



## Commutativity Theorems in Prime Rings with Involution Involving 3-Derivations

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**ABSTRACT:** In this paper, we examine the connection between 3-derivations and the commutativity of a prime ring  $R$  with an involution  $*$  that fulfills particular algebraic identities for symmetric and skew symmetric elements. In practice, certain well-known problems, have been studied in the setting of three derivations in involuted rings.

**Key Words:** Prime ring, involution, commutativity, derivation, generalized derivation, symmetric elements, skew symmetric elements.

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### 1. Introduction

$R$  will be used to refer to an associative ring with the center  $Z(R)$  throughout this paper. The commutator  $xy - yx$  is denoted as  $[x, y]$  for any  $x, y \in R$ , while the anti-commutator  $xy + yx$  is written as  $x \circ y$ . If  $aRb = 0$  indicates either  $a = 0$  or  $b = 0$ ,  $R$  is prime. As is common knowledge, a derivation is an additive mapping  $d: R \rightarrow R$  such that  $d(xy) = d(x)y + xd(y)$  for all  $x, y \in R$ . On the other hand, if  $d$  is a derivation, a generalized derivation of  $R$  with an attached derivation  $d$  is referred to as an additive mapping  $F: R \rightarrow R$  such that  $F(xy) = F(x)y + xd(y)$  for all  $x, y \in R$ . Numerous articles have been written about the connection between certain unique types of maps and the commutativity of a ring  $R$  in the literature ([1], [3], [5]). Posner's theorem, which states that a prime ring becomes commutative if it admits a nonzero centralizing derivation, is one of the most significant findings in this area.

A map's commutativity is preserved if all instances of  $[\psi(x), \psi(y)] = 0$  for all instances of  $[x, y] = 0$  for all  $x, y \in R$ . Matrix theory, operator theory, and ring theory have all explored the idea of commutativity preserving mapping (for references, see [6], [13]). If  $[\psi(x), \psi(y)] = [x, y]$  for all  $x, y \in S$ , then a map  $\psi: R \rightarrow R$  is said to be strong commutativity preserving (SCP) on a subgroup of  $R$ . In this regard, Bell and Daif [4] looked into the commutativity of rings that allow derivations that are SCP on nonzero right ideals. In fact, they demonstrated that  $I \subseteq Z(R)$  in the case where a semiprime ring  $R$  allows a derivation  $d$  satisfying  $[d(x), d(y)] = [x, y]$  for all  $x, y$  in a right ideal  $I$  of  $R$ . Additionally, Ali and Huang demonstrated that if  $R$  is a 2-torsion free semiprime ring and  $d$  is a derivation of  $R$  satisfying  $[d(x), d(y)] + [x, y] = 0$  for all  $x, y$  in nonzero ideal  $I$  of  $R$ , then  $R$  contains a nonzero central ideal. They specifically demonstrated that  $R$  is commutative if the ideal is  $I = R$ .

These findings have been expanded upon and investigated in the context of bands with involution as time goes on. An additive mapping  $*$ :  $R \rightarrow R$  satisfying  $(x^*)^* = x$  and  $(xy)^* = y^*x^*$  for all  $x, y \in R$  is called an involution. In a ring with the involution  $(R, *)$ , a member  $x$  is said to be hermitian if  $x^* = x$  and skew-hermitian if  $x^* = -x$ .  $H(R)$  will stand for the collections of all hermitian elements in  $R$ , while  $S(R)$  will stand for the set of skew-hermitian elements. If  $Z(R) \subseteq H(R)$ , the involution is said to be of the first kind, otherwise, it is said to be of the second kind.

Many authors have recently examined the commutativity of prime rings accepting pair of derivations

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meeting specific algebraic identities (see, for example, [2], [8], [9], [10]). In [9] Lanski has demonstrated that if  $R$  is a noncommutative prime ring and  $d, g$  are two derivations of  $R$  into itself with  $g$  is a nonzero derivation, then  $[d(x), g(x)] = 0$  holds for every  $x \in R$ . Then  $d = \lambda g$ , where  $\lambda$  is a component of  $C(R)$ . Abbassi et al. in [10], recently showed that if  $R$  is a prime ring with involution  $*$  of the second kind such that  $\text{char}(R) \neq 2$  and  $d_1, d_2$  are nonzero derivations of  $R$  such that  $[d_1(x); d_2(x^*)] = 0$  for all  $x \in R$ . Then  $R$  is commutative. Afterwards, El Mir et al. claimed in [8] that as long as  $R$  is a ring,  $P$  is a prime ideal, and  $R$  has  $d_1, d_2$  derivations, then  $d_1(x)d_2(y) - [x, y] \in P$  for every  $x, y \in R$  reveals that  $R/P$  is a commutative integral domain.

In order to progress this field of study, we look into the commutativity conditions for rings with involution enabling three derivations that meet specific algebraic identities as well as a few additional identities.

## 2. Preliminary results

**Lemma 2.1** [14] *Let  $R$  be a semiprime ring. Suppose that the relation  $axb + bxc = 0$  holds for all  $x \in R$  and some  $a, b, c \in R$ . In this case  $(a + c)xb = 0$  is satisfied for all  $x \in R$ .*

We begin by establishing the following fact, which will be used frequently.

**Fact 1 :** Let  $(R, *)$  be a 2-torsion free prime ring with involution of the second kind. If  $H(R) \subset Z(R)$  or  $S(R) \subset Z(R)$ , then  $R$  is commutative.

**Fact 2 :** Let  $(R, *)$  be a 2-torsion free prime ring with involution of the second kind. If  $d$  is a nonzero derivation of  $R$  such that  $d(h) \in Z(R)$  for all  $h \in H(R)$  or  $d(k) \in Z(R)$  for all  $k \in S(R)$ , then  $R$  is commutative.

## 3. Main theorems

**Proposition 3.1** *Let  $R$  be a prime ring. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $[d_1(x), d_2(y)] + [d_3(x), y] + [x, y] = 0$  for all  $x, y \in R$ , then  $R$  is commutative.*

**Proof:** Suppose that  $R$  is noncommutative, by the given assumption, we have

$$[d_1(x), d_2(y)] + [d_3(x), y] + [x, y] = 0 \quad \text{for all } x, y \in R. \quad (3.1)$$

Substituting  $yr$  for  $y$  in (3.1), we obtain

$$\begin{aligned} d_2(y)[d_1(x), r] + [d_1(x), y]d_2(r) + ([d_1(x), d_2(y)] + [d_3(x), y] + [x, y])r + \\ y([d_1(x), d_2(r)] + [d_3(x), r] + [x, r]) = 0 \quad \text{for all } r, x, y \in R. \end{aligned} \quad (3.2)$$

By invoking equation (3.1), the last equation yields

$$d_2(y)[d_1(x), r] + [d_1(x), y]d_2(r) = 0 \quad \text{for all } r, x, y \in R. \quad (3.3)$$

We replace  $r$  by  $rt$  in (3.3), we get

$$d_2(y)r[d_1(x), y] + [d_1(x), y]rd_2(t) + (d_2(y)[d_1(x), r] + [d_1(x), y]d_2(r))t = 0 \quad \text{for all } t, r, x, y \in R. \quad (3.4)$$

By equation (3.3) together with the last equation, it follows that

$$d_2(y)r[d_1(x), t] + [d_1(x), y]rd_2(t) = 0 \quad \text{for all } t, r, x, y \in R. \quad (3.5)$$

Taking  $y = t$ , we obtain

$$d_2(y)r[d_1(x), y] + [d_1(x), y]rd_2(y) = 0 \quad \text{for all } r, x, y \in R. \quad (3.6)$$

Applying Lemma 2.1, we obtain from the above relation

$$d_2(y)r[d_1(x), y] = 0 \quad \text{for all } r, x, y \in R. \quad (3.7)$$

Since  $R$  is prime, the last equation implies that  $d_2 = 0$  or  $[d_1(x), y] = 0$  for all  $x, y \in R$ . Our supposition forces  $d_2(R) = \{0\}$ , then (3.1) is reduced to  $[d_3(x), y] = 0$  for all  $x, y \in R$ . In accordance with Posner's Theorem [[12], Lemma 3],  $R$  is commutative, this is in conflict with our presumption. Therefore, we have  $d_3 = 0$ , hence (3.1) is reduced to  $[x, y] = 0$  for all  $x, y \in R$ . Which means that  $R$  is commutative, a contradiction. Thus  $R$  is commutative.  $\square$

**Theorem 3.1** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $[d_1(h), d_2(h')] + [d_3(h), h'] + [h, h'] = 0$  for all  $h, h' \in H(R)$ , then  $R$  is commutative.*

**Proof:** Take into account  $R$  is noncommutative, with the stated assumption, we have

$$[d_1(h), d_2(h')] + [d_3(h), h'] + [h, h'] = 0 \quad \text{for all } h, h' \in H(R). \quad (3.8)$$

Replacing  $h'$  by  $hh_0$ ,  $h_0 \in H(R) \cap Z(R)$  in (3.8), we find  $[d_1(h), h]d_2(h_0) = 0$  for all  $h \in H(R)$  and  $h_0 \in H(R) \cap Z(R)$ . The result of  $R$  being prime is that  $[d_1(h), h] = 0$  for all  $h \in H(R)$  or  $d_2(h_0) = 0$  for all  $h_0 \in H(R) \cap Z(R)$ . The first case  $R$  is commutative in light of [[7], Theorem 2.5], which goes against our selection of  $R$ . Thus we must have  $d_2(h_0) = 0$  for all  $h_0 \in H(R) \cap Z(R)$ , hence  $d_2(Z(R)) = 0$ . Substituting  $hh_0$  for  $h$  in (3.8), where  $h_0 \in Z(R) \cap H(R)$ , we obtain

$$[h, d_2(h')]d_1(h_0) + [h, h']d_3(h_0) = 0 \quad \text{for all } h, h' \in H(R). \quad (3.9)$$

Taking  $h = h'$  in (3.9), we arrive at  $[d_2(h), h]d_1(h_0) = 0$ . Primeness of  $R$  leads to  $[d_2(h), h] = 0$  for all  $h \in H(R)$  or  $d_1(h_0) = 0$  for all  $h_0 \in H(R) \cap Z(R)$ . When viewed in the context of [[7], Theorem 2.5], the first instance  $R$  is commutative. Which is at odds with the  $R$  we chose. Thus we must have  $d_1(h_0) = 0$  for all  $h_0 \in H(R) \cap Z(R)$ , then  $d_1(Z(R)) = \{0\}$ . So equation (3.9) gives  $[h, h']d_3(h_0) = 0$ , again using primeness of  $R$  and reasoning as above, we find  $d_3(Z(R)) = \{0\}$ . Replacing  $h$  by  $kk_0$  in (3.8), where  $k \in S(R)$  and  $k_0 \in Z(R) \cap S(R)$ , we get  $([d_1(k), d_2(h')] + [d_3(k), h'] + [k, h'])k_0 = 0$ . Since  $R$  is prime and  $Z(R) \cap S(R) \neq \{0\}$ , we find

$$[d_1(k), d_2(h')] + [d_3(k), h'] + [k, h'] = 0 \quad \text{for all } h' \in H(R) \text{ and } k' \in S(R). \quad (3.10)$$

Given that  $R$  is 2-torsion free, every  $x \in R$  can be expressed as  $2x = h + k$ , with  $h \in H(R)$  and  $k' \in S(R)$ . We have

$$\begin{aligned} 2([d_1(x), d_2(h')] + [d_3(x), h'] + [x, h']) &= [d_1(2x), d_2(h')] + [d_3(2x), h'] + [2x, h'] \\ &= [d_1(h + k), d_2(h')] + [d_3(h + k), h'] + [h + k, h'] \\ &= [d_1(h), d_2(h')] + [d_3(h), h'] + [h, h'] \\ &\quad + [d_1(k), d_2(h')] + [d_3(k), h'] + [k, h']. \end{aligned} \quad (3.11)$$

Using (3.8) and (3.10) gives that  $[d_1(x), d_2(h')] + [d_3(x), h'] + [x, h'] = 0$  for all  $x \in R$  and  $h' \in H(R)$ . Taking  $kk_0$  for  $h'$  in the last expression, where  $k \in S(R)$  and  $k_0 \in Z(R) \cap S(R)$ , we get  $[d_1(x), d_2(k)] + [d_3(x), k] + [x, k] = 0$ . Combining the last two expressions, we finally get  $[d_1(x), d_2(y)] + [d_3(x), y] + [x, y] = 0$  for all  $x, y \in R$ , then  $R$  is commutative in light of Proposition 3.1, a contradiction. Thus  $R$  is commutative.  $\square$

Following the same arguments and under the same conditions as in Theorem 3.1, we can readily obtain the following results.

**Theorem 3.2** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $[d_1(k), d_2(k')] + [d_3(k), k'] + [k, k'] = 0$  for all  $k, k' \in S(R)$ , then  $R$  is commutative.*

**Theorem 3.3** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $[d_1(h), d_2(k)] + [d_3(h), k] + [h, k] = 0$  for all  $h \in H(R)$  and  $k \in S(R)$ , then  $R$  is commutative.*

The following conclusions are drawn as uses of the aforementioned findings.

**Corollary 3.1** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $[d_1(x), d_2(y)] + [d_3(x), y] + [x, y] = 0$  for all  $x, y \in R$ , then  $R$  is commutative.*

**Corollary 3.2** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind and  $d$  is a derivation of  $R$ . If  $[d(x), d(y)] + [d(x), y] + [x, y] = 0$  for all  $x, y \in R$ , then  $R$  is commutative.*

Additionally, if we select  $d_3 = 0$  and  $d_2 = -d_1$ , we obtain the following finding.

**Corollary 3.3** *[[4], Theorem 1] Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind and  $d$  is a derivation of  $R$ . If  $[d(x), d(y)] = [x, y]$  for all  $x, y \in R$ , then  $R$  is commutative.*

We can come to the following result if either  $d_1 = 0$  or  $d_2 = 0$ .

**Corollary 3.4** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind and  $d$  is a derivation of  $R$ . If  $[d(x), y] + [x, y] = 0$  for all  $x, y \in R$ , then  $R$  is commutative.*

**Theorem 3.4** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $d_1(h) \circ d_2(h) + d_3(h) \circ h + h^2 \in Z(R)$  for all  $h \in H(R)$ , then  $R$  is commutative.*

**Proof:** Assume that  $R$  is not commutative. By hypothesis we have

$$d_1(h) \circ d_2(h) + d_3(h) \circ h + h^2 \in Z(R) \quad \text{for all } h \in H(R). \quad (3.12)$$

Linearizing the above equation, we get

$$d_1(h) \circ d_2(h') + d_1(h') \circ d_2(h) + d_3(h) \circ h' + d_3(h') \circ h + hh' + h'h \in Z(R) \quad \text{for all } h, h' \in H(R). \quad (3.13)$$

Replacing  $h'$  by  $h'h_0$  in (3.13), where  $h_0 \in Z(R) \cap H(R) \setminus \{0\}$  and making use equation (3.13), we arrive at

$$(d_1(h) \circ h')d_2(h_0) + (h' \circ d_2(h))d_1(h_0) + (h \circ h')d_3(h_0) \in Z(R) \quad \text{for all } h, h' \in H(R). \quad (3.14)$$

Taking  $h' \in Z(R) \cap H(R) \setminus \{0\}$  and using the fact that  $R$  is 2-torsion free prime, we obtain

$$d_1(h)d_2(h_0) + d_2(h)d_1(h_0) + hd_3(h_0) \in Z(R) \quad \text{for all } h \in H(R). \quad (3.15)$$

Replacing  $h$  by  $hh_0$ , where  $h_0 \in Z(R) \cap H(R) \setminus \{0\}$ , we find

$$(d_1(h)d_2(h_0) + d_2(h)d_1(h_0) + hd_3(h_0))h_0 + 2hd_1(h_0)d_2(h_0) \in Z(R) \quad \text{for all } h \in H(R). \quad (3.16)$$

Making use equation (3.15) together with 2-torsion freeness of  $R$ , we get  $hd_1(h_0)d_2(h_0) \in Z(R)$  for all  $h \in H(R)$ . Primeness of  $R$  leads to  $h \in Z(R)$  for all  $h \in H(R)$  or  $d_1(h_0)d_2(h_0) = 0$  for  $h_0 \in Z(R) \cap H(R)$ . In light of Fact 1 and contrary to what we had assumed, in the first instance,  $R$  is commutative. So in view of primeness of  $R$ , we can see  $d_1(h_0) = 0$  or  $d_2(h_0) = 0$  for  $h_0 \in Z(R) \cap H(R)$ . If  $d_1(h_0) = 0$  for  $h_0 \in Z(R) \cap H(R)$ , then  $d_1(Z(R)) = \{0\}$ . Taking  $h \in Z(R) \cap H(R) \setminus \{0\}$  in (3.14), we find that  $hh'd_3(h_0) \in Z(R)$  for all  $h' \in H(R)$ . Again using primeness of  $R$  and our supposition, we must have  $d_3(h_0) = 0$  for  $h_0 \in Z(R) \cap H(R)$ , thus  $d_3(Z(R)) = \{0\}$ . Therefore, equation (3.15) becomes  $d_1(h)d_2(h_0) \in Z(R)$  for all  $h \in H(R)$  and  $h_0 \in Z(R) \cap H(R)$ . Since  $R$  is prime, we arrive at  $d_1(h) \in Z(R)$  for all  $h \in H(R)$  or  $d_2(h_0) = 0$  for  $h_0 \in Z(R) \cap H(R)$ . The first case together with the Fact 2 forces  $R$  to be commutative which contradicts our supposition. Thus  $d_2(h_0) = 0$  for  $h_0 \in Z(R) \cap H(R)$ , hence  $d_2(Z(R)) = \{0\}$ . Taking  $h' \in Z(R) \setminus \{0\}$  in (3.13), we find  $d_3(h)h' + hh' \in Z(R)$  for all  $h \in H(R)$ . Primeness of  $R$  and

the fact that  $h' \neq 0$  leads to  $d_3(h) + h \in Z(R)$  for all  $h \in H(R)$ . Therefore,  $[d_3(h) + h, h] = 0$ , for all  $h \in H(R)$ , thus  $[d_3(h), h] = 0$ , for all  $h \in H(R)$ . When viewed in the context of [[7], Theorem 2.5],  $R$  is commutative, which runs counter to our assumption. Hence  $R$  is commutative.  $\square$

Using the same conditions, the following results can be reached by using the same justifications with necessary modifications as in Theorem 3.4 .

**Theorem 3.5** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $d_1(h) \circ d_2(h) + d_3(h) \circ h + h \circ h \in Z(R)$  for all  $h \in H(R)$ , then  $R$  is commutative.*

**Theorem 3.6** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $d_1(k) \circ d_2(k) + d_3(k) \circ k + k \circ k \in Z(R)$  for all  $k \in S(R)$ , then  $R$  is commutative.*

**Theorem 3.7** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $d_1(h) \circ d_2(k) + d_3(h) \circ k + h \circ k \in Z(R)$  for all  $h \in H(R)$  and  $k \in S(R)$ , then  $R$  is commutative.*

As a result of applying the aforementioned results, we derive the following corollaries.

**Corollary 3.5** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $d_1(x) \circ d_2(x^*) + d_3(x) \circ x^* + x \circ x^* \in Z(R)$  for all  $x \in R$ , then  $R$  is commutative.*

**Corollary 3.6** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $d_1(x) \circ d_2(y) + d_3(x) \circ y + x \circ y \in Z(R)$  for all  $x, y \in R$ , then  $R$  is commutative.*

**Corollary 3.7** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d$  is a derivation of  $R$ , satisfying  $d(x) \circ d(x^*) + d(x) \circ x^* + x \circ x^* \in Z(R)$  for all  $x \in R$ , then  $R$  is commutative.*

**Corollary 3.8** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d$  is a derivation of  $R$ , satisfying  $d(x) \circ d(y) + d(x) \circ y + x \circ y \in Z(R)$  for all  $x, y \in R$ , then  $R$  is commutative.*

The subsequent corollaries become clear if we select  $d_3 = 0$ . The next corollary is a generalization of [[11], Theorem 3.5].

**Corollary 3.9** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1$  and  $d_2$  are derivations of  $R$ , satisfying  $d_1(x) \circ d_2(x^*) + x \circ x^* \in Z(R)$  for all  $x \in R$ , then  $R$  is commutative.*

**Corollary 3.10** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1$  and  $d_2$  are derivations of  $R$ , satisfying  $d_1(x) \circ d_2(y) + x \circ y \in Z(R)$  for all  $x, y \in R$ , then  $R$  is commutative.*

**Corollary 3.11** [[11], Theorem 3.5] *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d$  is a derivation of  $R$ , such that  $d(x) \circ d(x^*) + x \circ x^* \in Z(R)$  for all  $x \in R$ , then  $R$  is commutative.*

**Corollary 3.12** [[11], Corollary 3.6] *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d$  is a derivation of  $R$ , such that  $d(x) \circ d(y) + x \circ y \in Z(R)$  for all  $x, y \in R$ , then  $R$  is commutative.*

We can come to the following result if either  $d_1 = 0$  or  $d_2 = 0$ .

**Corollary 3.13** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d$  is a derivation of  $R$ , such that  $d(x) \circ x^* + x \circ x^* \in Z(R)$  for all  $x \in R$ , then  $R$  is commutative.*

**Theorem 3.8** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $[d_1(h), d_2(h)] + d_3(h) \circ h + h^2 \in Z(R)$  for all  $h \in H(R)$ , then  $R$  is commutative.*

**Proof:** By the given assumption, we have

$$[d_1(h), d_2(h)] + d_3(h) \circ h + h^2 \in Z(R) \quad \text{for all } h \in H(R). \quad (3.17)$$

Linearizing, we obtain

$$[d_1(h), d_2(h')] + [d_1(h'), d_2(h)] + d_3(h) \circ h' + d_3(h') \circ h + hh' + h'h \in Z(R) \quad \text{for all } h, h' \in H(R). \quad (3.18)$$

Replacing  $h'$  by  $h_0$ , where  $h_0 \in Z(R) \cap H(R) \setminus \{0\}$ , and using the last equation, we obtain

$$d_3(h)h_0 + hd_3(h_0) + hh_0 \in Z(R) \quad \text{for all } h \in H(R). \quad (3.19)$$

Replacing  $h$  by  $hh_0$ , where  $h_0 \in Z(R) \cap H(R) \setminus \{0\}$ , the last equation yields

$$(d_3(h)h_0 + hd_3(h_0) + hh_0)h_0 + hd_3(h_0)h_0 \in Z(R) \quad \text{for all } h \in H(R). \quad (3.20)$$

Invoking (3.19) and (3.20), we get

$$hd_3(h_0)h_0 \in Z(R) \quad \text{for all } h \in H(R). \quad (3.21)$$

Hence

$$[h, r]Rd_3(h_0)h_0 = 0 \quad \text{for all } h \in H(R) \text{ and } r \in R. \quad (3.22)$$

Primeness of  $R$  and the fact that  $h_0 \neq 0$  leads to  $h \in Z(R)$  for all  $h \in H(R)$  or  $d_3(h_0) = 0$ , where  $h_0 \in Z(R) \cap H(R) \setminus \{0\}$ . In light of Fact 1, the first case forces  $R$  to be commutative. Otherwise if  $d_3(h_0) = 0$  for all  $h_0 \in Z(R) \cap H(R) \setminus \{0\}$ , we obtain  $d_3(Z(R)) = \{0\}$ . Therefore equation (3.19) becomes  $d_3(h)h_0 + hh_0 \in Z(R)$  for all  $h \in Z(R)$ . Since  $R$  is prime and  $h_0 \neq 0$ , we finally have  $d_3(h) + h \in Z(R)$  for all  $h \in H(R)$ . Therefore,  $[d_3(h) + h, h] = 0$ , for all  $h \in H(R)$ , thus  $[d_3(h), h] = 0$ , for all  $h \in H(R)$ . When viewed in the context of [[7], Theorem 2.5],  $R$  is commutative.  $\square$

Following the same arguments as in Theorem 3.8 with necessary modifications, we can easily prove the following results.

**Theorem 3.9** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $[d_1(h), d_2(h)] + d_3(h) \circ h + h \circ h \in Z(R)$  for all  $h \in H(R)$ , then  $R$  is commutative.*

**Theorem 3.10** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $[d_1(k), d_2(k)] + d_3(k) \circ k + k \circ k \in Z(R)$  for all  $k \in S(R)$ , then  $R$  is commutative.*

**Theorem 3.11** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $[d_1(h), d_2(k)] + d_3(h) \circ k + h \circ k \in Z(R)$  for all  $h \in H(R)$  and  $k \in S(R)$ , then  $R$  is commutative.*

As a result of the preceding discoveries, the following corollaries are acquired.

**Corollary 3.14** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $[d_1(x), d_2(x^*)] + d_3(x) \circ x^* + x \circ x^* \in Z(R)$  for all  $x \in R$ , then  $R$  is commutative.*

**Corollary 3.15** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , satisfying  $[d_1(x), d_2(y)] + d_3(x) \circ y + x \circ y \in Z(R)$  for all  $x, y \in R$ , then  $R$  is commutative.*

**Corollary 3.16** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d$  is a derivation of  $R$ , satisfying  $[d(x), d(x^*)] + d(x) \circ x^* + x \circ x^* \in Z(R)$  for all  $x \in R$ , then  $R$  is commutative.*



**Corollary 3.17** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d$  is a derivation of  $R$ , satisfying  $[d(x), d(y)] + d(x) \circ y + x \circ y \in Z(R)$  for all  $x, y \in R$ , then  $R$  is commutative.*

The following corollaries are found if we select  $d_3 = 0$ . The first corollary is an improved version of [[11], Theorem 3.8].

**Corollary 3.18** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1$  and  $d_2$  are derivations of  $R$ , satisfying  $[d_1(x), d_2(x^*)] + x \circ x^* \in Z(R)$  for all  $x \in R$ , then  $R$  is commutative.*

The next result is an improved version of [[11], Corollary 3.9].

**Corollary 3.19** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1$  and  $d_2$  are derivations of  $R$ , satisfying  $[d_1(x), d_2(y)] + x \circ y \in Z(R)$  for all  $x, y \in R$ , then  $R$  is commutative.*

**Corollary 3.20** [[11], Theorem 3.8] *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind and  $d$  is a derivation of  $R$ . If  $[d(x), d(x^*)] + x \circ x^* \in Z(R)$  for all  $x \in R$ , then  $R$  is commutative.*

**Corollary 3.21** [[11], Corollary 3.9] *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind and  $d$  is a derivation of  $R$ . If  $[d(x), d(y)] + x \circ y \in Z(R)$  for all  $x, y \in R$ , then  $R$  is commutative.*

**Theorem 3.12** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , where  $d_3$  is nonzero, satisfying  $d_1(h) \circ d_2(h') + [d_3(h), h'] = 0$  for all  $h, h' \in H(R)$ . Then  $R$  is commutative and  $(d_1 = 0 \text{ or } d_2 = 0)$ .*

**Proof:** Seeking a contradiction, assume that  $R$  is noncommutative. By the given assumption, we have

$$d_1(h) \circ d_2(h') + [d_3(h), h'] = 0 \quad \text{for all } h, h' \in H(R). \quad (3.23)$$

Replacing  $h'$  by  $hh_0$ ,  $h_0 \in H(R) \cap Z(R)$  in (3.23), we find  $(d_1(h) \circ h)d_2(h_0) = 0$  for all  $h \in H(R)$  and  $h_0 \in H(R) \cap Z(R)$ . Primeness of  $R$  leads to  $d_1(h) \circ h = 0$  for all  $h \in H(R)$  or  $d_2(h_0) = 0$  for all  $h_0 \in H(R) \cap Z(R)$ . The first case  $R$  is commutative in light of [[7], Theorem 2.7], a contradiction. Thus we must have  $d_2(h_0) = 0$  for all  $h_0 \in H(R) \cap Z(R)$ , hence  $d_2(Z(R)) = 0$ . Substituting  $hh_0$  for  $h$  in (3.23), where  $h_0 \in Z(R) \cap H(R)$ , we obtain

$$(h \circ d_2(h'))d_1(h_0) + [h, h']d_3(h_0) = 0 \quad \text{for all } h, h' \in H(R). \quad (3.24)$$

Taking  $h = h'$  in (3.24), we arrive at  $(d_2(h) \circ h)d_1(h_0) = 0$ . Again primeness of  $R$  leads to  $d_2(h) \circ h = 0$  for all  $h \in H(R)$  or  $d_1(h_0) = 0$  for all  $h_0 \in H(R) \cap Z(R)$ . In light of [[7], Theorem 2.7] the first case implied that  $R$  is commutative, which contradicts our choice of  $R$ . Therefore, we must have  $d_1(h_0) = 0$  for all  $h_0 \in H(R) \cap Z(R)$ , then  $d_1(Z(R)) = 0$ . Thereby, equation (3.24) gives  $[h, h']d_3(h_0) = 0$ , again using primeness of  $R$  and reasoning as above, we find  $d_3(Z(R)) = 0$ . Replacing  $h$  by  $kk_0$  in (3.23), where  $k \in S(R)$  and  $k_0 \in Z(R) \cap S(R)$ , we get  $(d_1(k) \circ d_2(h') + [d_3(k), h'])k_0 = 0$ . Since  $R$  is prime and  $Z(R) \cap S(R) \neq \{0\}$ , we find

$$d_1(k) \circ d_2(h') + [d_3(k), h'] = 0 \quad \text{for all } h' \in H(R) \text{ and } k' \in S(R). \quad (3.25)$$

In light of 2-torsion freeness of  $R$ , every  $x \in R$  can be written as  $2x = h + k$ , with  $h \in H(R)$  and  $k' \in S(R)$ . We have

$$\begin{aligned} 2(d_1(x) \circ d_2(h')) + [d_3(x), h'] &= d_1(2x) \circ d_2(h') + [d_3(2x), h'] \\ &= d_1(h + k) \circ d_2(h') + [d_3(h + k), h'] \\ &= d_1(h) \circ d_2(h') + [d_3(h), h'] + d_1(k) \circ d_2(h') + [d_3(k), h']. \end{aligned} \quad (3.26)$$

Using (3.23) and (3.25), gives that  $[d_1(x), d_2(h')] + [d_3(x), h'] = 0$  for all  $x \in R$  and  $h' \in H(R)$ . Putting  $kk_0$  for  $h'$  in the last expression, where  $k \in S(R)$  and  $k_0 \in Z(R) \cap S(R)$ , we get  $d_1(x) \circ d_2(k) + [d_3(x), k] = 0$ . Combining the last two expressions, we finally get

$$d_1(x) \circ d_2(y) + [d_3(x), y] = 0 \quad \text{for all } x, y \in R. \quad (3.27)$$

Substituting  $yr$  for  $y$  in (3.27), we obtain

$$\begin{aligned} (d_1(x) \circ d_2(y) + [d_3(x), y])r + y(d_1(x) \circ d_2(r) + [d_3(x), r]) \\ + [d_1(x), y]d_2(r) - d_2(y)[d_1(x), r] = 0 \\ \text{for all } x, y, r \in R. \end{aligned} \quad (3.28)$$

By invoking equation 3.27, the last equation yields

$$[d_1(x), y]d_2(r) - d_2(y)[d_1(x), r] = 0 \quad \text{for all } x, y, r \in R. \quad (3.29)$$

Writing  $d_1(x)$  instead of  $y$  in (3.29), one can obtain

$$d_2(d_1(x))[d_1(x), r] = 0 \quad \text{for all } x, r \in R. \quad (3.30)$$

Thus

$$d_2(d_1(x))R[d_1(x), r] = 0 \quad \text{for all } x, r \in R. \quad (3.31)$$

Since  $R$  is prime, hence either  $d_2(d_1(x)) = 0$  or  $[d_1(x), r] = 0$  for all  $x, r \in R$ . Thus  $R = R_1 \cup R_2$ , where  $R_1 = \{x \in R/[d_1(x), r] = 0, \text{ for all } r \in R\}$  and  $R_2 = \{x \in R/d_2(d_1(x)) = 0\}$ . Its well known that a group cannot be union of its subgroups, therefore, we have either  $R_2 \subset R_1$  and  $R = R_1$  or  $R_1 \subset R_2$  and  $R = R_2$ . As we are assuming  $R$  is noncommutative and in view of Posner's theorem [[12], Lemma 3], the first case gives a contradiction. Hence, we have just the case  $R = R_2$ , then  $d_2(d_1(R)) = \{0\}$ . Taking  $y = d_1(y)$  in equation (3.27), we can obtain  $[d_3(x), d_1(y)] = 0$  for all  $x, y \in R$  together with [2], Theorem 3.1] forces  $R$  to be commutative. Thus in both cases, we reached a contradiction, then  $R$  is commutative. Consequently, equation (3.27) reduces to  $d_1(x)d_2(y) = 0$ , then  $d_1(x)Rd_2(y) = 0$  for all  $x, y \in R$ . Accordingly,  $d_1 = 0$  or  $d_2 = 0$ .  $\square$

Following the same arguments as in Theorem 3.12, we can easily prove the following results.

**Theorem 3.13** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , where  $d_3$  is nonzero, satisfying  $d_1(k) \circ d_2(k') + [d_3(k), k'] = 0$  for all  $k, k' \in S(R)$ . Then  $R$  is commutative and ( $d_1 = 0$  or  $d_2 = 0$ ).*

**Theorem 3.14** *Let  $R$  be a 2-torsion free prime ring with involution  $*$  of the second kind. If  $d_1, d_2$  and  $d_3$  are derivations of  $R$ , where  $d_3$  is nonzero, satisfying  $d_1(h) \circ d_2(k) + [d_3(h), k] = 0$  for all  $h \in H(R)$  and  $k \in S(R)$ . Then  $R$  is commutative and ( $d_1 = 0$  or  $d_2 = 0$ ).*

#### 4. Examples

In this part, we go over a few illustrations that demonstrate how, in some circumstances, our findings do not hold. We start by demonstrating through the following instances that the condition " $*$ " is of the second kind is required.

**Example 4.1** *Let us consider  $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$ , and  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ . We have  $Z(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z} \right\}$ , therefore, it is simple to verify that  $R$  is a prime ring and  $*$  is an involution of the first kind. Moreover, we set  $d_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$ ,  $d_2 = d_1$  and  $d_3 = 0$ . Thus  $d_1, d_2$  and  $d_3$  satisfying the conditions of Theorem 3.1, Theorem 3.4, Theorem 3.5, Theorem 3.8, and Theorem 3.9, but  $R$  is not commutative. Consequently, the hypothesis of second kind involution is essential.*

In the following illustration, we show that the "primeness hypothesis" of  $R$  in our work is not a purely theoretical construct.

**Example 4.2** *Let  $R$  with involution  $*$  and  $d_1, d_2$ , and  $d_3$  be as in Example 4.1, and  $\mathbb{C}$  be the field of complex numbers. If we set  $R_1 = R \times \mathbb{C}$ , then  $R_1$  is a semi-prime ring provided with the involution of the second kind  $\tau : R_1 \rightarrow R_1$  where  $\tau(r, s) = (r^*, \bar{s})$  for all  $(r, s) \in R \times \mathbb{C}$ . Consider the derivation  $D_1 : R_1 \rightarrow R_1$  defined as  $D_1(x, s) = (d_1(x), 0)$ , the derivation  $D_2 : R_1 \rightarrow R_1$  defined as  $D_2(x, s) = (d_2(x), 0)$  and the derivation  $D_3 : R_1 \rightarrow R_1$  defined as  $D_3(x, s) = (0, 0)$ . Furthermore  $D_1, D_2$  and  $D_3$  satisfies the conditions of Theorem 3.1, but  $R_1$  is not commutative.*



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