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On a generalization of r-ideals in commutative rings

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ABSTRACT: Let R be a commutative ring with nonzero identity, J a proper ideal of R. This paper presents the concept of H_J-ideals in commutative rings. A proper ideal I of R is called an H_J-ideal if whenever a, $b \in R$ with $ab \in I$ and $(J : a) = J$ such that $(J : a) = \{x \in R : xa \in J\}$, we have $b \in I$. Our purpose is to extend the concept of r-ideals to H_J-ideals of commutative rings. Then, we investigate the basic properties of H_J-ideals and also, we give some examples about H_J-ideals.

Key Words: H_J-ideal, r-ideal, prime ideal, trivial ring extension, amalgamation of rings.

Contents

1. Introduction

Throughout this study, all rings are assumed to be commutative with nonzero identity. Let R be a ring. If I is an ideal of R with $I \neq R$, then I is called a proper ideal. Suppose that I is an ideal of R. We denote the radical of I by $\sqrt{I} = \{a \in R : a^n \in I \text{ for some } n \in \mathbb{Z}^+\}$. In particular, we mean by $\sqrt{0}$ the set of all nilpotents in R; i.e, $\{a \in R : a^n = 0 \text{ for some } n \in \mathbb{Z}^+\}$. Let S be a nonempty subset of R. Then the ideal $\{a \in R : aS \subseteq I\}$, which contains I, will be designated by $(I : S)$. Letting J be a proper ideal of R, we denote $H_J = \{h \in R : (J : h) \neq J\}$. An element $a \in R$ is said to be regular ($a \in \mathrm{Reg}(R)$) (resp., zerodivisor ($a \in Z(R)$)) if $a \notin H_0$ (resp., $a \in H_0$). Also, we denote $Ann(a) = \{b \in R : ab = 0\}$.

The notion of the prime ideal plays a key role in the theory of commutative algebra, and it has been widely studied. See, for example, [2,11]. Recall from [1] that a prime ideal P of R is a proper ideal having the property that $ab \in P$ implies either $a \in P$ or $b \in P$ for each $a, b \in R$. In [15], Mohamadian defined a proper ideal I of R to be an r-ideal if whenever $a, b \in R$ with $ab \in I$ and $Ann(a) = 0$, we have $b \in I$. Motivated by this concept, we give the notion of H_J-ideals and we investigate many properties of H_J-ideals which are analogous to r-ideals. Let J be a proper ideal of a ring R. A proper ideal I of R is said to be an H_J-ideal if the condition $ab \in I$ with $a \notin H_J$ implies $b \in I$ for every $a, b \in R$. It is clear that if $J = 0$, then I is an r-ideal if and only if I is an H_J-ideal. Among many results in this paper, it is shown (Proposition 2.1) that J is an H_J-ideal and a proper ideal I of R is an H_J-ideal of R if and only if $I = (I : h)$ for every $h \in R \setminus H_J$ (Theorem 2.1). In Proposition 2.9, we show that every maximal H_J-ideal of R is a prime ideal.

Let A be a ring and E an A-module. Then $A \ltimes E$, the trivial (ring) extension of A by E, is the ring whose additive structure is coordinate-wise addition and whose multiplication is defined by $(a, e)(b, f) := (ab, af + be)$ for all $a; b \in A$ and all $e, f \in E$. (This construction is also known by other terminology and other notation, such as the idealization $A(+)E$.) The basic properties of trivial ring extensions are summarized in the books [14,16]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf. [3,4,10,12]). In addition, for an ideal I of A and a submodule F of E, $I \ltimes F$ is an ideal of $A \ltimes E$ if and only if $IE \subseteq F$.

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Let A and B be two rings with unity, let $\mathcal{J}$ be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$:

$$A \bowtie^f \mathcal{J} := \{(a, f(a) + j) | a \in A, j \in \mathcal{J}\}$$

is called the *amalgamation of A and B along $\mathcal{J}$ with respect to f*. This construction is a generalization of the amalgamated duplication of a ring along an ideal denoted $A \bowtie I$ (introduced and studied by D'Anna and Fontana in [8]). In [6,7,9], D'Anna, Finocchiaro and Fontana introduced the more general context of amalgamations. They have studied these constructions in the context of pullbacks which allowed them to establish numerous results on the transfer of various ideals and ring-theoretic properties from A and $f(A)+\mathcal{J}$ to $A \bowtie^f \mathcal{J}$. The interest of amalgamations resides in their ability to cover basic constructions in commutative algebra, including classical pullbacks and trivial ring extensions. Moreover, other classical constructions (such as $A+XB[X]$, $A+XB[[X]]$ and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([6, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [5]) are strictly related to it ([6, Example 2.7 and Remark 2.8]). In [6], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f \mathcal{J}$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation.(cf. [13]). Finally, we give an idea about some H_J-ideals of the localization of rings, the trivial ring extensions, and the amalgamation of rings.

2. Main Results

Our aim in this section is to study the H_J-ideals in commutative rings. We begin with the following definition.

Definition 2.1 Let R be a commutative ring and I, J be proper ideals of R. We define I to be an H_J-ideal, if $ab \in I$ with $a \notin H_J$ implies that $b \in I$ for all $a, b \in R$.

Remark 2.1 Let R be a ring and J be a proper ideal of R.

1. The set H_J is a generalization of $Z(R)$ when $J = 0$.
2. Let $x \in \sqrt{0}$. Then, there is $n \in \mathbb{Z}^+$ such that $n = min\{m \in \mathbb{Z}^+ : x^m \in J\}$. We may assume that $n \geq 2$. So, $x^n = xx^{n-1} \in J$ with $x^{n-1} \notin J$ which implies that $x \in H_J$. Thus, $\sqrt{0} \subseteq H_J$.
3. Let $j \in J$ and $h \in H_J$. Then, $j + h \in H_J$. Indeed, there is $x \in R \setminus J$ such that $xh \in J$ which gives that $(j + h)x = jx + xh \in J$.
4. Let $J = 0$. Then, I is an r-ideal if and only if I is an H_J-ideal.

Next, we give the following result.

Proposition 2.1 *Let J be a proper ideal of a ring R. Then, J is an H_J-ideal.*

Proof: Let $ab \in J$ with $a \notin H_J$. Then, $(J : a) = J$. This implies that $b \in (J : a) = J$ and hence J is an H_J-ideal. □

By Proposition 2.1, we obtain an example of an H_J ideal that is not an r-ideal.

Example 2.1 *Let $R = \mathbb{Z}$ and $J = 4\mathbb{Z}$. By Proposition 2.1, J is an H_J-ideal of R. But J is not an r-ideal of R. Indeed. $2 \cdot 2 \in J$ and $ann(2) = 0$ but $2 \notin J$.*

Proposition 2.2 *Let J be a proper ideal of a ring R and let $\{I_i\}_{i\in\Delta}$ be a nonempty set of H_J-ideals of R. Then $\bigcap_{i\in\Delta} I_i$ is an H_J-ideal of R.*

Proof: Let $ab \in \bigcap_{i\in\Delta} I_i$ with $a \notin H_J$ for $a, b \in R$. Then, $ab \in I_i$ for every $i \in \Delta$. Since I_i is an H_J-ideal of R, we get the result that $b \in I_i$ and so $b \in \bigcap_{i\in\Delta} I_i$. This implies that $\bigcap_{i\in\Delta} I_i$ is an H_J-ideal of R. □

Proposition 2.3 *Let J be a proper ideal of a ring R.*

1. *If I is an H_J-ideal, then $I \subseteq H_J$.*
2. *For a prime ideal I of R, I is an H_J-ideal if and only if $I \subseteq H_J$.*

Proof: (1) Assume that I is an H_J-ideal but $I \nsubseteq H_J$. Then, there exists $a \in I$ such that $a \notin H_J$. Since $a.1 = a \in I$ and I is an H_J-ideal, we conclude that $1 \in I$, so that $I = R$, a contradiction. Hence $I \subseteq H_J$.
(2) Let I be a prime H_J-ideal. By (1) we have $I \subseteq H_J$. Conversely assume that I is a prime ideal such that $I \subseteq H_J$. Let $ab \in I$ with $a \notin H_J$ for $a, b \in R$. Then, $a \notin I$. This implies that $b \in I$ and this completes the proof. □

Remember that a proper ideal P of R is prime if and only if $P = (P : a)$ for every $a \notin P$. Then, $H_P = P$. By Proposition 2.3, we give an example to show an r-ideal that is not H_J-ideal.

Example 2.2 *Let $R = \mathbb{Z}_6$, $J = 2R$ and $I = 3R$. It is clear that $H_J = J$. Since $I \nsubseteq H_J$, we conclude by Proposition 2.3 that I is not an H_J-ideal of R. Clearly, $\mathrm{Reg}(R) = \{1, 5\}$ which gives that $\mathrm{Reg}(R) \cap I = \emptyset$. Now, let $ab \in I$ for $a, b \in R$ with $ann(a) = 0$. Then $a \in \mathrm{Reg}(R)$, which gives $a \notin I$. Since I is a prime ideal, then $b \in I$. Therefore, I is an r-ideal.*

Corollary 2.1 *Let R be a ring.*

1. *Let J be a prime ideal of R. If I is an H_J-ideal, then $I \subseteq J$. In particular, if I is prime, then, I is an H_J-ideal if and only if $I \subseteq J$.*
2. *Let P and Q be two prime ideals of a ring R. Then, P and Q are comparable if and only if P is an H_Q-ideal or Q is an H_P-ideal.*

Proof: (1) Let J be a prime ideal of R and I an H_J-ideal of R. It's clear that $H_J = J$. Then by Proposition 2.3 (1) we get that $I \subseteq J$. The in particular follows by Proposition 2.3 (2).
(2) This is clear by using (1). □

Proposition 2.4 *Let R be a ring, S a nonempty subset of R, and let J be a proper ideal of R. If I is an H_J-ideal of R with $S \nsubseteq I$, then $(I : S)$ is an H_J-ideal of R.*

Proof: It is easy to see that $(I : S) \neq R$. Let $ab \in (I : S)$ and $a \notin H_J$. Then, we have $abs \in I$ for every $s \in S$. Since I is an H_J-ideal of R, we conclude that $bs \in I$. Which implies that $b \in (I : S)$ for all $s \in S$. □

Next, we give the following characterization of H_J-ideals.

Theorem 2.1 *Let R be a ring and I, J be proper ideals of R. Then the following statements are equivalent:*

1. *I is an H_J-ideal.*
2. *$hR \cap I = hI$, for each $h \in (R \setminus H_J)$.*
3. *$I = (I : h)$, for each $h \in (R \setminus H_J)$.*
4. *For ideals L and K of R, $LK \subseteq I$ with $L \cap (R \setminus H_J) \neq \emptyset$ implies $K \subseteq I$.*

Proof: (1) ⇒ (2) It is clear that $hI \subset hR \cap I$. Now, let $x \in hR \cap I$, then $x = hr \in I$ for some $r \in R$. As, I is an H_J-ideal and $h \notin H_J$, we conclude that $r \in I$.
(2) ⇒ (3) It suffices to show that $(I : h) \subseteq I$. Let $b \in (I : h)$, then $hb \in I \cap hR$. Since $I \cap hR = hI$, we conclude that $b \in I$.
(3) ⇒ (4) Suppose that $LK \subseteq I$ with $L \cap (R \setminus H_J) \neq \emptyset$, for ideals L and K of R. Since $L \cap (R \setminus H_J) \neq \emptyset$, then there exists an $a \in L$ such that $a \notin H_J$. Then we have $aK \subseteq I$, and so $K \subseteq (I : a) = I$ by (3).
(4) ⇒ (1) Let $ab \in I$ with $a \notin H_J$ for $a, b \in R$. It is sufficient to take $L = aR$ and $K = bR$ to prove the result. □

Proposition 2.5 *Let J be a proper ideal of a ring R. If I is not an H_J-ideal such that $I \subseteq H_J$, then there exist tow ideals L and K such that $L \cap (R \setminus H_J) \neq \emptyset$, $I \subsetneq L, K$, and $LK \subseteq I$.*

Proof: Suppose that I is not an H_J-ideal. Then, there exists $a \notin H_J, b \in R$ with $ab \in I$ but $b \notin I$. Now put $L = (I : b)$ and $K = (I : L)$. Clearly, $a \in L \backslash I$, $L \cap (R \setminus H_J) \neq \emptyset$, $b \in K \backslash I$, and $LK \subseteq I$. □

Proposition 2.6 *Let R be a ring, J be a proper ideal of R and K an ideal of R with $K \cap (R \setminus H_J) \neq \emptyset$. Then the following hold:*

1. *If I_1, I_2 are H_J-ideals of R with $I_1K = I_2K$, then $I_1 = I_2$.*
2. *If I is a proper ideal such that IK is an H_J-ideal of R, then $IK = I$. In particular, I is an H_J-ideal.*

Proof: (1) Since I_1 is an H_J-ideal and $I_2K \subseteq I_1$, then by Theorem 2.1(4), we get the result that $I_2 \subseteq I_1$. Likewise, we get $I_1 \subseteq I_2$.
(2) Since IK is an H_J-ideal and $IK \subseteq IK$, we conclude by Theorem 2.1(4) that $I \subseteq IK$, so this completes the proof. □

Proposition 2.7 *Let R be a ring, and J, I be proper ideals of R with $I \cap (R \setminus H_J) \neq \emptyset$. If L and K are H_J-ideals of R such that $I \cap L = I \cap K$, then $L = K$.*

Proof: Let $x \in L$ and $y \in I \cap (R \setminus H_J)$. Then, $yx \in I \cap L$, and so $yx \in I \cap K$. Since K is an H_J-ideal with $y \notin H_J$, then $x \in K$. That implies $L \subseteq K$. Likewise, we get $K \subseteq L$. □

In Proposition 2.2, we observe that an intersection of H_J-ideals is an H_J-ideal. In the following proposition, we show that the converse is also true for prime ideals in the finite case. The result may not be true for an infinite number of primes. Take the intersection of nonzero prime ideals in $\mathbb{Z}$ and $J = 0$.

Proposition 2.8 *Suppose that $P_1, \cdots, P_n$ are prime ideals in a ring R, which are not comparable. Let J be a proper ideal of R. If $\bigcap_{i=1}^{n} P_i$ is an H_J-ideal, then P_i is an H_J-ideal, for $i = 1, \cdots, n$.*

Proof: Let $ab \in P_j$ with $a \notin H_J$ and take $x \in \left(\prod_{i \neq j} P_i\right) \backslash P_j$. Hence, $abx \in \bigcap_{i=1}^{n} P_i$. As $\bigcap_{i=1}^{n} P_i$ is an H_J-ideal, we get that $bx \in \bigcap_{i=1}^{n} P_i$, and so $bx \in P_j$. This implies that $b \in P_j$ and thus P_j is an H_J-ideal. □

Let R be a ring, we call an H_J-ideal M of R a maximal H_J-ideal if there is no H_J-ideal contains M properly.

Proposition 2.9 *Let R be a ring and J a proper ideal of R. Then, every maximal H_J-ideal of R is a prime ideal.*

Proof: Suppose that P is a maximal H_J-ideal of R, $xy \in P$ and $x \notin P$. Now show that $y \in P$. Clearly, $(P : x)$ is an H_J-ideal, $P \subseteq (P : x)$ and $y \in (P : x)$. Now by the maximality of P, we have $P = (P : x)$. This implies that $y \in P$. □

Proposition 2.10 *Let R be a ring and J a proper ideal of R.*

1. *Let $x, y \in R$ such that $x + y = 1$. Then, $I = (J : x) + (J : y)$ is an H_J-ideal of R.*
2. *Let e be an idempotent element of R and P a prime ideal such that $J \subseteq P$ and $(J : a) \cap R \setminus P \neq \emptyset$ for each $a \in P$. Then, $I = P + (J : e)$ is an H_J-ideal.*
3. *Let P be a prime ideal of R such that there is an H_J-ideal $I \subseteq P$ with $(I : a) \cap R \setminus P \neq \emptyset$ for each $a \in P$. Then, P is an H_J-ideal.*

Proof: (1) Let $a, b \in R$ such that $ab \in I$ and $a \notin H_J$. Thus, $ab = z + t$ for some $z \in (J : x)$ and $t \in (J : y)$ which implies that $xyab \in J$. Since $xyb \in (J : a) = J$, we conclude that $bx \in (J : y)$ and $by \in (J : x)$. Therefore, $b = b(x + y) = bx + by \in I$ and so I is an H_J-ideal of R.
(2) Assume that $ab \in I$ for $a, b \in R$ such that $a \notin H_J$. Hence, $ab = x + y$ for some $x \in P$ and $y \in (J : e)$ such that $ye \in J$. Thus, there exist $t \notin P$ such that $tx \in J$ and so $etab \in J$. Since $a \notin H_J$, we get that $etb \in J \subseteq P$. Therefore, $be \in P$. Now, the fact that $b = be + (1 - e)b \in P + (J : e)$ implies that I is an H_J-ideal.
(3) Let $ab \in P$ for some $a, b \in R$ with $a \notin H_J$. Hence, there is $x \in R \setminus P$ such that $xab \in I$ which implies that $xb \in I \subseteq P$ because I is an H_J-ideal. As $x \notin P$, we get that $b \in P$ and therefore P is an H_J-ideal. □

For a ring R and an ideal J of R, we say that R satisfies the condition $(*)$ if for every finitely generated ideal I of R there is an element $x \in I$ such that $(J : I) = (J : x)$.

Theorem 2.2 *Let R be a ring that satisfies the condition $(*)$ for an ideal J and let I be a proper ideal of R. Then, I is an H_J-ideal if and only if for each finitely generated ideal L and every ideal K of R with $LK \subseteq I$ and $(J : L) = J$, then $K \subseteq I$.*

Proof: Suppose that I is an H_J-ideal and let L be finitely generated ideal and K an ideal of R with $LK \subseteq I$ and $(J : L) = J$. Suppose that $K \nsubseteq I$. Then, there exists $b \in K \setminus I$. Since R satisfies the condition $(*)$, we have $(J : L) = (J : a) = J$ for some $a \in L$. It implies that $ab \in I$ with $a \notin H_J$, and so $b \in I$ because I is an H_J-ideal of R, a contradiction. Therefore, $K \subseteq I$. The converse is clear. □

Lemma 2.1 *Let $f : R \to S$ be a ring homomorphism and K be an ideal of S, and let $J = f^{-1}(K)$. Then:*

1. *If $a \in H_J$ then, $f(a) \in H_K$.*
2. *If f is an epimorphism, then, $a \in H_J$ if and only if $f(a) \in H_K$.*

Proof: (1) Let $a \in H_J$, then $(J : a) \neq J$, which gives that there exists $b \in R$ such that $ab \in J$ but $b \notin J$. That implies $f(ab) \in K$ and $f(b) \notin K$. Therefore $f(a) \in H_K$.
(2) If $a \in H_J$, (1) implies that $f(a) \in H_K$. Now suppose that $f(a) \in H_K$, then there exist $c' \in S$ such that $f(a)c' \in K$ but $c' \notin K$. Since f is epimorphism, then there exists $c \in R$ such that $f(c) = c'$. Therefore $f(ac) \in K$ but $f(c) \notin K$, then $ac \in J$ but $c \notin J$. □

Theorem 2.3 *Let $f : R \to S$ be a ring epimorphism and K be an ideal of S, and let $J = f^{-1}(K)$. Then, the following hold:*

1. *If L is an H_K-ideal of S, then $f^{-1}(L)$ is an H_J-ideal of R.*
2. *Assume that I is an ideal of R containing $\mathrm{Ker}(f)$. Then, I is an H_J ideal if and only if $f(I)$ is an H_K-ideal of S.*

Proof: (1) Let $ab \in f^{-1}(L)$ and $a \notin H_J$. Then, $f(ab) = f(a)f(b) \in L$. Since $a \notin H_J$ and f is a epimorphism, we get by Lemma 2.1 (2) that $f(a) \notin H_K$. Since L is an H_K-ideal of $S, f(b) \in L$ and so $b \in f^{-1}(L)$. Consequently, $f^{-1}(L)$ is an H_J-ideal of R.
(2) Suppose that I is an H_J-ideal of R. Let $a'b' \in f(I)$ with $a' \notin H_K$ for $a', b' \in S$. Since f is epimorphism, there exist $a, b \in R$ such that $a' = f(a)$ and $b' = f(b)$. Then, $a'b' = f(ab) \in f(I)$. As $\mathrm{Ker}(f) \subseteq I$, we conclude that $ab \in I$. Since $f(a) \notin H_K$, then by Lemma 2.1 (1) $a \notin H_J$. Now, the fact that I is an H_J-ideal of R implies that $b \in I$ and so $f(b) = b' \in f(I)$ as it is needed. For the converse it suffices to see that $I = f^{-1}(f(I))$. □

The author of [15] shows that for two r-ideals I and L of R, with $L \subseteq I$, the ideal I/L of R/L may not be an r-ideal in R/L. See [15, Example 2.21]. The next corollary shows when I/L is an H_K-ideal of R/L.

Corollary 2.2 *Let R be a ring, $L \subseteq I$ be two ideals of R, and let $\pi : R \to R/L$ be the natural homomorphism. Let K be an ideal of R/L and $J = \pi^{-1}(K)$. Then, I is an H_J-ideal of R if and only if I/L is an H_K-ideal of R/L.*

Proof: Assume that I is an ideal of R with $L \subseteq I$. Note that π is an epimorphism and $\mathrm{Ker}(\pi) = L \subseteq I$, and so by Theorem 2.3 it follows that I is an H_J-ideal of R if and only if I/L is an H_K-ideal of R/L. □

Lemma 2.2 *Let J be ideal and S a multiplicatively closed subset of a ring R such that $S \cap J = \emptyset$. If $\frac{a}{s} \in H_{S^{-1}J}$ for some $s \in S$, then $a \in H_J$. Moreover, if at last one of the following conditions holds:*

1. *J is prime;*
2. *$S = \mathrm{Reg}(R)$ and $Z(R) \subseteq J$;*

then, $a \in H_J$ if and only if $\frac{a}{1} \in H_{S^{-1}J}$.

Proof: Suppose that $\frac{a}{s} \in H_{S^{-1}J}$ and $a \notin H_J$. Then, there exists $\frac{b}{t} \in S^{-1}R$ such that $\frac{b}{t}\frac{a}{s} \in S^{-1}J$, but $\frac{b}{t} \notin S^{-1}J$. Thus, there exists $u \in S$ such that $uba \in J$ but $ub \notin J$, a contradiction. Moreover, let $a \in H_J$. Then, there exists $b \in R$ such that $ab \in J$ but $b \notin J$ which gives that $\frac{a}{1}\frac{b}{1} \in S^{-1}J$. Assume that J is prime, then for all $s \in S$, we get $bs \notin J$. Now, assume that $S = \mathrm{Reg}(R)$ and $Z(R) \subseteq J$, then $b \in \mathrm{Reg}(R)$, which gives that for all $s \in S$ we get $bs \notin J$. Therefore in all cases we have $\frac{b}{1} \notin S^{-1}J$ and thus $\frac{a}{1} \in H_{S^{-1}J}$. □

Let R be a ring and S a multiplicatively closed subset of R, and let $f : R \to S^{-1}R$ be a ring homomorphism defined by $f(a) = \frac{a}{1}$. If J an ideal of $S^{-1}R$ then $f^{-1}(J)$ is always an ideal of R, called the contraction J^c of J to R. The following proposition shows that: If K is an $H_{S^{-1}J}$-ideal of $S^{-1}R$, then K^c is an H_J-ideal of R.

Proposition 2.11 *Let R be a ring, J a proper ideal of R, and S a multiplicatively closed subset of R such that $J \cap S = \emptyset$. Then the following hold:*

1. *If K is an $H_{S^{-1}J}$-ideal of $S^{-1}R$, then K^c is an H_J-ideal of R.*
2. *Assume that at last one of the following conditions holds:*
   (a) *J is prime;*
   (b) *$S = \mathrm{Reg}(R)$ and $Z(R) \subseteq J$;*

   then. If I is an H_J-ideal of R, then $S^{-1}I$ is an $H_{S^{-1}J}$-ideal of $S^{-1}R$.

Proof: (1) Let $ab \in K^c$ and $a \notin H_J$ for some $a, b \in R$. Hence, we have $\frac{a}{1}\frac{b}{1} \in K$. By Lemma 2.2, we get that $\frac{a}{1} \notin H_{S^{-1}J}$. This implies that $\frac{b}{1} \in K$ because K is an $H_{S^{-1}J}$-ideal of $S^{-1}R$, which gives that $b \in K^c$.
(2) Let $\frac{a}{s}\frac{b}{t} \in S^{-1}I$ with $\frac{a}{s} \notin H_{S^{-1}J}$, where $a, b \in R$ and $s, t \in S$. Then we have $uab \in I$ for some $u \in S$. By Lemma 2.2 we get $a \notin H_J$. Since I is an H_J-ideal of R, we conclude that $ub \in I$ and so $\frac{b}{t} = \frac{ub}{ut} \in S^{-1}I$. Consequently, $S^{-1}I$ is an $H_{S^{-1}J}$-ideal of $S^{-1}R$. □

By the above observations, we have the following definition of H_J-multiplicatively closed subsets.

Definition 2.2 Let R be a ring and J a proper ideal of R. Let S be a nonempty subset of R with $R \setminus H_J \subseteq S$. Then, S is called an H_J-multiplicatively closed subset of R if $xy \in S$ for all $x \in R \setminus H_J$ and $y \in S$.

Proposition 2.12 *For a proper ideal I of R, I is an H_J-ideal of R if and only if $R \setminus I$ is an H_J-multiplicatively closed subset of R.*

Proof: Suppose that I is an H_J-ideal of R. Then, by Proposition 2.3 (1) we have $I \subseteq H_J$ and so $R \setminus H_J \subseteq R \setminus I$. Let $x \in R \setminus H_J$ and $y \in R \setminus I$. Assume that $xy \in I$. Since $x \notin H_J$ and I is an H_J-ideal, we conclude that $y \in I$, is a contradiction. Thus, we get that $xy \in R \setminus I$, and so $R \setminus I$ is an H_J-multiplicatively closed subset of R. For the converse, suppose that I is an ideal and $R \setminus I$ is an H_J-multiplicatively closed subset of R. Now, we show that I is an H_J-ideal. Let $ab \in I$ with $a \notin H_J$ for $a, b \in R$. Then, we have $b \in I$, or else we would have $ab \in R \setminus I$ since $R \setminus I$ is an H_J-multiplicatively closed subset of R. So, it follows that I is an H_J-ideal of R. □

We remind the reader that if I is an ideal which is disjoint from a multiplicatively closed subset S of R, then there exists a prime ideal P of R containing I such that $P \cap S = \emptyset$: The following Theorem states that a similar result is true for H_J-ideals.

Theorem 2.4 *Let I be an ideal of a ring R such that $I \cap S = \emptyset$ where S is an H_J-multiplicatively closed subset of R for a proper ideal J of R. Then, there exists an H_J-ideal K containing I such that $K \cap S = \emptyset$.*

Proof: Consider the set $\Omega = \{I' : I' \text{ is an ideal of } R \text{ with } I' \cap S = \emptyset\}$. Since $I \in \Omega$, we have $\Omega \neq \emptyset$. By using Zorn's lemma, we get a maximal element K of Ω. Now, we show that K is an H_J-ideal of R. Suppose not. Then, we have $ab \in K$ for some $a \notin H_J$ and $b \notin K$. Thus, we get $b \in (K : a)$ and $K \subsetneq (K : a)$. By the maximality of K, we have $(K : a) \cap S \neq \emptyset$; and thus there exists an $s \in S$ such that $s \in (K : a)$. So, we have $as \in K$. Also $as \in S$, because $a \in R \setminus H_J$, $s \in S$ and S is an H_J-multiplicatively closed subset of R. Thus, we get that $S \cap K \neq \emptyset$, and this contradicts by $K \in \Omega$. Hence K is an H_J-ideal of R. □

Proposition 2.13 *Let R_1, R_2 be two ring and J_1, J_2 two proper ideals of R_1 and R_2 respectively. Set $J := J_1 \times J_2$ and let $I := I_1 \times I_2$ be ideal of $R = R_1 \times R_2$. Then, I is an H_J-ideal if and only if $I_1 = R_1$ and I_2 is an H_{J_2}-ideal of R_2 or $I_2 = R_2$ and I_1 is an H_{J_1}-ideal of R_1 or I_i is an H_{J_i}-ideal of R_i for $i = 1, 2$.*

Proof: First, we show that for an element $(x, y) \in R$ we have $(x, y) \in H_J$ if and only if $x \in H_{J_1}$ or $y \in H_{J_2}$. Indeed. As $(x, y) \in H_J$, then there exists $(x', y') \in R \setminus J$ such that $(x, y)(x', y') \in J$. By using the fact that $x' \notin J_1$ or $y' \notin J_2$, we conclude that $x \in H_{J_1}$ or $y \in H_{J_2}$. Conversely, we may assume that $x \in H_{J_1}$. So, there is $x' \in R_1 \setminus J_1$ such that $x'x \in J_1$. Thus, $(x', 0)(x, y) = (x'x, 0) \in J$ and $(x', 0) \notin J$ which implies that $(x, y) \in H_J$. Now, without loss of generality, we may assume that I_1 and I_2 are proper ideals of R_1 and R_2 respectively. Suppose that I is an H_J-ideal. Let $ab \in I_1$ with $a \notin H_{J_1}$, let $c \in R_2 \setminus H_{J_2}$, then $(a, c)(b, 0) \in I$ with $(a, c) \notin H_J$. Since I is an H_J-ideal, then $(b, 0) \in I$, Which gives $b \in I_1$. Therefore I_1 is an H_{J_1}-ideal of R_1. Likewise we get I_2 is an H_{J_2}-ideal of R_2. For the converse assume that I_i is an H_{J_i}-ideal of R_i for all $i = 1, 2$. Let $(a_1, a_2)(b_1, b_2) \in I$ with $(a_1, a_2) \notin H_J$, then for all $i = 1, 2$, we get $a_i b_i \in I_i$ and $a_i \notin H_{J_i}$. Which gives $b_i \in I_i$. Therefore $(b_1, b_2) \in I$, as desired. □

Remark 2.2 Let $a \in H_J$ and $x \in \sqrt{0}$, then $a + x \in H_J$. Indeed. There is $b \notin J$ and $n \in \mathbb{Z}^+$ such that $ab \in J$ and $x^n = 0$. Then, $(a + x)bx^{n-1} \in J$ (we may assume that $n \geq 2$). If $bx^{n-1} \notin J$, thus $(a+x) \in H_J$. If $bx^{n-1} \in J$, then let k be the maximal integer such that $bx^k \notin J$. So, $(a+x)bx^k \in J$ which implies that $(a + x) \in H_J$.

Let A be a ring and N be a submodule of an A-module E, we denote by $H_N = \{h \in A : (N : h) \neq N\}$. We next study when certain ideals of $A \ltimes E$ are H_J-ideals.

Lemma 2.3 *Let A be a ring, E an A-module, J a proper ideal of A and N a submodule of E such that $JE \subseteq N$. Then $H_{J \ltimes N} = \{(a, e) \in A \ltimes E \mid a \in H_J \cup H_N\}$.*

Proof: Let $(a, e) \in H_{J \ltimes N}$. Hence, there exists $(b, f) \notin J \ltimes N$ such that $(a, e)(b, f) = (ab, af + be) \in J \ltimes N$. If $b \in A \setminus J$, then $a \in H_J$. Assume that $b \in J$. Thus, $f \notin N$ and $af \in N$ since $af + be \in N$ and $be \in N$. It implies that $a \in H_N$. Now, let $(a, e) \in \{(a', e') \in A \ltimes E \mid a \in H_J \cup H_N\}$. If $a \in H_J$, then there is $b \in A \setminus J$ such that $ab \in J$. So, $(a, 0)(b, 0) = (ab, 0) \in J \ltimes N$ but $(b, 0) \notin J \ltimes N$ which gives that

$(a,0) \in H_{J\ltimes N}$. Now, suppose that $a \in H_N$. Then, the fact that $af \in N$ for some $f \notin N$ implies that $(a,0)(0,f) = (0,af) \in J \ltimes N$. As $(0,f) \notin J \ltimes N$, we get that $(a,0) \in H_{J\ltimes N}$. Then, in all cases we have $(a,0) \in H_{J\ltimes N}$. By using Remark 2.2 and since $(a,e) = (a,0) + (0,e)$, we conclude that $(a,e) \in H_{J\ltimes N}$, as desired. □

Definition 2.3 Let R be a ring, E an R-module and F a proper submodule of E.
(1) F is called an H_J-submodule if whenever $am \in F$ for $a \in R$, $m \in E$ then either $a \in H_J$ or $m \in F$. In particular, when $J = 0$, we say that F is an r-submodule of E.
(2) Let N be a proper submodule of E. Then, F is called an $H_{(J,N)}$-submodule if whenever $am \in F$ for $a \in R$, $m \in E$ with $a \notin H_J \cup H_N$, then $m \in F$.

Theorem 2.5 *Let A be a ring, E an A-module and I, J be proper ideals of A. Let F be a submodule of E such that $IE \subseteq F$ and N a proper submodule with $JE \subseteq N$. Then, $I \ltimes F$ is an $H_{J\ltimes N}$-ideal of $A \ltimes E$ if and only if I is an $H_{(J,N)}$-ideal of A and either $F = E$ or F is an $H_{(J,N)}$-submodule of E.*

Proof: Assume that $I \ltimes F$ is an $H_{J\ltimes N}$-ideal of $A \ltimes E$. Let $ab \in I$ with $a \notin H_J$ and $a \notin H_N$ for a, b two elements of A. Thus, $(a,0)(b,0) = (ab,0) \in I \ltimes F$ and by Lemma 2.3, $(a,0) \notin H_{J\ltimes N}$. This implies that $(b,0) \in I \ltimes F$. Therefore, $b \in I$. Now, suppose that $F \neq E$ and $am \in F$ with $a \notin H_J \cup H_N$. Then, $(a,0)(0,m) = (0,am) \in I \ltimes F$ and by Lemma 2.3, $(a,0) \notin H_{J\ltimes N}$. This implies that $(0,m) \in I \ltimes F$ and so $m \in F$. For the converse, assume that I is an $H_{(J,N)}$-ideal of A and either $F = E$ or F is an $H_{(J,N)}$-submodule of E. Let $(a,e)(b,f) = (ab, af + be) \in I \ltimes F$, with $(a,e) \notin H_{J\ltimes N}$. Then we have $ab \in I$ with $a \notin H_J \cup H_N$. It implies that $b \in I$. If $F = E$, then we have $(b,f) \in I \ltimes E$. Assume that $F \neq E$. Since $IE \subseteq F$ and $af + be \in F$, we conclude that $af \in F$, which gives that $f \in F$. Therefore $(b,f) \in I \ltimes F$, as desired. □

Moreover, we give the following corollaries as consequences of the previous theorem.

Corollary 2.3 *Let A be a ring, E an A-module and I, J be proper ideals of A. Let F be a submodule of E such that $IE \subseteq F$ and N a proper submodule of E with $JE \subseteq N$.*

1. *If I is an H_J-ideal of A and F is an H_J-submodule of E, then $I \ltimes F$ is an $H_{J\ltimes N}$-ideal of $A \ltimes E$.*
2. *Assume that $H_J = H_N$. Then, $I \ltimes F$ is an $H_{J\ltimes N}$-ideal of $A \ltimes E$ if and only if I is an H_J-ideal of A and either $F = E$ or F is an H_J-submodule of E.*

Proof: (1) Clear because every H_J-submodule is an $H_{(J,N)}$-submodule.
(2) Since $H_J = H_N$, then we have the equivalence between H_J-submodule and $H_{(J,N)}$-submodule. □

Corollary 2.4 *Let A be a ring, E an A-module and I be an ideal of A. Let F be a submodule of E such that $IE \subseteq F$.*

1. *$I \ltimes F$ is an r-ideal of $A \ltimes E$ if and only if I is an $H_{(0,0)}$-ideal of A and either $F = E$ or F is an $H_{(0,0)}$-submodule of E.*
2. *Assume that $Z(R) = Z(M)$. Then, $I \ltimes F$ is an r-ideal of $A \ltimes E$ if and only if I is an r-ideal of A and either $F = E$ or F is an r-submodule of E.*

Let $f : A \to B$ be a ring homomorphism and $\mathcal{J}$ be an ideal of B. Let J (resp., H) be an ideal of A (resp., $f(A)+\mathcal{J}$) such that $f(J)\mathcal{J} \subseteq H \subseteq \mathcal{J}$. Observe that $J \bowtie^f H := \{(i, f(i)+h) \mid i \in J, h \in H\}$ is an ideal of $A \bowtie^f \mathcal{J}$. Set $I \bowtie^f \mathcal{J} := \{(i, f(i)+j) \mid i \in I, j \in \mathcal{J}\}$ and $\overline{K}^f := \{(a, f(a)+j) \mid a \in A, j \in \mathcal{J}, f(a)+j \in K\}$ where I and K are subsets of A and $f(A)+\mathcal{J}$ respectively. Notice that if I and K are ideals, then $I \bowtie^f \mathcal{J}$ and $\overline{K}^f$ are ideals of $A \bowtie^f \mathcal{J}$. Next, we investigate the transfer of H_J-ideals to the amalgamation $A \bowtie^f \mathcal{J}$.

Proposition 2.14 *Let $f : A \to B$ be a ring homomorphism and $\mathcal{J}$ be an ideal of B. Let J (resp., H) be an ideal of A (resp., $f(A)+\mathcal{J}$) such that $f(J)\mathcal{J} \subseteq H \subseteq \mathcal{J}$. Let $L_1 = \{(a, f(a)+j) \in A \bowtie^f \mathcal{J} \mid a \in H_J\}$ and $L_2 = \{(a, f(a)+j) \in A \bowtie^f \mathcal{J} \mid (f(a)+j)k \in H$ for some $k \in \mathcal{J} \setminus H\}$. Then, the following statements hold:*

1. *$H_{J\bowtie^f H} \subseteq L_1 \cup L_2$.*

2. *Assume that $f(A \setminus H_J) = (f(A) + \mathcal{J}) \setminus H_H$ and $\mathcal{J}^2 \subseteq H$. Then, $H_{J\bowtie^f H} = L_1$.*

3. *Suppose that J is prime, $f(J) \subseteq H$ and $\mathcal{J} \subseteq H_H$. Then, $H_{J\bowtie^f H} = L_2$.*

Proof: (1) Let $(a, f(a) + j) \in H_{J\bowtie^f H}$. Then, there is $(b, f(b) + k) \in A \bowtie^f \mathcal{J} \setminus J \bowtie^f H$ such that $(a, f(a) + j)(b, f(b) + k) = (ab, f(ab) + jf(b) + k(f(a) + j)) \in J \bowtie^f H$. If $b \notin J$, then as $ab \in J$, we conclude that $a \in H_J$ and so $(a, f(a)+j) \in L_1$. Now, suppose that $b \in J$. Since $jf(b)+k(f(a)+j) \in H$ and $f(J)\mathcal{J} \subseteq H$, we then have $k(f(a) + j) \in H$. As $k \notin H$, we conclude that $(a, f(a) + j) \in L_2$.
(2) Let $(a, f(a)+j) \in H_{J\bowtie^f H}$. By (1), we may assume that $(a, f(a)+j) \in L_2$. Then, there exists $k \notin H$ such that $(f(a) + j)k \in H$. As $k \notin H$ and $\mathcal{J}^2 \subseteq H$, we get that $f(a) \in H_H$. Now, using the fact that $f(A \setminus H_J) = B \setminus H_H$, we conclude that $a \in H_J$. Hence, in all cases we have $(a, f(a) + j) \in H_J \bowtie^f \mathcal{J}$. Therefore, $H_{J\bowtie^f H} \subseteq L_1$. For the converse, let $(a, f(a) + j) \in L_1$. Then $ab \in J$ for some $b \in A \setminus J$. If $jf(b) \in H$, then $(a, f(a)+j)(b, f(b)) = (ab, f(ab)+jf(b)) \in J \bowtie^f H$ with $(b, f(b)) \notin J \bowtie^f H$. Therefore, $(a, f(a) + j) \in H_{J\bowtie^f H}$. In the remaining case, we have $(a, f(a) + j)(0, jf(b)) = (0, jf(ab) + j^2 f(b)) \in J \bowtie^f H$ because $\mathcal{J}^2 \subseteq H$ and $f(J)\mathcal{J} \subseteq H$. As $(0, jf(b)) \notin J \bowtie^f H$ since $jf(b) \notin H$, we conclude that $(a, f(a) + j) \in H_{J\bowtie^f H}$.
(3) Let $(a, f(a)+j) \in H_{J\bowtie^f H}$. Hence, there exists $(b, f(b)+k) \notin J \bowtie^f H$ such that $(a, f(a)+j)(b, f(b)+k) = (ab, f(ab) + jf(b) + k(f(a) + j)) \in J \bowtie^f H$. If $a \notin J$, then the fact that $ab \in J$ implies that $b \in J$ because J is prime. So, $k \notin H$ and thus $(f(a) + j)k \in H$ since $jf(b) + k(f(a) + j) \in H$ and $f(J)\mathcal{J} \subseteq H$. Suppose that $a \in J$. Since $\mathcal{J} \subseteq H_H$, there exists $f(c) + t \notin H$ such that $j(f(c) + t) \in H$. Thus, $(a, f(a)+j)(c, f(c)+t) = (a, f(a))(c, f(c)+t)+(0, j(f(c)+t)) \in J \bowtie^f H$ with $(c, f(c)+t) \notin J \bowtie^f H$. In all cases we have $(a, f(a)+j) \in L_2$. For the converse, let $(a, f(a)+j) \in L_2$, then $(f(a)+j)k \in H$ for some $k \notin H$. Hence, $(a, f(a)+j)(0, k) \in J \bowtie^f H$ and $(0, k) \notin J \bowtie^f H$ which implies that $(a, f(a)+j) \in H_{J\bowtie^f H}$. Therefore, $H_{J\bowtie^f H} = L_2$. □

Theorem 2.6 *Let $f : A \to B$ be a ring homomorphism and $\mathcal{J}$ be an ideal of B. Let J (resp., H) be an ideal of A (resp., $f(A) + \mathcal{J}$) such that $f(J)\mathcal{J} \subseteq H \subseteq \mathcal{J}$. Let I be an ideal of A and K an ideal of $f(A) + \mathcal{J}$. Then, the following statements hold:*

1. *Assume that $H_{J\bowtie^f H} = H_J \bowtie^f \mathcal{J}$. Then, $I \bowtie^f \mathcal{J}$ is an $H_{J\bowtie^f H}$-ideal of $A \bowtie^f \mathcal{J}$ if and only if I is an H_J-ideal of A.*

2. *Assume that $H_{J\bowtie^f H} = \overline{H_H}^f$. Then, $\overline{K}^f$ is an $H_{J\bowtie^f H}$-ideal of $A \bowtie^f \mathcal{J}$ if and only if K is an H_H-ideal of $f(A) + \mathcal{J}$.*

Proof: (1) Let $a, b \in A$ such that $ab \in I$ and $a \notin H_J$. So, it is clear that $(a, f(a))(b, f(b)) = (ab, f(ab)) \in I \bowtie^f \mathcal{J}$ and $(a, f(a)) \notin H_{J\bowtie^f H}$. Thus, since $I \bowtie^f \mathcal{J}$ is an $H_{J\bowtie^f H}$-ideal, we conclude that $(b, f(b)) \in I \bowtie^f \mathcal{J}$ and so $b \in I$. Conversely, assume that I is an H_J-ideal of A and let $(a, f(a)+j)(b, f(b)+k) \in I \bowtie^f \mathcal{J}$ for $(b, f(b) + k) \in A \bowtie^f \mathcal{J}$ and $(a, f(a) + j) \in A \bowtie^f \mathcal{J} \setminus H_{J\bowtie^f H}$. Thus, by assumption, we have $ab \in I$ such that $a \notin H_J$. This implies that $b \in I$ and so $(b, f(b) + k) \in I \bowtie^f \mathcal{J}$. Therefore, $I \bowtie^f \mathcal{J}$ is an $H_{J\bowtie^f H}$-ideal of $A \bowtie^f \mathcal{J}$.
(2) Let $(f(a) + j)(f(b) + k) \in K$ for some $f(a) + j \in f(A) + \mathcal{J} \setminus H_H$ and $f(b) + k \in f(A) + \mathcal{J}$. Hence, $(a, f(a) + j)(b, f(b) + k) \in \overline{K}^f$ and $(a, f(a) + j) \notin H_{J\bowtie^f H}$ because $H_{J\bowtie^f H} = \overline{H_H}^f$. It implies that $(b, f(b) + k) \in \overline{K}^f$ since $\overline{K}^f$ is an $H_{J\bowtie^f H}$-ideal. Therefore, $f(b) + k \in K$ and thus K is an H_H-ideal of $f(A) + \mathcal{J}$. For the converse, let $(a, f(a) + j)(b, f(b) + k) \in \overline{K}^f$ such that $(b, f(b) + k) \in A \bowtie^f \mathcal{J}$ and $(a, f(a) + j) \in A \bowtie^f \mathcal{J} \setminus H_{J\bowtie^f H}$. As $H_{J\bowtie^f H} = \overline{H_H}^f$, we then have $(f(a) + j)(f(b) + k) \in K$ with $f(a) + j \notin H_H$. Therefore, $f(b) + k \in K$ and hence $(b, f(b) + k) \in \overline{K}^f$. □

In the next example, we show that the conditions that $H_{J\bowtie^f H} = H_J \bowtie^f \mathcal{J}$ in (1) of Theorem 2.6 cannot be eliminated.

Example 2.3 *Let $A = \mathbb{Z}$, $J = \{0\}$, and $I = 2\mathbb{Z}$. Let $B = \mathbb{Z}_4$, $H = \{\overline{0}\}$ and $\mathcal{J} = \{\overline{0}, \overline{2}\}$ be two ideals of B. Consider the ring homomorphism $f : A \to B$, defined by $f(a) = \overline{a}$. It's clear that $f(J)\mathcal{J} \subseteq H \subseteq \mathcal{J}$. Which implies that $J \bowtie^f H$ is an ideal of $A \bowtie^f \mathcal{J}$. Now, we can show that $H_J = \{0\}$ and $H_{J\bowtie^f H} = 2\mathbb{Z} \bowtie^f \mathcal{J}$. Since I is a prime ideal of A, then $I \bowtie^f \mathcal{J}$ is prime ideal of $A \bowtie^f \mathcal{J}$. As $H_{J\bowtie^f H} = I \bowtie^f \mathcal{J}$, by Proposition 2.3(2) we get that $I \bowtie^f \mathcal{J}$ is an $H_{J\bowtie^f H}$-ideal of $A \bowtie^f \mathcal{J}$. But, $I \nsubseteq H_J$ and so I is not an H_J-ideal by Proposition 2.3(1).*

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