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Some Inequalities for Polynomials with a Multiple Zero at Origin *

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ABSTRACT: If $P(z)$ is a polynomial of degree n with origin as a zero of some multiplicity, then the estimate for $|P'(z)|$ in terms of $|P(z)|$ on $|z|=1$, when all other zeros of $P(z)$ lie in $|z|\le k,\ k\le 1$, is known [A. Aziz and W. M. Shah, *Math. Inequal. Appl.,* **7**(3)(2004), 379-391]. In this paper we prove some results in case $P(z)$ has a zero of multiplicity $s\ge 0$ at origin and all other zeros in $|z|\ge k, k\ge 1$, as well as on $|z|=k, k\le 1$. We also consider lacunary-type polynomials of degree n, by which we generalize some earlier well known results.

Key Words: Polynomial, Inequalities, Restricted zeros.

Contents

1. Introduction

For each positive integer n, let $\mathcal{P}_n$ denote the linear space of all polynomials $P(z) := \sum_{j=0}^{n} c_j z^j$ of degree atmost n over the field $\mathbb{C}$ of complex numbers. If $P\in\mathcal{P}_n$ and P' be its derivative, then concerning the estimate of $|P'(z)|$ in terms of $|P(z)|$ on $|z|=1$ for $z\in\mathbb{C}$, we have the following famous result due to Bernstein [4].

If $P\in\mathcal{P}_n$, then

$$\max_{|z|=1}|P'(z)|\le n\max_{|z|=1}|P(z)|. \tag{1.1}$$

The result is sharp and equality holds only when $P(z)$ is a constant multiple of z^n. Inequality (1.1) is an immediate consequence of a famous result due to Bernstein on the derivative of a trigonometric polynomial (for reference see [4], [11]- [13]). This inequality can be sharpened if there is a restriction on the zeros of $P(z)$. In fact, if $P(z)\ne 0$ in $|z|<1$, then

$$\max_{|z|=1}|P'(z)|\le \frac{n}{2}\max_{|z|=1}|P(z)|, \tag{1.2}$$

whereas in reverse direction, if $P(z)\ne 0$ in $|z|>1$, then (1.2) can be replaced by

$$\max_{|z|=1}|P'(z)|\ge \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.3}$$

Both the inequalities are sharp and equality in each holds for the polynomials $P(z)=\alpha z^n+\beta$, where $|\alpha|=|\beta|$. Inequality (1.2) was conjectured by Erdös and latter verified by Lax [8], whereas inequality (1.3) is due to Turán [14].

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If $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)| \tag{1.4}$$

and in reverse direction, if $P(z) \neq 0$ in $|z| > k,\ k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.5}$$

Both these inequalities are sharp and extremal polynomial is $P(z) = (z+k)^n$. The above inequalities (1.4) and (1.5) are due to Malik [9].

Concerning the estimate of $|P'(z)|$ in terms of $|P(z)|$ on $|z| = 1$ for a polynomial $P(z)$ having some zeros at the origin and the rest in $|z| \leq k,\ k \leq 1$, Aziz and Shah [3] proved the following:

Theorem 1.1 *If $P \in \mathcal{P}_n$ have all zeros in the disc $|z| \leq k,\ k \leq 1$, with s-fold zero at the origin, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n+ks}{1+k} \max_{|z|=1} |P(z)|. \tag{1.6}$$

The result is sharp and equality holds for $P(z) = z^s(z+k)^{n-s}$.

Inequality (1.6) was further refined by various authors from time to time (for reference see [10]). For lacunary-type polynomials Dewan and Hans [5] proved.

Theorem 1.2 *If $P(z) = c_n z^n + \sum\limits_{i=\eta}^{n} c_{n-i} z^{n-i}, 1 \leq \eta \leq n$, is a polynomial of degree n having all zeros on $|z| = k, k \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^{n-\eta+1}} \left(\frac{n|c_n|k^{2\eta} + \eta|c_{n-\eta}|k^{\eta-1}}{\eta|c_{n-\eta}|(1+k^{\eta-1}) + n|c_n|k^{\eta-1}(1+k^{\eta+1})} \right) \max_{|z|=1} |P(z)|. \tag{1.7}$$

2. Basic Lemmas

The following lemma, which we require for the proof of first two theorems is due to Aziz [1].

Lemma 2.1 *If $P \in \mathcal{P}_n$ and $P^*(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then for $|z| = 1$ and for every real α,*

$$|P'(z)|^2 + |(P^*(z))'|^2 \leq \frac{n^2}{2}(M_\alpha^2 + M_{\alpha+\pi}^2), \tag{2.1}$$

where $M_\alpha = \max\limits_{1 \leq k \leq n} |P(e^{\frac{i(\alpha+2k\pi)}{n}})|$ and $M_{\alpha+\pi}$ is obtained from M_α by replacing α by $\alpha + \pi$..

3. Main Results

It is natural to ask what happens to the bound in case of discrete norm for the polynomial $P(z)$ of degree n having a zero of some multiplicity at origin and the rest of the zeros in $|z| \leq k$ or $|z| \geq k, k > 0$. As an answer to this question, here we prove the following results which generalize a results due to Aziz [1].

Theorem 3.1 *If $P \in \mathcal{P}_n$ have origin as its zero of multiplicity $s \geq 0$ and all other zeros lie in the disc $|z| \geq 1$, then for $|z| = 1$*

$$|P'(z)| \leq \frac{n}{2}\left[M_\alpha^2 + M_{\alpha+\pi}^2 + \frac{2}{n}s|P(z)|^2\right]^{\frac{1}{2}}, \tag{3.1}$$

where M_α and $M_{\alpha+\pi}$ are defined in Lemma 2.1.

As a generalization of Theorem 3.1, we prove the following :

Theorem 3.2 *If $P \in \mathcal{P}_n$ have all zeros in the disc $|z| \geq k,\ k \geq 1$ except s-fold zero at origin, then for $|z| = 1$*

$$|P'(z)| \leq \frac{n}{2}\left\{M_\alpha^2 + M_{\alpha+\pi}^2 + \frac{2}{n}\left(\frac{n+2ks-nk}{1+k}\right)|P(z)|^2\right\}^{\frac{1}{2}}, \tag{3.2}$$

where M_α and $M_{\alpha+\pi}$ are defined in Lemma 2.1.

Next we prove the following result which is an extension of Theorem 1.2 to the polynomials of the type $P(z) = z^s\big(c_{n-s}z^{n-s} + \sum\limits_{t=\eta}^{n-s} c_{n-s-t}z^{n-s-t}\big), 1 \leq \eta \leq n-s, 0 \leq s \leq n-1$.

Theorem 3.3 *Let $P(z) = z^s\big(c_{n-s}z^{n-s} + \sum\limits_{t=\eta}^{n-s} c_{n-s-t}z^{n-s-t}\big), 1 \leq \eta \leq n-s, 0 \leq s \leq n-1$ be a polynomial of degree n, having all zeros on $|z| = k, k \leq 1$ except s-fold zeros at origin, then*

$$\max_{|z|=1}|P'(z)| \leq \frac{1}{k^{n-s-\eta+1}}\left(\frac{(n-s)L + sT}{\eta|c_{n-s-\eta}|(1+k^{\eta-1}) + (n-s)|c_{n-s}|k^{\eta-1}(1+k^{\eta+1})}\right)\max_{|z|=1}|P(z)|,$$

where

$$L = (n-s)|c_{n-s}|k^{2\eta} + \eta|c_{n-s-\eta}|k^{\eta-1}$$

and

$$T = k^{n-s-\eta+1}\left(\eta|c_{n-s-\eta}|(1+k^{\eta-1}) + (n-s)|c_{n-s}|k^{\eta-1}(1+k^{\eta+1})\right).$$

Remark 3.1 For $s = 0$, Theorem 3.3 reduces to the result due to Dewan and Hans [5] and for $\eta = 1, s = 0$, Theorem 3.3 reduces to the result of Dewan and Mir [6].

4. Proofs of Theorems

Proof of Theorem 3.1: Since all the zeros of the polynomial $P(z)$ of degree n lie in $|z| \geq 1$ and origin is its zero of multiplicity $s \geq 0$, therefore we can write

$$P(z) = z^s h(z), \tag{4.1}$$

where $h(z)$ is a polynomial of degree $n-s$ having all zeros in $|z| \geq 1$. We write

$$h(z) = \prod_{j=1}^{n-s}(z - z_j),$$

where $z_1, z_2, \ldots, z_{n-s}$ are zeros of $h(z)$, $|z_j| \geq 1$, $j = 1, \ldots, n-s$. Therefore

$$\frac{h'(z)}{h(z)} = \sum_{j=1}^{n-s}\frac{1}{z - z_j}$$

From (4.1), we have

$$\begin{aligned}\frac{zP'(z)}{P(z)} &= s + \frac{zh'(z)}{h(z)},\\ &= s + \sum_{j=1}^{n-s}\frac{z}{z - z_j}.\end{aligned}$$

This gives,

$$\Re e\frac{zP'(z)}{P(z)} = s + \Re e\sum_{j=1}^{n-s}\frac{z}{z-z_j}, \quad |z_j| \geq 1. \tag{4.2}$$

It can be easily verified that for $|z| = 1$ and $|z_j| \geq 1,\ j = 1, 2, \dots, n-s$,

$$\Re e\sum_{j=1}^{n-s}\frac{z}{z-z_j} \leq \frac{n-s}{2}.$$

Using this in (4.2), we get

$$\Re e\frac{zP'(z)}{P(z)} \leq \frac{n+s}{2}. \tag{4.3}$$

Now if $P^*(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then it can be easily verified that for $|z| = 1$

$$|(P^*(z))'| = |nP(z) - zP'(z)|. \tag{4.4}$$

This gives by using inequality (4.3),

$$\begin{aligned}\left|\frac{(P^*(z))'}{P(z)}\right|^2 &= \left|n - \frac{zP'(z)}{P(z)}\right|^2\\ &= n^2 + \left|z\frac{P'(z)}{P(z)}\right|^2 - 2n\Re e\left(z\frac{P'(z)}{P(z)}\right)\\ &\geq \left|z\frac{P'(z)}{P(z)}\right|^2 - ns, \quad |z| = 1.\end{aligned}$$

That is

$$\left\{|P'(z)|^2 - ns|P(z)|^2\right\} \leq |(P^*(z))'|^2 \quad |z| = 1. \tag{4.5}$$

Therefore using Lemma 2.1, we get

$$\begin{aligned}2|P'(z)|^2 - ns|P(z)|^2 &\leq |(P^*(z))'|^2 + |P'(z)|^2\\ &\leq \frac{n^2}{2}(M_\alpha^2 + M_{\alpha+\pi}^2).\end{aligned}$$

Equivalently, we get for $|z| = 1$

$$|P'(z)| \leq \frac{n}{2}\left[M_\alpha^2 + M_{\alpha+\pi}^2 + \frac{2}{n}s|P(z)|^2\right]^{\frac{1}{2}}.$$

This completes the proof of Theorem 3.1.

Proof of Theorem 3.2: Since $P(z)$ has all zeros in $|z| \geq k, \quad k \geq 1$, except $s-fold$ zero at origin, therefore

$$P(z) = z^s f(z).$$

Let $z_1, z_2, \dots, z_{n-s}$ be the zeros of $f(z)$, so that we can write

$$f(z) = \prod_{j=1}^{n-s}(z - z_j), |z_j| \geq k \geq 1.$$

This gives as in the proof of Theorem 3.1

$$\Re e\frac{zP'(z)}{P(z)} = s + \Re e\sum_{j=1}^{n-s}\frac{z}{z-z_j}. \tag{4.6}$$

Since $k \geq 1$, it can be easily verified that

$$\Re e\left\{\frac{z}{z-z_j}\right\} \leq \frac{1}{1+k},$$

Using this in (4.6) we get

$$\Re e\frac{zP'(z)}{P(z)} \leq \frac{n+ks}{1+k}. \tag{4.7}$$

Again as in the proof of Theorem 3.1, we obtain by using inequality (4.7) for $|z| = 1$

$$\begin{aligned}\left|\frac{(P^*(z))'}{P(z)}\right|^2 &= n^2 + \left|\frac{zP'(z)}{P(z)}\right|^2 - 2n\Re e\left(\frac{zP'(z)}{P(z)}\right)\\ &\geq n^2 + \left|\frac{zP'(z)}{P(z)}\right|^2 - 2n\frac{n+ks}{1+k}\\ &= \left|\frac{zP'(z)}{P(z)}\right|^2 + \frac{n(nk-n-2ks)}{1+k}.\end{aligned}$$

Equivalently,

$$|P'(z)|^2 + \left\{\frac{n(nk-n-2ks)}{1+k}\right\}|P(z)|^2 \leq |(P^*(z))'|^2. \tag{4.8}$$

From inequality (4.8), it easily follows by using Lemma 2.1

$$\left\{2|P'(z)|^2 + \left\{\frac{n(nk-n-2ks)}{1+k}\right\}|P(z)|^2\right\} \leq \frac{n^2}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2\right). \tag{4.9}$$

From (4.9) we conclude that for $|z| = 1$,

$$|P'(z)| \leq \frac{n}{2}\left\{M_\alpha^2 + M_{\alpha+\pi}^2 + \frac{2}{n}\left(\frac{n+2ks-nk}{1+k}\right)|P(z)|^2\right\}^{\frac{1}{2}}.$$

This completes proof of Theorem 3.2.

Proof of Theorem 3.3: Let $P(z) = z^s\phi(z)$, where

$$\phi(z) = \left(c_{n-s}z^{n-s} + \sum_{t=\eta}^{n-s} c_{n-s-t}z^{n-s-t}\right),$$

is a polynomial of degree $n-s$, having all its zeros on $|z| = k, k \leq 1$. Using inequality (1.7) for $\phi(z)$, we get

$$\max_{|z|=1}|\phi'(z)| \leq \frac{n-s}{k^{n-s-\eta+1}}\left(\frac{(n-s)|c_{n-s}|k^{2\eta} + \eta|c_{n-s-\eta}|k^{\eta-1}}{\eta|c_{n-s-\eta}|(1+k^{\eta-1}) + (n-s)|c_{n-s}|k^{\eta-1}(1+k^{\eta+1})}\right)\max_{|z|=1}|\phi(z)|. \tag{4.10}$$

Now we have

$$\begin{aligned}zP'(z) &= sz^s\phi(z) + z^{s+1}\phi'(z)\\ &= sP(z) + z^{s+1}\phi'(z).\end{aligned}$$

For $|z| = 1$, we have

$$|P'(z)| \leq s|P(z)| + |\phi'(z)|.$$

This implies

$$\max_{|z|=1}|P'(z)| \leq s\max_{|z|=1}|P(z)| + \max_{|z|=1}|\phi'(z)|. \tag{4.11}$$

Combining (4.10) and (4.11), we get using $\max_{|z|=1}|\phi(z)| = \max_{|z|=1}|P(z)|$

$$\max_{|z|=1}|P'(z)|$$
$$\leq s\max_{|z|=1}|P(z)| + \frac{n-s}{k^{n-s-\eta+1}}\left(\frac{(n-s)|c_{n-s}|k^{2\eta}+\eta|c_{n-s-\eta}|k^{\eta-1}}{\eta|c_{n-s-\eta}|(1+k^{\eta-1})+(n-s)|c_{n-s}|k^{\eta-1}(1+k^{\eta+1})}\right)\max_{|z|=1}|P(z)|.$$

On solving the above inequality, we get

$$\max_{|z|=1}|P'(z)| \leq \frac{1}{k^{n-s-\eta+1}}\left(\frac{(n-s)L+sT}{\eta|c_{n-s-\eta}|(1+k^{\eta-1})+(n-s)|c_{n-s}|k^{\eta-1}(1+k^{\eta+1})}\right)\max_{|z|=1}|P(z)|,$$

where L, T are defined in the statement of theorem.
This completely proves Theorem 3.3.
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