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Planarity and Genus of Co-Unit Graphs in direct product of local rings

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ABSTRACT: This research article focuses on the co-unit graph, denoted as $G_{nu}(R)$, which is associated with a commutative ring R. The vertex set of $G_{nu}(R)$ is $U(R)$, which represents the set of units of R, and two distinct vertices x and y of $U(R)$ are considered to be adjacent if and only if $x + y \notin U(R)$. The objective of the research article is to characterize the ring R, where R is taken as the direct product of local rings $\mathbb{Z}_n$, and determine whether $G_{nu}(R)$ is planar, outerplanar, a tree, or a split graph. Moreover, the study aims to identify the rings for which $G_{nu}(R)$ has a genus of one. Overall, this research article investigates the properties of $G_{nu}(R)$ and explores the characteristics of rings that lead to specific graph structures.

Key Words: Unit graph, co-unit graph, planar graph, genus of a graph.

Contents

1. Introduction

The investigation of graphs associated with algebraic structures is a rapidly growing field, with a focus on classifying the graphs of algebraic structures and vice versa. Researchers are particularly interested in understanding the relationship between the algebraic structure of a known object and the graph-theoretic properties of a similar graph. When a combinatorial object is assigned to an algebraic structure, it often leads to intriguing problems in both algebra and combinatorics. Currently, one of the most active areas of research in this field is the study of graphs associated with commutative rings. Commutative rings are a fundamental algebraic structure in mathematics and have many applications in diverse areas such as coding theory, cryptography, and algebraic geometry. In this paper, we will explore the relationship between commutative rings and their associated graphs, focusing on characterizing the properties of these graphs for specific types of commutative rings.

A finite commutative ring R has a set of units called $U(R)$. If $R \cong R_1 \times R_2 \times \cdots \times R_n$ and each R_i has a unit e_i, then the tuple $(e_1, e_2, e_3, \ldots, e_n)$ is the unit element of $R_1 \times R_2 \times \cdots \times R_n$.

Grimaldi [14] introduced the concept of the unit graph $G(\mathbb{Z}_n)$, where $\mathbb{Z}_n$ represents the ring of integers modulo n. The vertices of $G(\mathbb{Z}_n)$ are the elements of $\mathbb{Z}_n$, and two vertices x and y are connected by an edge if and only if their sum is a unit of $\mathbb{Z}_n$. Building upon this concept, Ashrafi et al. [11] extended the idea to $G(R)$, where R is any associative ring with a nonzero identity. Thus, the unit graph $G(\mathbb{Z}_n)$ is a specific example of the more general unit graph $G(R)$. The paper references previous literature on the topic and aims to contribute to the existing body of knowledge on co-unit graphs, and several related works have been published in the literature, see [5,6,7,8,10,16].

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2. Preliminaries

In this section, we summarize notation, concepts and some results related to the graph which will be needed in the subsequent sections.

A graph $G(V, E)$ is a mathematical structure consisting of a set of vertices V (nodes) and a set of edges E connecting these vertices. Graphs are commonly used to represent relationships between objects or entities. A path P_n in a graph is a sequence of edges that connects a sequence of vertices. P_n is said to be simple if it does not contain any repeated vertices. A cycle in a graph is a closed path that starts and ends at the same vertex. A cycle is said to be simple if it does not contain any repeated vertices or edges. A connected graph is a graph in which there is a path between every pair of vertices. In other words, every vertex is reachable from every other vertex. A disconnected graph is a graph that is not connected. It consists of two or more connected components, each of which is a connected subgraph. A complete graph is a graph in which every pair of distinct vertices is connected by an edge. A bipartite graph is a graph in which the vertices can be partitioned into two disjoint sets such that every edge connects a vertex from one set to a vertex in the other set. A tree is a connected graph that does not contain any cycles. Topological graph theory is concerned with finding ways to embed a graph onto a surface, with the minimum number of handles, called the genus of the graph. The surface is represented as S_k, where k is the number of handles. The goal is to draw the graph in a way that the edges only intersect at their vertices. A planar graph is one with genus 0, while a toroidal graph has genus 1. The genus of a subgraph is less than or equal to that of the main graph. Euler's formula states that for a finite, connected graph with n vertices, e edges, and genus γ, the equation $n - e + f = 2 - 2\gamma$ holds true, where f is the number of faces created when the graph is embedded in S_γ. This formula, along with combinatorial identities and inequalities, can be used to determine if certain embeddings exist. For more information on embedding graphs onto surfaces we refer the reader to [1,2,12,13].

The units of a ring R are the elements that have multiplicative inverse in R. A local ring R is a ring that has a unique maximal ideal. The decomposition theorem on finite commutative rings states that every finite commutative ring can be decomposed as a direct product of local rings. This decomposition is unique up to isomorphism.

Recently, Pirzada et al. [15] introduced the *co-unit graph* associated to a ring R, denoted by $G_{nu}(R)$, with vertex set $U(R)$ and two vertices $x, y \in U(R)$ are adjacent if and only if $x + y \notin U(R)$.

In this paper, we focus on the properties of $G_{nu}(R)$ for finite commutative ring R where R is taken as the direct products of finite local rings $\mathbb{Z}_n$. We investigate when $G_{nu}(R)$ is planar, outerplanar, a tree, or a split graph. In addition, we identify the rings for which $G_{nu}(R)$ has a genus of one. Our analysis contributes to the broader understanding of algebraic structures and their associated combinatorial objects and provides insights into the relationship between commutative rings and their associated graphs.

3. Planarity of co-unit graphs

In this section, we provide a characterization of the finite commutative ring with unity that satisfies certain properties such as being a planar, outer planar, tree, or a split graph with respect to its $G_{nu}(R)$ structure.

Theorem 3.1 *[4] (Kuratowski's Theorem) A graph G is planar if and only if it does not contains subdivision of K_5 or $K_{3,3}$.*

Theorem 3.2 *Let R tative ring with unity. Then $G_{nu}(R)$ is a planar graph if and only one of the following holds:*

(1) $R \cong \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

(2) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_4$.

(3) $R \cong \mathbb{Z}_8$, $\mathbb{Z}_4$ *or* $\mathbb{Z}_p$, *where p is prime.*

Proof: Suppose $G_{nu}(R)$ is planar. Since R is finite commutative ring with unity. Then by the Structure Theorem $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a local ring for each $1 \leq i \leq n$ and $n \geq 1$. Hence, $|R_i| = p_j^m$, where p_j is prime for each $j = 1, 2, \ldots, n$ and $m \geq 1$ is a positive integer. For $n \geq 3$ and if $p_j^m \geq 3$ for atleast $3i$'s, then $\{1, p_j^m - 1\} \in U(R_i)$ such that $G_{nu}(R)$ has K_5 induced by the set $S = \{(1, 1, \ldots, 1), (p_1^m - 1, 1, \ldots, 1), (1, p_2^m - 1, \ldots, 1), (1, 1, p_3^m - 1, 1, \ldots, 1), (p_1^m - 1, p_2^m - 1, p_3^m - 1, 1 \ldots, 1)\}$, a contradiction. Hence, $p_j^m \leq 2$ i,e., $|R_i| \leq 2$ for some i's. So, fix $|R_3| = |R_4| = \cdots = |R_n| = 2$, i,e., $R \cong R_1 \times R_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. If R_1 and R_2 are fields and $|R_i| \geq 7$ for atleast one i where $i = 1, 2$, say $|R_1| \geq 7$, then $|U(R_1)| \geq 6$. Let $\{1, f_1, f_2, f_3, f_4, f_5\} \in U(R_1)$ then $G_{nu}(R)$ has K_5 generated by the set $S = \{(1, 1, \ldots, 1), (f_1, 1, \ldots, 1), (f_2, 1, \ldots, 1), (f_3, 1, \ldots, 1), (f_4, 1, 1, \ldots, 1)\}$, a contradiction. Hence, $|R_i| \leq 5$, where $i = 1, 2$. Implies $|R_i| = 5$ or 3 or 2 for $i = 1, 2$. If $|R_1| = 5$ and $3 \leq |R_2| \leq 5$, then $|U(R_2)| \geq 2$. Let $\{1, x\} \in U(R_2)$, then $G_{nu}(R)$ has K_5 generated by the set $S = \{(1, 1, \ldots, 1), (e_1, 1, \ldots, 1), (e_2, 1, \ldots, 1), (e_3, 1, \ldots, 1), (1, x, 1, \ldots, 1)\}$, a contradiction, where $\{1, e_1, e_2, e_3\} \in U(R_1)$. Hence, $R \cong \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. Also, if R_1 and R_2 are not fields and suppose $|R_i| = p_j^m \geq 9$ for atleast one i, say $|R_1| \geq 9$, then $|U(R_1)| \geq 6$. Let $\{1, x_1, x_2, x_3, x_4, x_6\} \in U(R_1)$, then $G_{nu}(R)$ has K_5 generated by the set $S = \{(1, 1, \ldots, 1), (x_1, 1, \ldots, 1), (x_2, 1, \ldots, 1), (x_3, 1, \ldots, 1), (x_4, 1, \ldots, 1)\}$, a contradiction. Hence, $|R_i| \leq 8$, where $i = 1, 2$, implies R_i is $\mathbb{Z}_8$ or $\mathbb{Z}_4$. If $R_1 \times R_2 \cong \mathbb{Z}_8 \times \mathbb{Z}_8$ or $\mathbb{Z}_8 \times \mathbb{Z}_4$, then $G_{nu}(R)$ has K_5 by [15, Theorem 6], a contradiction. Hence, $R \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. Moreover, suppose R_1 is not a field and R_2 is a field, if $|R_2| \geq 7$, then $|U(R_2)| \geq 6$. Let $\{1, e_1, e_2, e_3, e_4, e_5\} \in U(R_2)$, then $G_{nu}(R)$ has K_5 induced by the set $S = \{(1, 1, \ldots, 1), (1, e_1, 1, \ldots, 1), (1, e_2, 1, \ldots, 1), (1, e_3, 1, \ldots, 1), (1, e_4, 1, \ldots, 1)\}$, a contradiction. Hence, $|R_2| \leq 5$, implies $R_2 \cong \mathbb{Z}_5$, $\mathbb{Z}_3$ or $\mathbb{Z}_2$. If $|R_1| \geq 9$, then $|U(R_1)| \geq 6$. Thus with the similar arguments as above, $G_{nu}(R)$ has K_5, a contradiction. Hence, $|R_1| \leq 8$, implies $R_1 \cong \mathbb{Z}_8$ or $\mathbb{Z}_4$. Moreover, if $R_1 \times R_2 \cong \mathbb{Z}_8 \times \mathbb{Z}_5$, $\mathbb{Z}_8 \times \mathbb{Z}_3$, or $\mathbb{Z}_4 \times \mathbb{Z}_3$, then $G_{nu}(R)$ has K_5 by [15, Theorem 6], a contradiction. Hence, $R \cong \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.
For $n = 2$ i,e., $R \cong R_1 \times R_2$. Then the following cases arises:
Case(i): If R_1 and R_2 are fields and $|R_i| = p_j \geq 7$ for atleast one i, say $|R_1| = p_j \geq 7$ where p_j is prime for each $j = 1, 2$, then $|U(R_1)| \geq 6$. Let $\{1, x, y, p_1 - x, p_1 - y, p_1 - 1\} \in U(R_i)$. Then $G_{nu}(R)$ has a subgraph formed by the set $\{(1, 1), (x, 1), (y, 1), (1, p_2 - 1), (x, p_2 - 1), (y, p_2 - 1)\} \cong K_{3,3}$ where $p_2 \neq 2$, a contradiction. In case $p_2 = 2$, then $G_{nu}(R)$ has K_5 generated by $\{(1, 1), (x, 1), (y, 1), (p_1 - x, 1), (p_1 - y, 1)\}$, again a contradiction. Hence, $|R_i| \leq 5$, where $i = 1, 2$. Implies R_i is one of the rings $\mathbb{Z}_5$, $\mathbb{Z}_3$ or $\mathbb{Z}_2$. If $R \cong \mathbb{Z}_5 \times \mathbb{Z}_5$, then by Lemma 5.3, it can be seen that $\gamma(G){>}0$, a contradiction. Hence, $R \cong \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2$.
Case(ii): If R_1 and R_2 are not fields. Since R_1 and R_2 are finite local rings, therefore $|R_i| = p_j^s$ for each $1 \leq i \leq n$, where p_j is prime number for each $j = 1, 2, \ldots, n$ and $s \geq 2$ is a positive integer. If $p_j^s \geq 9$ for atleast one i, say $|R_1| = p_j^s \geq 9$ then $U(R_1) \geq 6$. Let $\langle m \rangle \neq \langle 2 \rangle$ be the non-zero maximal ideal of R_1, then $\{1, m - 1, 2m - 1, 3m - 1\} \in U(R_1)$ such that $G_{nu}(R)$ has $K_{3,3}$ of the form $S = \{(m-1, 1), (2m-1, 1), (3m-1, 1), (m-1, p_2^s - 1), (2m-1, p_2^s - 1), (3m-1, p_2^s - 1)\}$, a contradiction. Also, in case $\langle m \rangle = \langle 2 \rangle$, then for $p_j^s \geq 16$ for atleast one i, say $|R_1| = p_j^s \geq 16$, $G_{nu}(R)$ has K_5 by [15, Theorem 6], a contradiction. Hence, $|R_i| = p_j^s \leq 8$ for each $i = 1, 2$. Implies R_i is $\mathbb{Z}_8$ or $\mathbb{Z}_4$ for each $i = 1, 2$. If $R \cong \mathbb{Z}_8 \times \mathbb{Z}_8$ or $R \cong \mathbb{Z}_8 \times \mathbb{Z}_4$, then $G_{nu}(R)$ has K_5 by [15, Theorem 6], a contradiction. Hence, $R \cong \mathbb{Z}_4 \times \mathbb{Z}_4$.
Case(iii): If R_1 is a field and R_2 is not a field. Suppose $|R_1| = p_j \geq 5$, where p_j is prime for each $j = 1, 2, \ldots, n$. If $|R_2| = p_j^s \geq 9$, where $s \geq 2$ is a positive integer, then $|U(R_2)| \geq 6$. Let $\langle m \rangle \neq \langle 2 \rangle$ be the non-zero maximal ideal of R_2, then $\{1, m - 1, 2m - 1, 3m - 1\} \in U(R_2)$ such that $G_{nu}(R)$ has $K_{3,3}$ generated by the set $T = \{(1, m-1), (1, 2m-1), (1, 3m-1), (p^j-1, m-1), (p^j-1, 2m-1), (p^j-1, 3m-1)\}$, a contradiction. Also, in case $\langle m \rangle = \langle 2 \rangle$ then for $|R_2| = p_j^s \geq 4$, $m+1 \in U(R_2)$ such that $G_{nu}(R)$ has K_5 generated by the set $S = \{(1, 1), (x_1, 1), (x_2, 1), (x_3, 1), (1, m+1)\}$, a contradiction. Hence, $|R_1| = p_j \leq 3$. Fix $|R_1| = 2$ and if $|R_2| \geq 9$, then $G_{nu}(R)$ has K_5 by [15, Theorem 6], a contradiction. Hence, $|R_2| \leq 8$, implies R_2 is $\mathbb{Z}_8$ or $\mathbb{Z}_4$. Hence, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_8$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$. Also, fix $|R_1| = p_j = 3$, then for $|R_2| = p_j^s \geq 9$, $U(R_2) \geq 6$. Let $\langle m \rangle \neq \langle 2 \rangle$ be the non-zero maximal ideal of R_2, then $\{1, m-1, 2m-1, 3m-1\} \in U(R_2)$ such that $G_{nu}(R)$ has $K_{3,3}$ generated by the set $T = \{(1, m - 1), (1, 2m - 1), (1, 3m - 1), (p^j - 1, m - 1), (p^j - 1, 2m - 1), (p^j - 1, 3m - 1)\}$, a contradiction. Also, in case $\langle m \rangle = \langle 2 \rangle$ then for $|R_2| = p_j^s \geq 8$, $m+1 \in U(R_2)$ such that $G_{nu}(R)$ has K_5 generated by the set $S = \{(1, 1), (x_1, 1), (x_2, 1), (x_3, 1), (1, m+1)\}$,

a contradiction. Hence, $|R_2| = p_j^s \leq 4$, implies R_2 is $\mathbb{Z}_4$, and hence $R \cong \mathbb{Z}_3 \times \mathbb{Z}_4$.
Finally, for $n = 1$, implies R is a finite local ring. If R is a field. Then by [15, Theorem 1], $R \cong \mathbb{Z}_p$, where p is prime. If R is not a field and $|R| = p^s \geq 16$, then for $p = 2$, $G_{nu}(R)$ has K_5 by [15, Theorem 6], a contradiction. And if $p \neq 2$, then for $|R| = p^s \geq 9$, $|U(R)| \geq 6$. Let $\langle m \rangle$ be the non-zero maximal ideal of R_2 and $\langle m \rangle \neq \langle 2 \rangle$, then $G = \{1, m+1, 2m+1, m-1, 2m-1, 3m-1\} \in U(R)$ such that $G_{nu}(R)$ has $K_{3,3}$ generated by the set G, a contradiction. Hence, $R \cong \mathbb{Z}_8$ or $\mathbb{Z}_4$.
Conversely, if $R \cong \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2, \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_8$, then $G_{nu}(R) \cong K_4$ and the planar embedding of K_4 is given in Figure 1(b). If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_4$, then $G_{nu}(R) \cong K_2$ is a planar graph, and if $R \cong Z_p$, where p is odd prime, then $G_{nu}(R) \cong \frac{(p-1)}{2} K_2$ and is a planar graph. Also, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2$ then $G_{nu}(R) \cong K_1$ is a planar graph. Moreover, the planar embedding of $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_5)$ is given in Figure 1(a) and hence, the proof is complete. □

Theorem 3.3 *[4] The graph G is outerplanar if and only if it contains no subgraph that is a subdivision of either K_4 or $K_{2,3}$.*

Theorem 3.4 *Let R be a finite commutative ring with unity. Then $G_{nu}(R)$ is an outerplanar graph if and only if R is one of the following rings:*

(1) $R \cong \mathbb{Z}_i \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where $i = 2, 3, 4$.

(2) $R \cong \mathbb{Z}_4$, $\mathbb{Z}_p$, where p is prime.

Proof: Suppose $G_{nu}(R)$ is outerplanar graph. Since R is finite commutative ring with unity. Then by the Structure Theorem $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a local ring for each $1 \leq i \leq n$ and $n \geq 1$. Now, for $n \geq 2$, if $|R_i| \geq 3$ for atleast two i's, then $|U(R_i)| \geq 2$. Let $\langle m_i \rangle$ be the non-zero maximal ideal of R_i. If $\langle m_i \rangle = \langle 2 \rangle$, then $\{1, m_i + 1\} \in U(R_i)$ such that $G_{nu}(R)$ has a subgraph generated by the set $S = \{(1, 1, \ldots, 1), (m_i + 1, 1, \ldots, 1), (1, m_2 + 1, 1, \ldots, 1), (m_1 + 1, m_2 + 1, 1, \ldots, 1)\} \cong K_4$, a contradiction. Hence, $|R_i| = 2$ for some i. Fix $|R_2| = |R_3| = \cdots = |R_n| = 2$ i,e., $R \cong R_1 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. Now, suppose $|R_1| \geq 5$, then $|U(R_1)| \geq 4$. Let $\{1, x_1, x_2, x_3\} \in U(R_1)$, then $G_{nu}(R)$ has K_4 generated by the set $S = \{(1, 1, \ldots, 1), (x_1, 1, \ldots, 1), (x_2, 1, \ldots, 1), (x_3, 1, \ldots, 1)\}$, a contradiction. Hence, $|R_1| \leq 4$, implies R_1 is one of the rings $\mathbb{Z}_4, \mathbb{Z}_3$ or $\mathbb{Z}_2$. Thus, $R \cong \mathbb{Z}_i \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2, \quad i = 2, 3, 4$.
Finally, for $n = 1$, implies R is a local ring and $|R| = p^s$, where p is a prime and $s \geq 1$ is a positive integer. Suppose R is not a field. Let $\langle m \rangle$ be the non-zero maximal ideal of R. If $\langle m \rangle = \langle 2 \rangle$, then for $|R| \geq 8$, $G_{nu}(R)$ has K_4 by [15, Theorem 6], a contradiction. Also, if $\langle m \rangle \neq \langle 2 \rangle$, then for $|R| \geq 9$, $G_{nu}(R)$ has $K_{2,3}$ generated by the set $S = \{1, m+1, m-1, 2m-1, 3m-1\}$, where $\{1, m+1, m-1, 2m-1, 3m-1\} \in U(R)$, again a contradiction. Hence, $R \cong \mathbb{Z}_4$. Moreover, if R is a field, then by [15, Theorem 1], $R \cong \mathbb{Z}_p$, where p is a prime.

Converse follows from Figure 2. □

Theorem 3.5 *Let R be a finite commutative ring with unity. Then $G_{nu}(R)$ is a tree if and only if R is one of the following rings:*

(1) $R \cong \mathbb{Z}_i \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where $i = 2, 3, 4$.

(2) $R \cong \mathbb{Z}_2$, $\mathbb{Z}_3$, or $\mathbb{Z}_4$.

Proof: Suppose $G_{nu}(R)$ is tree. Since R is finite commutative ring with unity. Then by the Structure Theorem $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a local ring for each $1 \leq i \leq n$ ans $n \geq 1$. For $n \geq 2$, suppose $|R_i| \geq 3$ for atleast two i's, let $\langle m_i \rangle$ be the maximal ideal of R_i and $\langle m_i \rangle \neq \langle 2 \rangle$, then $m_i - 1 \in U(R_i)$ such that $G_{nu}(R)$ has C_3 generated by the set $S = \{(1, 1, \ldots, 1), (m_i - 1, 1, \ldots, 1), (1, m_i - 1, 1, \ldots, 1)\}$,

a contradiction. Also, in case $\langle m_i \rangle = \langle 2 \rangle$, then $m_i + 1 \in U(R_i)$ such that $G_{nu}(R)$ has cycle formed by the set $F = \{(1, 1, \ldots, 1), (m_i + 1, 1, \ldots, 1), (1, m_i + 1, 1, \ldots, 1)\}$, a contradiction. Hence, $|R_i| = 2$ for some i and so fix $|R_i| = 2$ for $i = 2, 3, \ldots, n$. i,e., $R_i \cong \mathbb{Z}_2$ for $i = 2, 3, \ldots, n$. Moreover, if $|R_1| \geq 5$, then $|U(R_1)| \geq 4$. Let $\{1, g_1, g_2, g_3\} \in U(R_1)$, clearly $G_{nu}(R)$ has a triangle formed by $W = \{(1, 1, \ldots, 1), (g_1, \ldots, 1), (g_2, 1, \ldots, 1)\}$, a contradiction. This implies that $|R_1| \leq 4$, and hence, R_1 is one of the rings $\mathbb{Z}_4$, $\mathbb{Z}_3$ or $\mathbb{Z}_2$. Therefore, $R \cong \mathbb{Z}_i \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where $i = 2, 3, 4$.
Finally, for $n = 1$, implies R is a finite local ring. If R is a field and $|R| \geq 5$, then $|U(R)| \geq 4$. Therefore, for every $R \setminus \{0\} \in U(R)$, there exists x and y such that $x + y = 0$ and contains more than one copies of K_2, a contradiction. Hence, $|R| \leq 4$, implies R is one of the rings $\mathbb{Z}_3$ or $\mathbb{Z}_2$. Also, if R is a local ring but not field and if $|R| = 2^s$, where $s \geq 3$ is a positive integer, then by [15, Theorem 6] $G_{nu}(R)$ has K_3, a contradiction. If $|R|$ is not of the form 2^s, then for $|R| \geq 9$, $|U(R)| \geq 6$ with $\langle m \rangle$ as non zero maximal ideal, it can be seen that $G_{nu}(R)$ has a cycle formed by the set of units $S = \{1, m-1, 2m-1, 2m+1\}$, a contradiction. Hence, $|R| \leq 4$ implies $R \cong \mathbb{Z}_4$.

Conversely, $G_{nu}(\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2) \cong K_1$ is a tree and $G_{nu}(\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2) \cong K_2$ is a tree. Hence, the proof is complete.

□

Theorem 3.6 *[4] Let G be a connected graph, then G is a split graph if and only if G contains no induced subgraph isomorphic to $2K_2, C_4$ or C_5.*

Theorem 3.7 *Let R be a finite commutative ring with unity. Then $G_{nu}(R)$ is a split graph if and only if R is one of the following rings:*

(1) $R \cong \mathbb{Z}_4$, $\mathbb{Z}_p$, where $p = 2$ or 3 are only primes.

(2) $R \cong \mathbb{Z}_i \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where $i = 2, 3, 4$.

Proof: Suppose $G_{nu}(R)$ is a split graph. Since R is finite commutative ring with unity. Then by the Structure Theorem $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a local ring for each $1 \leq i \leq n$ and $n \geq 1$. Now, for $n \geq 2$, if $|R_i| \geq 3$ for atleast two i's, then $|U(R_i)| \geq 2$. Let $\langle m_i \rangle$ be the maximal ideal of R_i. Clearly $m_i + 1, m_i - 1 \in U(R_i)$. Now, when $\langle m_i \rangle \neq \langle 2 \rangle$, then $m_i - 1 \in U(R_i)$ such that $G_{nu}(R)$ has C_4 generated by the set $S = \{(1, 1, \ldots, 1), (m_1 - 1, 1, \ldots, 1), (1, m_2 - 1, 1, \ldots, 1), (m_1 - 1, m_2 - 1, 1, \ldots, 1)\}$, a contradiction. Also, when $\langle m_i \rangle = \langle 2 \rangle$, then $m_i + 1 \in U(R_i)$ such that the subgraph generated by the set $S = \{(1, 1, \ldots, 1), (m_1 + 1, 1, \ldots, 1), (1, m_2 + 1, 1, \ldots, 1), (m_1 + 1, m_2 + 1, 1, \ldots, 1)\}$ in $G_{nu}(R)$ is C_4, a contradiction. Therefore, $|R_i| \leq 2$ for some i, and hence fix $|R_2| = |R_3| = \cdots = |R_n| = 2$, i,e., $R \cong R_1 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. If $|R_1| \geq 5$, then $|U(R_1)| \geq 4$. Let $\{1, f_1, f_2, f_3\} \in U(R_1)$ then $G_{nu}(R)$ has C_4 induced by the set $S = \{(1, 1, \ldots, 1), (f_1, 1, \ldots, 1), (f_2, 1, \ldots, 1), (f_3, 1, \ldots, 1)\}$, a contradiction. Hence, $|R_1| \leq 4$, implies R_1 is one of the rings $\mathbb{Z}_4$, $\mathbb{Z}_3$ or $\mathbb{Z}_2$. Therefore, $R \cong \mathbb{Z}_i \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where $i = 2, 3, 4$. Also, for $n = 1$, implies R is a finite local ring. Suppose R is not a field. Since $|R| = p^s$, where p is a prime and $s \geq 2$ is a positive integer, since $\langle p \rangle$ is maximal ideal of R. In case $\langle p \rangle \neq \langle 2 \rangle$, then $\{1, p-1, p+1, 2p-1\} \in U(R)$ such that $G_{nu}(R)$ has C_4 generated by the set $S = \{1, p-1, p+1, 2p-1\}$, a contradiction. And when $\langle p \rangle = \langle 2 \rangle$ then for $s \geq 3$, i,e., $|R| \geq 8$, $G_{nu}(R)$ has C_4 generated by the set $S = \{1, p+1, 2p+1, 3p+1\}$, a contradiction. Hence, $|R| \leq 4$, implies $R \cong \mathbb{Z}_4$. Finally, for $s = 1$ i,e., $|R| = p$, then $R \cong \mathbb{Z}_p$. Suppose $p \geq 5$, then according to [15, Theorem 1] $G_{nu}(R)$ induces atleast 2 copies of k_2, leads to a contradiction. Hence, $p = 2$ or 3.

Conversely, $G_{nu}(\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2) \cong K_1$ is a split graph and $G_{nu}(\mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2) \cong K_2$, $G_{nu}(\mathbb{Z}_3) \cong K_2$, are split graphs. Hence, the proof is complete. □

4. Figures

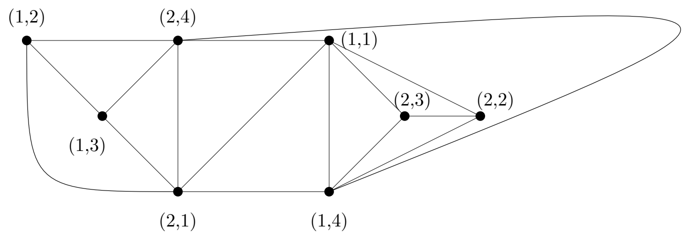


Figure 1(a). $G_{nu}(\mathbb{Z}_3 \times \mathbb{Z}_5)$.

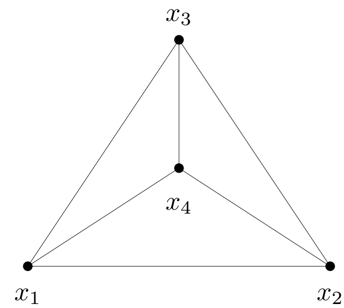


Figure 1(b). $G_{nu}(R) \cong K_4$.

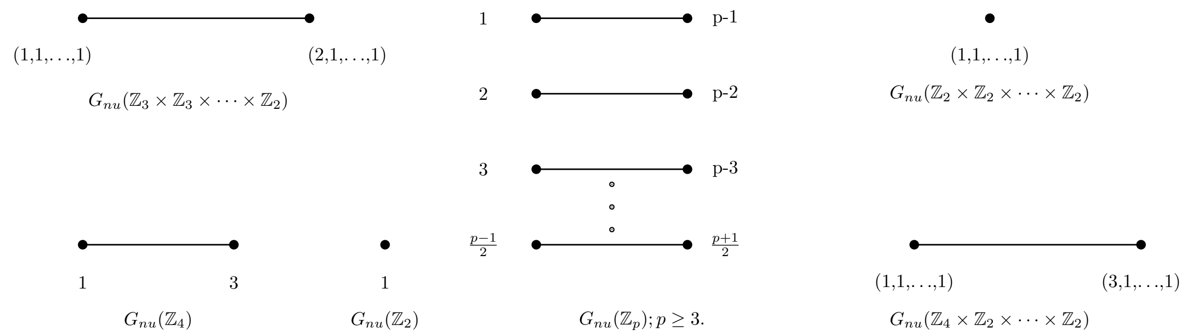
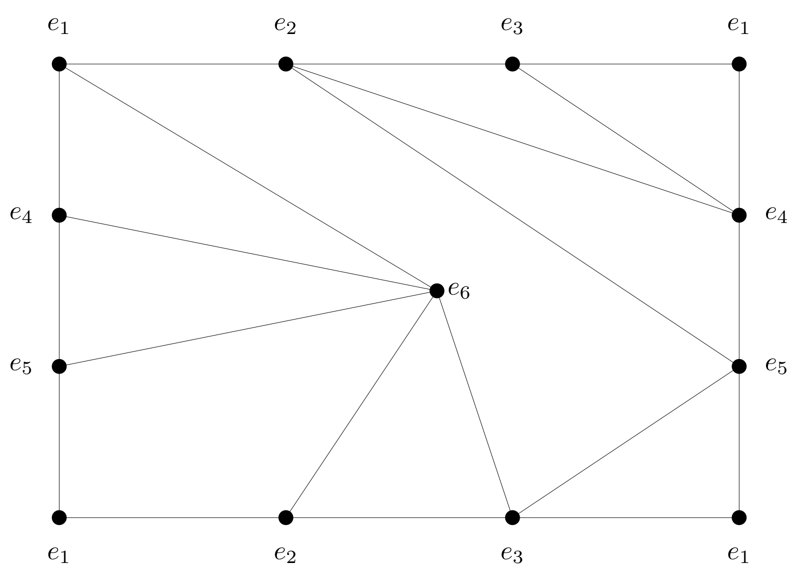


Figure 2

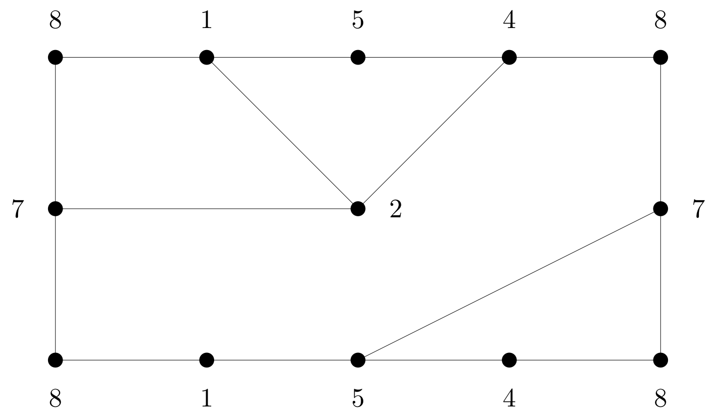


Toroidal embedding of $G_{nu}(\mathbb{Z}_7 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2, \mathbb{Z}_9 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2,\ \ \mathbb{Z}_2 \times \mathbb{Z}_7$ **or** $\mathbb{Z}_2 \times \mathbb{Z}_9)$

Figure 3

where $e_1 = (1, 1, \ldots, 1)$ or $(1, 1)$, $e_2 = (2, 1, \ldots, 1)$ or $(1, 2)$, $e_3 = (3, 1, \ldots, 1)$ or $(7, 1, \ldots, 1)$ or $(1, 3)$ or $(1, 7)$, $e_4 = (4, 1, \ldots, 1)$ or $(1, 4)$, $e_5 = (5, 1, \ldots, 1)$ or $(1, 5)$, and $e_6 = (6, 1, \ldots, 1)$ or

(8, 1, . . . , 1) or (1, 6), or (1, 8).



Toroidal embedding of $G_{nu}(\mathbb{Z}_9)$

Figure 4

5. Genus of $G_{nu}(R)$

In this section, we will analyze the finite commutative rings R and determine the conditions under which the genus of the graph $G_{nu}(R)$ is equal to one.

The genus of a graph G, denoted by $\gamma(G)$, is defined as the minimum integer k such that the graph can be drawn on a sphere with k handles without any self-crossing. We will also consider the genus of complete and complete bipartite graphs in our analysis. Our findings will contribute to a better understanding of the algebraic and topological properties of finite commutative rings.

Lemma 5.1 *[1] Let $n, m \geq 2$. Then*

$$\gamma(K_{m,n}) = \lceil \frac{1}{4}(m-2)(n-2) \rceil.$$

In particular, $\gamma(K_{4,4}) = \gamma(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$. Also, $\gamma(K_{5,4}) = \gamma(K_{6,4}) = \gamma(K_{3,m}) = 2$, if $m = 7, 8, 9, 10$.

Lemma 5.2 *[1] Let $n \geq 3$. Then*

$$\gamma(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil.$$

In particular, $\gamma(K_n) = 1$ if $n = 5, 6, 7$ and $\gamma(K_8) = 2$.

Lemma 5.3 *[3, Proposition 4.4.4] Let G be a connected graph with q edges and $m \geq 3$ vertices. Then*

$$\gamma(G) \geq \left\lceil \frac{q}{6} - \frac{m}{2} + 1 \right\rceil.$$

Theorem 5.1 *[9] The genus of a graph is the sum of the genera of its blocks.*

We are now able to describe finite commutative rings R in which the co-unit graph $G_{nu}(R)$ has genus one.

Theorem 5.2 *Let R be a finite local commutative ring. Then $\gamma(G_{nu}(R)) = 1$ if and only if one of the following holds:*

(1) $R \cong \mathbb{Z}_7 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, or $\mathbb{Z}_9 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

(2) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_7$ or $\mathbb{Z}_2 \times \mathbb{Z}_9$.

(3) $R \cong \mathbb{Z}_9$.

Proof: Suppose $\gamma(G_{nu}(R)) = 1$. Since R is finite commutative ring with unity. Then by the Structure Theorem $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a finite local ring for each $1 \leq i \leq n$, $n \geq 1$ and $|R_i| = q_i^m$ for each i where q_i is prime for each i and m is a positive integer. If $|R_i| = q_i^m \geq 3$ for atleast $3i$'s, then for $n \geq 3$, $G_{nu}(R)$ has K_8 generated by the set $S = \{(1,1,\ldots,1),(1,1,q_3^m - 1,\ldots,1),(1,q_2^m-1,1,\ldots,1),(1,q_2^m-1,q_3^m-1,1,\ldots,1),(q_1^m-1,1,\ldots,1),(q_1^m-1,1,q_3^m-1,1,\ldots,1),(q_1^m-1,q_2^m-1,1,\ldots,1),(q_1^m-1,q_2^m-1,q_3^m-1,1,\ldots,1)\}$, a contradiction. Therefore, $|R_i| = 2$ for some i and hence, fix $|R_3| = |R_4| = \cdots = |R_n| = 2$, i,e., $R \cong R_1 \times R_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.
When $|R_i| \geq 11$ for atleast one i, where $i = 1, 2$, then $|U(R_i)| \geq 10$ for each $i = 1, 2$. Let $\{1, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} \in U(R_1)$ then $G_{nu}(R)$ has K_8 induced by the set $S = \{(1,1,\ldots,1),(x_1,1,\ldots,1),(x_2,1,\ldots,1),(x_3,1,\ldots,1),(x_4,1\ldots,1),(x_5,1,\ldots,1),(x_6,1,\ldots,1),(x_7,1\ldots,1),\}$, a contradiction. Hence, $|R_i| \leq 9$ for each $i = 1, 2$. Implies R_i is one of the rings $\mathbb{Z}_9$, $\mathbb{Z}_8$ $\mathbb{Z}_7$, $\mathbb{Z}_5$, $\mathbb{Z}_4$, $\mathbb{Z}_3$ or $\mathbb{Z}_2$. If $R_1 \times R_2 \cong \mathbb{Z}_9\times\mathbb{Z}_9, \mathbb{Z}_9\times\mathbb{Z}_8, \mathbb{Z}_8\times\mathbb{Z}_8, \mathbb{Z}_9\times\mathbb{Z}_7, \mathbb{Z}_9\times\mathbb{Z}_5, \mathbb{Z}_9\times\mathbb{Z}_4, \mathbb{Z}_9\times\mathbb{Z}_3, \mathbb{Z}_8\times\mathbb{Z}_7, \mathbb{Z}_7\times\mathbb{Z}_5, \mathbb{Z}_8\times\mathbb{Z}_4, \mathbb{Z}_8\times\mathbb{Z}_3, \mathbb{Z}_7\times\mathbb{Z}_7, \mathbb{Z}_7\times\mathbb{Z}_5, \mathbb{Z}_7\times\mathbb{Z}_4, \mathbb{Z}_7\times\mathbb{Z}_3, \mathbb{Z}_5\times\mathbb{Z}_5, \mathbb{Z}_5\times\mathbb{Z}_4, \mathbb{Z}_5\times\mathbb{Z}_3$, then $G_{nu}(R)$ has K_8 by [15, Theorem 6], a contradiction. Moreover, if $R \cong \mathbb{Z}_2\times\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$, $\mathbb{Z}_3\times\mathbb{Z}_2\times\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$, $\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$, $\mathbb{Z}_8\times\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$, and $\mathbb{Z}_4\times\mathbb{Z}_3\times\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$, then by Theorem 3.2, $\gamma(G_{nu}(R)) = 0$, a contradiction. Hence, $R \cong \mathbb{Z}_7\times\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$ or $\mathbb{Z}_9\times\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$.

Now, for $n = 2$ i,e., $R \cong R_1 \times R_2$, then we can encounter through the following cases :
Case(i): If R_1 and R_2 are fields and $|R_i| \geq 11$ for atleast one i, say $i = 1$. Then $|U(R_1)| \geq 10$ and hence we assume $|R_1| = p$, where p is prime and $p \geq 11$. Then $\{1, u_1, u_2, u_3, u_4, p-u_1, p-u_2, p-u_3, p-u_4\} \in U(R_1)$ such that $G_{nu}(R)$ has $K_{5,4}$ induced by the set $S = \{(1,1),(u_1,1),(u_2,1),(u_3,1),(u_4,1),(1,p-1),(u_1,p-1),(u_2,p-1),(u_3,p-1)\}$, a contradiction. Hence, $|R_i| \leq 7$ for each i. Implies R_i is one of the rings $\mathbb{Z}_7$, $\mathbb{Z}_5$, $\mathbb{Z}_3$ or $\mathbb{Z}_2$. If $R \cong \mathbb{Z}_2\times\mathbb{Z}_5$, $\mathbb{Z}_3\times\mathbb{Z}_5$, $\mathbb{Z}_5\times\mathbb{Z}_5$, $\mathbb{Z}_2\times\mathbb{Z}_3$, $\mathbb{Z}_3\times\mathbb{Z}_3$, $\mathbb{Z}_2\times\mathbb{Z}_2$, then by Theorem 3.2, $\gamma(G_{nu}(R)) = 0$, a contradiction. Hence, the possibility for R is $\mathbb{Z}_3\times\mathbb{Z}_7$, $\mathbb{Z}_5\times\mathbb{Z}_7$, $\mathbb{Z}_7\times\mathbb{Z}_7$ or $\mathbb{Z}_2\times\mathbb{Z}_7$. If $R \cong \mathbb{Z}_3\times\mathbb{Z}_7$, $\mathbb{Z}_5\times\mathbb{Z}_7$, $\mathbb{Z}_7\times\mathbb{Z}_7$ then by Lemma 5.3, it can be seen that $\gamma(G_{nu}(R))>1$, a contradiction. Hence, $R \cong \mathbb{Z}_2\times\mathbb{Z}_7$.
Case(ii): If both R_1 and R_2 are not fields, then $|R_i| = p^j$ for each i, where p is prime number and j is a positive integer with $j \geq 2$. Now, if $p \neq 2$ and $p^j \geq 27$ for atleast one i say $i = 1$, then $|U(R_i)| \geq 18$. Let $\{1, m_1-1, 2m_1-1, 3m_1-1, 4m_1-1, p^j-1\} \in U(R_i)$, where $\langle m_i\rangle$ is the non-zero maximal ideal of R_i. It can be seen that $G_{nu}(R)$ has $K_{5,4}$ generated by the set $S = \{(m_1-1,1),(2m_1-1),(3m_1-1),(4m_1-1),(p^j-1,1),(m_1-1,p^j-1),(2m_1-1,p^j-1),(3m_1-1,p^j-1),(4m_1-1,p^j-1)\}$, a contradiction. Hence, $|R_i| = p^j \leq 9$ for all $i = 1, 2$. If $R \cong \mathbb{Z}_9\times\mathbb{Z}_9$, then by Lemma 5.3, it can be seen that $\gamma(G)>1$, a contradiction. Also, in case $p = 2$, and $|R_i| \geq 8$, then by applying [15, Theorem 6], $G_{nu}(R) \cong K_{\phi(p^j),\phi(p^j)}$ and hence K_8 is a subgraph of $G_{nu}(R)$, a contradiction. Thus R_i is a ring $\mathbb{Z}_4$. If $R \cong \mathbb{Z}_4\times\mathbb{Z}_4$, then by Theorem 3.2, $\gamma(G_{nu}(R)) = 0$, a contradiction.
Case(iii): If one of the R_i is a field and other is not a field say R_1 is field and R_2 is not a field. Since both R_1 and R_2 are finite local rings. If $|R_1| = 2$ and $|R_2| \geq 16$, then $|U(R_2)| \geq 8$, let $\{1, f_1, f_2, f_3, f_4, f_5, f_6, f_7\} \in U(R_2)$ then $G_{nu}(R)$ has K_8 generated by the set $S = \{(1,1),(1,f_1),(1,f_2),(1,f_3),(1,f_4),(1,f_5),(1,f_6),(1,f_7)\}$, a contradiction. Hence, $|R_2| \leq 9$, implies R_2 is one the rings $\mathbb{Z}_9$, $\mathbb{Z}_8$ or $\mathbb{Z}_4$. If $R \cong \mathbb{Z}_2\times\mathbb{Z}_8$ or $\mathbb{Z}_2\times\mathbb{Z}_4$, then by Theorem 3.2, $\gamma(G_{nu}(R)) = 0$, a contradiction. Hence, $R \cong \mathbb{Z}_2\times\mathbb{Z}_9$. Also, if $|R_1| = 3$ and $|R_2| = p^n, n \geq 2$, where p is prime, if p is odd prime and $p^n \geq 25$ then $|U(R_2)| \geq 20$. Let $\langle m\rangle$ be the non-zero maximal ideal of R_2, then $\{m-1, 2m-1, 3m-1, 4m-1, 5m-1\} \in U(R_2)$ such that $G_{nu}(R)$ has $K_{5,4}$ generated by the set $S = \{(1,m-1),(1,2m-1),(1,3m-1),(1,4m-1),(1,5m-1),(2,m-1),(2,2m-1),(2,3m-1),(2,4m-1)\}$, a contradiction. Hence, $|R_2| \leq 9$ implies $R_2 \cong \mathbb{Z}_9$. Also, in case when p is even prime, then for $n \geq 3$, i,e., $p^n \geq 8$, $G_{nu}(R)$ has K_8 by [15, Theorem 6], a contradiction. Hence, R_2 is $\mathbb{Z}_4$ implies $R \cong \mathbb{Z}_2\times\mathbb{Z}_4$, again by Theorem 3.2, $\gamma(G_{nu}(R)) = 0$, a contradiction. Moreover, if $|R_1| = q$, where q is prime and $q \geq 11$, and $|R_2| = p^n, n \geq 2$ where p is prime and $p \neq 2$. Then $|U(R_1)| \geq 10$ and $|U(R_2)| \geq 6$. Let $\{1, f_1, f_2, f_3, f_4, q-f_1, q-f_2, q-f_3, q-f_4, q-1\} \in U(R_1)$ and $\{1, p^n-1\} \in U(R_2)$, then clearly $G_{nu}(R)$ has $K_{5,4}$ induced by the set $S = \{(1,1),(f_1,1),(f_2,1),(f_3,1),(f_4,1),(1,p^n-1),(f_1,p^n-1),(f_2,p^n-1),(f_3,p^n-1)\}$, a contradiction. Also, when $p = 2$ and $n \geq 2$, then by [15, Theorem 6], $G_{nu}(R) \cong K_{\phi(q)\times\phi(p^n)}$ and contains K_8 as induced subgraph, a contradiction. Hence, $|R_1| \leq 7$ implies $R_1 \cong \mathbb{Z}_q$, where $q = 5, 7$. Also, $|R_2| = p^n, n \geq 2$

if p is odd prime and $p^n \geq 25$, then $|U(R_2)| \geq 20$. Let $\langle m \rangle$ be the non-zero maximal ideal of R_2. Then $\{1, m-1, 2m-1, 3m-1, 4m-1, 5m-1\} \in U(R_2)$ such that $G_{nu}(R)$ has $K_{5,4}$ induced by the set $S = \{(1, m-1), (1, 2m-1), (1, 3m-1), (1, 4m-1), (1, 5m-1), (q-1, m-1), (q-1, 2m-1), (q-1, 3m-1)\}$, a contradiction. Hence, $|R_2| \leq 9$, implies $R_2 \cong \mathbb{Z}_9$. Moreover, when $p = 2$,then $G_{nu}(R)$ has K_8 by [15, Theorem 6], a contradiction. Hence, the possibility of R is $\mathbb{Z}_5 \times \mathbb{Z}_9$ or $\mathbb{Z}_7 \times \mathbb{Z}_9$. If $R \cong \mathbb{Z}_5 \times \mathbb{Z}_9$ or $\mathbb{Z}_7 \times \mathbb{Z}_9$, then by Lemma 5.3, it can be seen that $\gamma(G_{nu}(R))>1$, a contradiction.
Finally, for $n = 1$, implies R is a finite local ring. Suppose R is not a field and $|R| = p^n$, where p is prime and $n \geq 2$. If $p^n \geq 25$, then $|U(R)| \geq 20$. Let $\langle m \rangle$ be the non-zero maximal ideal of R_2 and $\langle m \rangle \neq \langle 2 \rangle$, then $G = \{1, m+1, 2m+1, 3m+1, 4m+1, m-1, 2m-1, 3m-1, 4m-1\} \in U(R)$ such that $G_{nu}(R)$ has $K_{5,4}$ generated by the set G, a contradiction. Hence, $R \cong \mathbb{Z}_9$. In case $\langle m \rangle = \langle 2 \rangle$, if $p^n \geq 16$, then $G_{nu}(R)$ has K_8 by [15, Theorem 6], a contradiction. Hence, $R \cong \mathbb{Z}_8$ or $\mathbb{Z}_4$, again by using Theorem 3.2, $\gamma(G_{nu}(R)) = 0$, a contradiction. Also, if R is a field, then $R \cong \mathbb{Z}_p$ by [15, Theorem 1], where p is prime, then by Theorem 3.2, $\gamma(G_{nu}(R)) = 0$, a contradiction.

Converse follows from Figure 3 and Figure 4 and the proof is complete. □

Conclusion:

In conclusion, finding planarity, outer planarity, split graph, and genus of a graph G can provide important insights into its properties and structure. These concepts have applications in a wide range of fields, including circuit design, network optimization, social network analysis, algorithm design, map coloring, chemical compounds, visualization, transportation networks, image processing, and game theory. Knowing the planarity or outer planarity of a graph can help to optimize network routing algorithms, simplify the layout of road networks and transit systems, and create visually appealing and informative visualizations of complex graphs. Split graphs can be used to cluster data and segment images, while the genus of a graph can provide information about its complexity, the number of subgroups or communities within the graph, and the optimal strategies for players in game theory.

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References

1. A. T. White, *Graphs, Groups and Surfaces*, North-Holland, Amsterdam, 1973.
2. A. M. Alanazi, M. Nazim, and N. Rehman., *Classification of rings with toroidal and projective coannihilator graph*, J. Math., 2021, 1-7.
3. B. Mohar and C. Thomassen, *Graphs on Surfaces*, The Johns Hopkins University Press, Baltimore and London, (1956).
4. D. B. West, *Introduction to Graph Theory*, 2nd edn. (Prentice-Hall of India, New Delhi, 2002).
5. H. R. Maimani, M. R. Pournaki and S. Yassemi, *Necessary and sufficient conditions for unit graphs to be Hamiltonian*, Pacific J. Math. **249, 2** (2011) 419-429.
6. H. Su, G. Tang and Y. Zhou, *Rings whose unit graphs are plannar*, Publ. Math. Debrecen, **86,3-4** (2015) 363-376.
7. H. Su and Y. Wei, *The diameter of unit graphs of rings*, Taiwanese J. Math. **23, 1** (2019) 1-10.
8. H. Su and Y. Zhou, *On the girth of the unit graph of a ring*, J. Algebra Appl., **13, 2** (2014) 1350082, 12 pp.
9. J. Battle, F. Harary, Y. Kodama and J.W.T. Youngs, *Additivity of the genus of a graph*, Bull. Amer. Math. Soc., 68 (1962), 565–568.
10. M. Afkhami and F. Khosh-Ahang, *Unit graphs of rings of polynomials and power series*, Arab. J. Math., **2,3** (2013) 233-246.
11. N. Ashrafi, H. R. Maimani, M. R. Pournaki and S. Yassemi, *Unit graphs associated to rings*, Comm. Algabra, **38, 8** (2010) 2851-2871.
12. N. Rehman, M. Nazim and S. A. Mir, *On the planarity, genus and crosscap of weakly zero-divisor graph of commutative rings*, Rev. Un. Mat. Argentina. DOI: 10.33044/revuma.2837
13. N. Rehman, M. Nazim and K. Selvakumar, *On the planarity, genus and crosscap of new extension of zero-divisor graph of commutative rings*, AKCE Int. J. Graphs Comb., 19(1),(2022) 61-68.

14. R. P. Grimaldi, Graphs from rings, *Proceedings of the 20th Southeastern Conference on Combinatorics, Graph Theory, and Computing* (Boca Raton, FL, 1989). Congr. Numer. **71** (1990) 95-103.

15. S. Pirzada, and A. Altaf, *Co-unit graphs associated to ring of integers modulo*. Acta Univ. Sapientiae, Mathematica, **14(2)**, pp.308-316.

16. S. Kiani, H. R. Maimani, M. R. Pournaki and S. Yassemi, *Classification of rings with unit graphs having domination number less than four*, *Rend. Sem. Mat. Univ. Padova*, **133** (2015) 173-195.

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