ and $t > a$. Then, ψ-fractional integral and derivative of a power function are given by*

(i) $I_{a^+}^{\alpha;\psi}(\psi(s)-\psi(a))^{\upsilon-1}(t) = \dfrac{\Gamma(\upsilon)}{\Gamma(\upsilon+\alpha)}(\psi(s)-\psi(a))^{\upsilon+\alpha-1}(t).$

(ii) $^{H}D_{a^+}^{\alpha;\psi}(\psi(s)-\psi(a))^{\upsilon-1}(t) = \dfrac{\Gamma(\upsilon)}{\Gamma(\upsilon+\alpha)}(\psi(s)-\psi(a))^{\upsilon-\alpha-1}(t)$, $n-1 < \alpha < n, \upsilon > n.$

Lemma 2.3 *[10] If $f \in C^n([a,b],\mathbb{R})$, $n-1 < \alpha < n$, $\quad 0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(n-\alpha)$, then*

$$I_{a^+}^{\alpha;\psi}(^{H}D_{a^+}^{\alpha,\beta;\psi} f)(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(a), \tag{2.7}$$

for all $t \in [a,b]$, where $f_{\psi}^{[n-k]} f(t) := \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t)$.

Lemma 2.4 *Let $a \geq 0$, $0 < \alpha_j < 1$, $\gamma_j = \alpha_j + \beta_j - \alpha_j\beta_j$, $j = 1,2$ and $f \in C([a,b],\mathbb{R})$. Then the function x is a solution of the boundary value problem:*

$$\begin{cases} ^{H}D^{\alpha_1,\beta_1;\psi}(^{H}D^{\alpha_2,\beta_2;\psi} + \lambda)x(t) = f(t), \quad a \leq t \leq b, \\ x(a) = 0 \quad , \qquad x(b) = \displaystyle\sum_{i=1}^{n} \mu_i (I^{\nu_i;\psi}(x))(\eta)ds, \quad a < \eta < b, \end{cases} \tag{2.8}$$

if and only if

$$x(t) = I^{\alpha_1+\alpha_2;\psi}h(t) - \lambda I^{\alpha_2;\psi}x(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[I^{\alpha_1+\alpha_2;\psi}h(b) - \lambda I^{\alpha_2;\psi}x(b) - \sum_{i=1}^{n}\mu_i I^{\alpha_1+\alpha_2+\nu_i;\psi}h(\eta) + \lambda\sum_{i=1}^{n}\mu_i I^{\alpha_2+\nu_i;\psi}x(\eta)\Bigg], \tag{2.9}$$

where

$$\Lambda = \sum_{i=1}^{n}\mu_i\frac{(\psi(\eta_i)-\psi(a))^{\gamma_1+\alpha_2+\nu_i-1}}{\Gamma(\gamma_1+\alpha_2+\nu_i)} - \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Gamma(\gamma_1+\alpha_2)} \neq 0. \tag{2.10}$$

Proof: The problem (2.8) can be written as

$$I^{\gamma_1-\alpha_1;\psi}D^{\gamma_1,\beta_1;\psi}\Big(D^{\alpha_2,\beta_2;\psi}+\lambda\Big)x(t) = f(t).$$

Applying the ψ-Riemann-Liouville fractional integral of order α_1 to both sides we obtain by using Lemma 2.3

$${}^{H}D^{\alpha_2,\beta_2;\psi}x(t) + \lambda x(t) = I^{\alpha_1;\psi}f(t) + \frac{c_0}{\Gamma(\gamma_1)}((\psi(t)-\psi(a))^{\gamma_1-1}, \tag{2.11}$$

where c_0 constant and $\gamma_1 = \alpha_1+\beta_1-\alpha_1\beta_1$. Applying the ψ-Riemann-Liouville fractional integral of order α_2 to both sides of (2.11) we obtain by using Lemma 2.3

$$x(t) = I^{\alpha_1+\alpha_2;\psi}f(t) - \lambda I^{\alpha_2;\psi}x(t) + \frac{c_0}{\Gamma(\gamma_1+\alpha_2)}((\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1} + \frac{c_1}{\Gamma(\gamma_2)}((\psi(t)-\psi(a))^{\gamma_2-1}. \tag{2.12}$$

From using the boundary condition $x(a) = 0$ in (2.12) we obtain that $c_1 = 0$. Then, we get

$$x(t) = I^{\alpha_1+\alpha_2;\psi}f(t) - \lambda I^{\alpha_2;\psi}x(t) + \frac{c_0}{\Gamma(\gamma_1+\alpha_2)}((\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}. \tag{2.13}$$

From using the boundary condition $x(b) = \sum_{i=1}^{n}\mu_i(I^{\nu_i;\psi}(x))(\eta_i)$, in (2.13) we find

$$c_0 = \frac{1}{\Lambda}\Bigg[I^{\alpha_1+\alpha_2;\psi}f(b) - \lambda I^{\alpha_2;\psi}x(b) - \sum_{i=1}^{n}\mu_i I^{\alpha_1+\alpha_2+\nu_i;\psi}f(\eta_i) + \lambda\sum_{i=1}^{n}\mu_i I^{\alpha_2+\nu_i;\psi}x(\eta_i)\Bigg]. \tag{2.14}$$

Substituting the value of c_0 in (2.13) we obtain the solution. The converse follows by direct computation. This completes the proof. □

2.2. Multivalued Analysis

For a normed space $(X, \|.\|)$, we define :
$\mathcal{P}(X) = \{Y \subset X : Y \neq \varnothing\}$, $\mathcal{P}_{c,cp}(X) = \{Y \subset X : Y \quad \textit{is} \quad \textit{convex} \quad \textit{and} \quad \textit{compact}\}$
For the basic concepts of multivalued analysis, we refer to [9]

Definition 2.4 *A multivalued map $F : [a,b]\times\mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if :*

(i) $t \longrightarrow F(t,x)$ is measurable for each $x \in \mathbb{R}$.

(ii) $x \longrightarrow F(t,x)$ is upper semicontinuous for almost all $t \in [a,b]$.
Furthermore, a Carathéodory function F is called $\mathbb{L}^1$-Carathéodory if :

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in \mathbb{L}^1([a,b];\mathbb{R})$ such that

$$\|F(t,x)\| = sup\{|v| : v \in F(t,x)\} \leq \varphi_\rho(t)$$

for all $x \in \mathbb{R}$ with $\|x\| \leq \rho$ and for a.e. $t \in [a,b]$.

Theorem 2.1 *(Leray–Schauder nonlinear alternative [6])*
Let X be a Banach space, C a closed, convex subset of X, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \longrightarrow C$ is a continuous, compact ($F(\overline{U})$ s a relatively compact subset of C) map. Then either

(i) F has a fixed point in $\overline{U}$, or

(ii) there exists a $x \in \partial U$ (the boundary of U in C) and $\theta \in (0,1)$ with $x = \theta F(x)$.

3. Main Results

In this section, we deals with the existence of solution for the boundary value problem (1.1). By Lemma 2.4 we define an operator $\mathcal{A} : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$\begin{aligned}(\mathcal{A}x)(t) &= I^{\alpha_1+\alpha_2;\psi} f(t,x(t)) - \lambda I^{\alpha_2;\psi} x(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[I^{\alpha_1+\alpha_2;\psi} f(b,x(b)) \\ &\quad - \lambda I^{\alpha_2;\psi} x(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i;\psi} f(\eta_i, x(\eta_i)) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i;\psi} x(\eta_i)\Bigg], \qquad (3.1)\end{aligned}$$

where $\mathcal{C} = C([a,b],\mathbb{R})$ denotes the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm $\|x\| := sup\{|x(t)|; t \in [a,b]\}$. the boundary value problem (1.1) has a solution if and only if the operator $\mathcal{A}$ has fixed point.
To simplify the computations, we use the following notations:

$$\Omega_1 = \frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)}\left[\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n} |\mu_i| \frac{(\psi(\eta_i)-\psi(a))^{\alpha_1+\alpha_2+\nu_i}}{\Gamma(\alpha_1+\alpha_2+\nu_i+1)}\right], \qquad (3.2)$$

and

$$\Omega_2 = |\lambda|\left\{\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)}\left[\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n} |\mu_i| \frac{(\psi(\eta_i)-\psi(a))^{\alpha_2+\nu_i}}{\Gamma(\alpha_2+\nu_i+1)}\right]\right\}. \qquad (3.3)$$

3.1. Existence results

Definition 3.1 *A continuous function x is said to be a solution of problem (1) if $x(a) = 0; x(b) = \sum_{i=1}^{n} \mu_i (I^{\nu_i;\psi}(x))(\eta_i)$ and there exists a function $v \in \mathbb{L}^1([a,b],\mathbb{R})$ with $v \in F(t,x(t))$ a.e, on $[a,b]$ such that*

$$\begin{aligned}x(t) &= I^{\alpha_1+\alpha_2;\psi} v(t) - \lambda I^{\alpha_2;\psi} x(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[I^{\alpha_1+\alpha_2;\psi} v(b) \\ &\quad - \lambda I^{\alpha_2;\psi} x(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i;\psi} v(\eta) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i;\psi} x(\eta)\Bigg], \qquad (3.4)\end{aligned}$$

for each $x \in \mathcal{C}([a,b],\mathbb{R}$, define the set of selections of F by

$$\mathcal{S}_{F,x} := \big\{v \in \mathbb{L}^1([a,b],\mathbb{R}) : v \in F(t,x(t)) \quad on \quad [a,b]\big\}. \qquad (3.5)$$

Lemma 3.1 *([11]) Let X be a Banach space, and $F : [a,b] \times \mathbb{R} \longrightarrow \mathcal{P}_{c,cp}$ be a $\mathbb{L}^1$-Carathéodory multivalued map. And let Ξ be a linear continuous mapping from $\mathbb{L}^1([a,b],X)$ to $\mathcal{C}([a,b],X)$. Then the operator :*

$$\Xi \circ \mathcal{S}_{F} : \mathcal{C}([a,b],X) \longrightarrow \mathcal{P}_{c,cp}\big(\mathcal{C}([a,b],X)\big); \quad x \longrightarrow \big(\Xi \circ \mathcal{S}_{F}\big)(x) = \Xi\big(\mathcal{S}_{F,x}\big),$$

is a closed graph operator in $\mathcal{C}([a,b],X) \times \mathcal{C}([a,b],X)$.

Assume that

(H1). $F : [a,b] \times \mathbb{R} \longrightarrow \mathcal{P}_{c,cp}(\mathbb{R})$ is $\mathbb{L}^1$-Carathéodory and has nonempty compact and convex values, and for each fixed $x \in \mathcal{C}([a,b],\mathbb{R})$ the set :

$$\mathcal{S}_{F,x} = \{v \in \mathbb{L}^1([a,b],X) : v(t) \in F(t,x(t)); \quad t \in [a,b]\},$$

is nonempty.

(H2). $\|F(t,x)\| := sup\{|v| : v \in F(t,x)\} \leq p(t)\Phi(\|x\|)\}$ for all $t \in [a,b]$ and all $x \in \mathcal{C}([a,b],X)$, where $p \in \mathbb{L}^1([a,b],\mathbb{R}^+)$ and $\Phi : \mathbb{R}^+ \longrightarrow [0,+\infty)$ is continuous and nondecreasing function.

(H3). there exists a constant $M > 0$ such that :

$$\frac{(1-\Omega_2)M}{\|p\|\Phi(M)\Omega_1} > 1,$$

where Ω_1, Ω_2 are respectively given by (17), (18) and $\Omega_2 < 1$.

The existence result, is based on the nonlinear alternative of Leray-Schauder for multivalued maps [6]

Theorem 3.1 *Assume that $(H1)-(H3)$ holds, then there exists at least one solution for problem (1) on $[a,b]$.*

Proof: Let us introduce the multivalued map $\mathcal{A} : \mathcal{C}([a,b],\mathbb{R}) \longrightarrow \mathcal{P}_{c,cp}([a,b],\mathbb{R})$, in order to transform Problem (1) into a fixed point problem, we define $\mathcal{A}$ by:

$$\mathcal{A}(x) := \left\{ h \in \mathcal{C}([a,b],\mathbb{R}) : h(t) = \left\{ \begin{array}{l} I^{\alpha_1+\alpha_2;\psi}v(t) - \lambda I^{\alpha_2;\psi}x(t) + \dfrac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \\ \times \Bigg[I^{\alpha_1+\alpha_2;\psi}v(b) - \lambda I^{\alpha_2;\psi}x(b) - \displaystyle\sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i;\psi}v(\eta_i) \\ +\lambda \displaystyle\sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i;\psi}x(\eta_i) \Bigg]; \quad t \in [a,b], \quad v \in \mathcal{S}_{F,x} \end{array} \right. \right\}$$

We will prove that the multivalued map $\mathcal{A}$ satisfies all conditions of the nonlinear alternative of Leray-Schauder type [6], we give the poof in several steps:

Step 1 : $\mathcal{A}(x)$ is convex for each $x \in \mathcal{C}([a,b],\mathbb{R})$.

Indeed, if h_1, h_2 belong to $\mathcal{A}(x)$, then there exist $v_1, v_2 \in \mathcal{S}_{F,x}$ such that for each $t \in [a,b]$ we have for $j = 1,2.$:

$$\begin{aligned} h_j(t) = I^{\alpha_1+\alpha_2;\psi}v_j(t) - \lambda I^{\alpha_2;\psi}x(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \Bigg[I^{\alpha_1+\alpha_2;\psi}v_j(b) \\ - \lambda I^{\alpha_2;\psi}x(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i;\psi}v_j(\eta_i) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i;\psi}x(\eta_i) \Bigg]. \end{aligned} \tag{3.6}$$

Let $0 \leq k \leq 1$ then, for each $t \in [a,b]$, we have

$$\begin{aligned} kh_1(t) + (1-k)h_2(t) = {} & I^{\alpha_1+\alpha_2;\psi}\Big[kv_1(s) + (1-k)v_2(s)\Big] - \lambda I^{\alpha_2;\psi}x(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \\ & \times \Bigg[I^{\alpha_1+\alpha_2;\psi}\Big[kv_1(s) + (1-k)v_2(s)\Big] - \lambda I^{\alpha_2;\psi}x(b) \\ & - \sum_{i=1}^{n}\mu_i I^{\alpha_1+\alpha_2+\nu_i;\psi}\Big[kv_1(\eta) + (1-k)v_2(\eta_i)\Big] + \lambda\sum_{i=1}^{n}\mu_i I^{\alpha_2+\nu_i;\psi}x(\eta_i)\Bigg], \end{aligned}$$

thus $kv_1 + (1-k)v_2 \in \mathcal{A}(x)$ $\big($ because $\mathcal{S}_{F,x}$ is convex $\big)$, then $\mathcal{A}(x)$ is convex for each $x \in \mathcal{C}([a,b],\mathbb{R})$

Step 2 : $\mathcal{A}(x)$ maps bounded set into bounded set in $\mathcal{C}([a,b],\mathbb{R})$.

Indeed, it is enough to show that there exists a positive constant l such that for each $h \in \mathcal{A}(x)$; $x \in \mathcal{B}_\rho = \{x \in \mathcal{C}([a,b],\mathbb{R} : \|x\| \leq \rho\}$ we have $\|h\| \leq l$.
If $h \in \mathcal{A}(x)$ then there exist $v \in \mathcal{S}_{F,x}$, such that :

$$\begin{aligned} h(t) = {} & I^{\alpha_1+\alpha_2;\psi}v(t) - \lambda I^{\alpha_2;\psi}x(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[I^{\alpha_1+\alpha_2;\psi}v(b) \\ & - \lambda I^{\alpha_2;\psi}x(b) - \sum_{i=1}^{n}\mu_i I^{\alpha_1+\alpha_2+\nu_i;\psi}v(\eta_i) + \lambda\sum_{i=1}^{n}\mu_i I^{\alpha_2+\nu_i;\psi}x(\eta_i)\Bigg], \end{aligned} \tag{3.7}$$

then for every $t \in [a,b]$ we have

$$\begin{aligned} \big|h(x)(t)\big| \leq {} & \sup_{t\in[a,b]}\Bigg\{I^{\alpha_1+\alpha_2;\psi}|v(t)| + |\lambda|I^{\alpha_2;\psi}|x(t)| + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\ & \times \Bigg[I^{\alpha_1+\alpha_2;\psi}|v(b)| + \sum_{i=1}^{n}|\mu_i|I^{\alpha_1+\alpha_2+\nu_i;\psi}|v(\eta_i)| \\ & + |\lambda|I^{\alpha_2;\psi}|x(b)| + |\lambda|\sum_{i=1}^{n}|\mu_i|I^{\alpha_2+\nu_i;\psi}|x(\eta_i)|\Bigg]\Bigg\}, \\ \leq {} & \|p\|\Phi(\|x\|)\Bigg\{\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\ & \times \Bigg[\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_1+\alpha_2+\nu_i}}{\Gamma(\alpha_1+\alpha_2+\nu_i+1)}\Bigg]\Bigg\} \\ & + \|x\||\lambda|\Bigg\{\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\ & \times \Bigg[\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_2+\nu_i}}{\Gamma(\alpha_2+\nu_i+1)}\Bigg]\Bigg\} \\ \leq {} & \|p\|\Phi(\|x\|)\Omega_1 + \|x\|\Omega_2, \\ \leq {} & \|p\|\Phi(\rho)\Omega_1 + \rho\Omega_2, \end{aligned}$$

then

$$\|h\| \leq \|p\|\Omega_1\Phi(\rho) + \rho\Omega_2 := l,$$

where Ω_1, Ω_2 are respectively given by (17) and (18).

Step 3 : $\mathcal{A}$ maps bounded set into equicontinuous sets of $\mathcal{C}([a,b],\mathbb{R})$.

Let $t_1, t_2 \in [a,b]; t_1 < t_2$, and $x \in \mathcal{B}_\rho$ where $\mathcal{B}_\rho$, as above, is a bounded set of $\mathcal{C}([a,b],\mathbb{R})$, for each $x \in \mathcal{B}_\rho$ and $h \in \mathcal{A}(x)$, there exist $v \in \mathcal{S}_{F,x}$ then we obtain :

$$
\begin{aligned}
\Big|h(t_2) - h(t_1)\Big| \leq{}& \frac{1}{\Gamma(\alpha_1+\alpha_2)}\Big| \int_a^{t_1} \psi'(s)\Big((\psi(t_2)-\psi(s))^{\alpha_1+\alpha_2-1} - (\psi(t_1)-\psi(s))^{\alpha_1+\alpha_2-1}\Big)v(s)ds \\
&+ \int_{t_1}^{t_2} \psi'(s)\Big(\psi(t_2)-\psi(s)\Big)^{\alpha_1+\alpha_2-1} v(s)ds\Big| \\
&+ \frac{|\lambda|}{\Gamma(\alpha_2)}\Big| \int_a^{t_1} \psi'(t)\Big((\psi(t_2)-\psi(s))^{\alpha_2-1} - (\psi(t_1)-\psi(s))^{\alpha_2-1}\Big)x(s)ds \\
&+ \int_{t_1}^{t_2} \psi'(s)\Big(\psi(t_2)-\psi(s)\Big)^{\alpha_2-1} x(s)ds\Big| + \frac{(\psi(t_2)-\psi(a))^{\gamma_1+\alpha_2-1} - (\psi(t_1)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\
&\times \Bigg[\|v(s)\|\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \|v(s)\|\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_1+\alpha_2+\nu_i}}{\Gamma(\alpha_1+\alpha_2+\nu_i+1)} \\
&+ \|x(b)\||\lambda|\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \|x(\eta_i)\||\lambda|\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_2+\nu_i}}{\Gamma(\alpha_2+\nu_i+1)}\Bigg], \\
\leq{}& \frac{\|p\|\Phi(\rho)}{\Gamma(\alpha_1+\alpha_2)}\Big| \int_a^{t_1} \psi'(s)\Big((\psi(t_2)-\psi(s))^{\alpha_1+\alpha_2-1} - (\psi(t_1)-\psi(s))^{\alpha_1+\alpha_2-1}\Big)ds \\
&+ \int_{t_1}^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha_1+\alpha_2-1}ds\Big| \\
&+ \frac{\rho|\lambda|}{\Gamma(\alpha_2)}\Big| \int_a^{t_1} \psi'(s)\Big((\psi(t_2)-\psi(s))^{\alpha_2-1} - (\psi(t_1)-\psi(s))^{\alpha_2-1}\Big)ds \\
&+ \int_{t_1}^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha_2-1}ds\Big| + \frac{(\psi(t_2)-\psi(a))^{\gamma_1+\alpha_2-1} - (\psi(t_1)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\
&\times \Bigg[\|p\|\Phi(\rho)\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \|p\|\Phi(\rho)\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_1+\alpha_2+\nu_i}}{\Gamma(\alpha_1+\alpha_2+\nu_i+1)} \\
&+ \rho|\lambda|\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \rho|\lambda|\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_2+\nu_i}}{\Gamma(\alpha_2+\nu_i+1)}\Bigg],
\end{aligned}
$$

as $t_2 \longrightarrow t_1$ the right-hand side of the above inequality tends to zero, implies that $\mathcal{A}(x)$ is equicontinuous. Therefore it follows by Arzelà-Ascoli theorem that $\mathcal{A} : \mathcal{C}([a,b],\mathbb{R}) \longrightarrow \mathcal{P}_{c,cp}(\mathcal{C}([a,b],\mathbb{R}))$ is relatively compact then $\mathcal{A}$ is completely continuous.

Now, to show that the operator $\mathcal{A}$ is upper semicontinuous, is enough to show that $\mathcal{A}$ has a closed graph.

Step 4 : $\mathcal{A}$ has a closed graph.

Let $x_n \longrightarrow x_*$, $h_n \in \mathcal{A}(x_n)$ and $h_n \longrightarrow h_*$, we shall prove that $h_* \in \mathcal{A}(x_*)$.
$h_n \in \mathcal{A}(x_n)$ then there exists $v_n \in \mathcal{S}_{F,x_n}$ such that for each $t \in [a,b]$,

$$h_n(t) = I^{\alpha_1+\alpha_2;\psi} v_n(t) - \lambda I^{\alpha_2;\psi} x_n(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \Bigg[I^{\alpha_1+\alpha_2;\psi} v_n(b) - \lambda I^{\alpha_2;\psi} x_n(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i;\psi} v_n(\eta_i) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i;\psi} x_n(\eta_i) \Bigg], \tag{3.8}$$

we should prove that $v_* \in \mathcal{S}_{F,x_*}$ such that for each $t \in [a,b]$:

$$h_*(t) = I^{\alpha_1+\alpha_2;\psi} v_*(t) - \lambda I^{\alpha_2;\psi} x_*(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \Bigg[I^{\alpha_1+\alpha_2;\psi} v_*(b) - \lambda I^{\alpha_2;\psi} x_*(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i;\psi} v_*(\eta_i) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i;\psi} x_*(\eta_i) \Bigg], \tag{3.9}$$

we have that:

$$\Bigg\| \Bigg(h_n(t) + \lambda I^{\alpha_2;\psi} x_n(t) - \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \Bigg[-\lambda I^{\alpha_2;\psi} x_n(b) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i;\psi} x_n(\eta) \Bigg] \Bigg) - \Bigg(h_*(t) + \lambda I^{\alpha_2;\psi} x_*(t) - \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \Bigg[-\lambda I^{\alpha_2;\psi} x_*(b) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i;\psi} x_*(\eta) \Bigg] \Bigg) \Bigg\| \longrightarrow 0,$$

as $n \longrightarrow \infty$

Consider the linear operator :

$$\Xi : \mathbb{L}^1([a,b],\mathbb{R}) \longrightarrow \mathcal{C}([a,b],\mathbb{R})$$
$$v \longrightarrow \Xi(v)(t).$$

With

$$\Xi(v)(t) = I^{\alpha_1+\alpha_2;\psi} v(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \Bigg[I^{\alpha_1+\alpha_2;\psi} v(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i;\psi} v(\eta_i) \Bigg],$$

from Lemma 3.2, $\Xi \circ \mathcal{S}_F$ is a closed graph operator then we have that :

$$\Bigg(h_n(t) + \lambda I^{\alpha_2;\psi} x_n(t) - \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \Bigg[-\lambda I^{\alpha_2;\psi} x_n(b) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i;\psi} x_n(\eta_i) \Bigg] \Bigg) \in \Xi\Big(\mathcal{S}_{F,x_n}\Big). \tag{3.10}$$

Since $x_n \longrightarrow x_*$, and $h_n \longrightarrow h_*$ then :

$$\Bigg(h_*(t) + \lambda I^{\alpha_2;\psi} x_*(t) - \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \Bigg[-\lambda I^{\alpha_2} x_*(b) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i;\psi} x_*(\eta_i) \Bigg] \Bigg) = \Xi(v_*) \in \Xi\Big(\mathcal{S}_{F,x_*}\Big). \tag{3.11}$$

It follows that $v_* \in \mathcal{S}_{F,x_*}$ such that

$$h_*(t) = I^{\alpha_1+\alpha_2;\psi} v_*(t) - \lambda I^{\alpha_2;\psi} x_*(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \Bigg[I^{\alpha_1+\alpha_2;\psi} v_*(b) - \lambda I^{\alpha_2;\psi} x_*(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i;\psi} v_*(\eta_i) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i;\psi} x_*(\eta_i) \Bigg]. \tag{3.12}$$

Finally, $\mathcal{A}$ has a closed graph.

Step 5: $\mathcal{A}$ has a fixed point.

We show that (ii) from Theorem 2.9 is not possible. Then if $x \in \theta\mathcal{A}x$ for $\theta \in [0,1]$ there exist $v \in \mathcal{S}_{F,x}$ such that $x(t) = \theta h(t)$ implies $|x(t)| \leq |h(t)|$ then

$$\|x\| \leq \|p\|\Phi(\|x\|)\Omega_1 + \|x\|\Omega_2,$$

then

$$(1-\Omega_2)\|x\| \leq \|p\|\Phi(\|x\|), \tag{3.13}$$

if (ii) from Theorem 2.9 hold then there exist $\theta \in [0,1]$ and $x \in \partial\mathcal{B}_M$ with $x = \theta\mathcal{A}$, wich means that x is solution to the problem (1) with $\|x\| = M$ then we have from (28) that :

$$(1-\Omega_2)M \leq \|p\|\Phi(M), \tag{3.14}$$

then

$$\frac{(1-\Omega_2)M}{\|p\|\Phi(M)} \leq 1, \tag{3.15}$$

which contredicts $(H3)$. Consequently $\mathcal{A}$ has afixed point in $[a,b]$.

By the nonlinear alternative of Leray–Schauder we deduce that the problem (1) has at least one solution. □

4. Example

Consider the following problem

$$\begin{cases} {}^{H}D^{\frac{1}{5},\frac{3}{5};\frac{e^t}{6}}\Big({}^{H}D^{\frac{2}{5},\frac{4}{5};\frac{e^t}{6}} + \dfrac{1}{9}\Big)x(t) \in F(t,x), \quad 0 \leq t \leq 1 \\ x(0) = 0 \quad , \qquad x(1) = \dfrac{3}{8}I^{\frac{5}{2};\frac{e^t}{6}}x(\dfrac{1}{3}) + \dfrac{5}{8}I^{\frac{7}{2};\frac{e^t}{6}}x(\dfrac{1}{2}). \end{cases} \tag{4.1}$$

Where $\alpha_1 = \frac{1}{5}$, $\alpha_2 = \frac{2}{5}$, $\beta_1 = \frac{3}{5}$, $\beta_2 = \frac{4}{5}$, $\lambda = \frac{1}{3}$, $a = 0$, $b = 1$, $n = 2$, $\nu_1 = \frac{5}{2}$, $\nu_2 = \frac{7}{2}$, $\mu_1 = \frac{3}{8}$, $\mu_2 = \frac{5}{8}$, $\eta_1 = \frac{1}{3}$, $\eta_2 = \frac{1}{2}$ and $\psi(t) = \frac{e^t}{6}$.

Set, $F : [0,1] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$, is a multivalued map defined by

$$(t,x) \longrightarrow F(t,x) = \left[\frac{|x|^3}{5(|x|^3+3)} + \frac{t+1}{10}; \frac{|sinx|}{5(|sinx|+1)} + \frac{t}{5}\right].$$

For $f \in F(t,x)$ we have

$$|f| \leq max\left\{\frac{|x|^3}{5(|x|^3+3)} + \frac{t+1}{10}; \frac{|sinx|}{5(|sinx|+1)} + \frac{t}{5}\right\} \leq \frac{2}{5}.$$

Thus

$$\|F(t,x)\| = sup\{|f| : f \in F(t,x)\} \leq \frac{2}{5} = p(t)\Phi(\|x\|), x \in \mathbb{R}.$$

With $p(t) = 1$, $\Phi(\|x\|) = \frac{2}{5}$. With the given data, we get $\gamma_1 = \alpha_1 + \beta_1 - \alpha_1\beta_1 = \frac{17}{25}$, $|\Lambda| \simeq 1,25119$, $\Omega_1 = 2,94613$ and $\Omega_2 = 0,19594 < 1$.

Then

$$M > \frac{\Phi(\|x\|)\|p\|\Omega_1}{1-\Omega_2} \simeq 1.46562.$$

Finally, all the conditions of Theorem 3,3 are satisfied, thus the problem (4.1) has at least one solution defined on $[0,1]$

5. Conclusion

The present paper examined the existence of solution for a new kind of Langevin inclusion involving ψ-Hilfer fractional derivative. The challenges and the novelty of this work is generalize the types of fractional derivatives. With the assistance of the nonlinear alternative of Leray–Schauder type, we investigate the existence results of solution for the multivalued problem. In the end, we illustrate our result with an example.

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